SIXTH EDITION

Mathematical Applications

For the MANAGEMENT, LIFE, AND SOCIAL SCIENCES

HARSHBARGER

REYNOLDS

VOLUME TWO

Index of Selected Applications

SIXTH EDITION

Mathematical Applications

For the MANAGEMENT, LIFE, AND SOCIAL SCIENCES

RONALD J. HARSHBARGER

UNIVERSITY OF SOUTH CAROLINA BEAUFORT

JAMES J. REYNOLDS

CLARION UNIVERSITY OF PENNSYLVANIA

VOLUME TWO

HOUGHTON MIFFLIN COMPANY

BOSTON NEW YORK

Sponsoring Editor: Paul Murphy
Associate Editor: Mary Beckwith
Senior Project Editor: Maria A. Morelli
Editorial Assistant: Lauren M. Gagliardi
Senior Production/Design Coordinator: Jennifer Waddell
Senior Manufacturing Coordinator: Marie Barnes

Custom Publishing Editor: Gina Montoya
Custom Publishing Production Manager: Kathleen McCourt
Project Coordinator: Kayla Whittet

Cover Design: Galen B. Murphy
Cover Art: PhotoDisc, Inc.

Printed in the United States of America.

ISBN: 0-618-25801-9
N00959

1 2 3 4 5 6 7 8 9 – DMI – 04 03 02

Houghton Mifflin
Custom Publishing

222 Berkeley Street • Boston, MA 02116

Address all correspondence and order information to the above address.

trol pill have, on average, different blood pressure from the women who are not using the pill. That is, there is statistically significant evidence that the pill affects blood pressure of women.

Because 95% of all normally distributed data points lie within 2 standard deviations of the mean, researchers at Bering Labs need to determine whether the sample mean $\bar{x} = 112.5$ is more than 2 standard deviations from $\mu = 110$. To determine how many standard deviations $\bar{x} = 112.5$ is from $\mu = 110$, they compute the z-score for $\bar{x}$ with the formula

$$z_{\bar{x}} = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

Chapter 9

Derivatives

If a firm receives \$30,000 in revenue during a 30-day month, its average revenue per day is \$30,000/30 = \$1000. This does not necessarily mean that the actual revenue was \$1000 on any one day, just that the average is \$1000 per day. Similarly, if a person drives a car 50 miles in an hour's time, the car's average velocity is 50 miles per hour, but the driver could have gotten a speeding ticket for traveling 70 miles per hour on this trip. When we say a car is moving at a velocity of 50 miles per hour, we are talking about the velocity of the car at an instant in time (the instantaneous velocity). We can use the average velocity to find the instantaneous velocity, as follows.

If a car travels in a straight line from position y_1 *at time* x_1 *and arrives at position* y_2 *at time* x_2*, then it has traveled the distance* $y_2 - y_1$ *in the elapsed time* $x_2 - x_1$*. If we represent the distance traveled by* Δy *and the elapsed time by* Δx*, the average velocity is given by*

$$V_{av} = \frac{\Delta y}{\Delta x}$$

The smaller the time interval, the nearer the average velocity will be to the instantaneous velocity. For example, knowing that a car traveled 50 miles in an hour does not tell us much about its instantaneous velocity at any time during that hour. But knowing that it traveled 1 mile in 1 minute, or 50 feet in one second, tells us much more about the velocity at a given time. Continuing to decrease the length of the time interval (Δx) will get us closer and closer to the instantaneous velocity.

Some police departments have equipment that measures how fast a car is traveling by measuring how much time elapses while the car travels between two sensors placed 60 inches apart on the road. This is not the instantaneous velocity of the car, but it is an excellent approximation.

We define the ***instantaneous velocity*** *to be the* limit *of* $\Delta y/\Delta x$ *as* Δx *approaches 0. We write this as*

$$V = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Thus we may think of velocity as the instantaneous rate of change of distance with respect to time.

This chapter is concerned with limits *and* rates of change. *We will see that the* derivative *of a function can be used to determine the rate of change of the dependent variable with respect to the independent variable. In this chapter the derivative will be used to find the marginal profit, marginal cost, and marginal revenue, given the respective profit, total cost, and total revenue functions, and we will find other rates of change, such as rates of change of populations and velocity. We will also use the derivative to determine the slope of a tangent to a curve at a point on the curve. In the next chapter, more applications of the derivative will be discussed. For example, we will use differentiation to minimize average cost, maximize total revenue, maximize profit, and find the maximum dosage for certain medications.*

9.1 Limits

OBJECTIVES

- *To use graphs and numerical tables to find limits of functions, when they exist*
- *To find limits of polynomial functions*
- *To find limits of rational functions*

APPLICATION PREVIEW

Although everyone recognizes the value of eliminating any and all particulate pollution from smokestack emissions of factories, company owners are concerned about the cost of removing this pollution. Suppose that USA Steel has shown that the cost C of removing p percent of the particulate pollution from the emissions at one of its plants is

$$C = C(p) = \frac{7300p}{100 - p}$$

To investigate the cost of removing as much of the pollution as possible, we can evaluate the **limit** as p (the percent) approaches 100 from values less than 100. Using a limit is important in this case, because we cannot evaluate this function at $p = 100$.

We have used the notation $f(c)$ to indicate the value of a function $f(x)$ at $x = c$. If we need to discuss a value that $f(x)$ approaches as x approaches c, we use the idea of a **limit.** For example, if

$$f(x) = \frac{x^2 - x - 6}{x + 2}$$

then we know that -2 is not in the domain of $f(x)$ so that $f(-2)$ does not exist. Figure 9.1 shows the graph of $y = f(x)$ with an open circle where $x = -2$. Even though $f(-2)$ is not defined, the figure shows that as x approaches -2 from either side of -2, the graph approaches the open circle at $(-2, -5)$ and the values of $f(x)$ approach -5. Thus -5 is the limit of $f(x)$ as x approaches -2, and we write

$$\lim_{x \to -2} f(x) = -5, \quad \text{or} \quad f(x) \to -5 \quad \text{as} \quad x \to -2$$

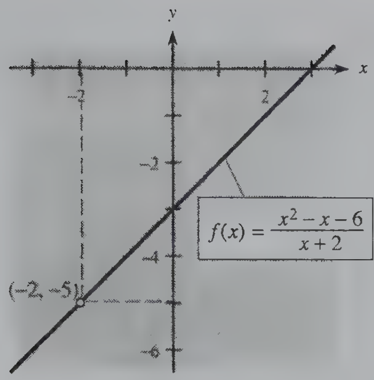


Figure 9.1

TABLE 9.1

Left of −2	
x	$f(x) = \frac{x^2 - x - 6}{x + 2}$
−3.000	−6.000
−2.500	−5.500
−2.100	−5.100
−2.010	−5.010
−2.001	−5.001
Right of −2	
x	$f(x) = \frac{x^2 - x - 6}{x + 2}$
−1.000	−4.000
−1.500	−4.500
−1.900	−4.900
−1.990	−4.990
−1.999	−4.999

This conclusion is fairly obvious from the graph, but it is not so obvious from the equation for $f(x)$.

We can use the values near $x = -2$ in Table 9.1 to help verify that $f(x) \to -5$ as $x \to -2$. Note that to the left of -2, the values of x increase from -3.000 to -2.001 in small increments, and in the corresponding column for $f(x)$, the values of the function $f(x)$ increase from -6.000 to -5.001. To the right of -2, the values of x decrease from -1.000 to -1.999 while the corresponding values of $f(x)$ decrease from -4.000 to -4.999. Hence, Table 9.1 and Figure 9.1 indicate that the value of $f(x)$ approaches -5 as x approaches -2 from both sides of $x = -2$.

From our discussion of the graph in Figure 9.1 and Table 9.1, we see that as x approaches -2 from either side of -2, the limit of the function is the value L that the function approaches. This limit L is not necessarily the value of the function at $x = -2$.

EXAMPLE 1

Figure 9.2 shows three functions for which the limit exists as x approaches 2. Use this figure to find the following.

(a) $\lim_{x\to 2} f(x)$ and $f(2)$ (if it exists)
(b) $\lim_{x\to 2} g(x)$ and $g(2)$ (if it exists)
(c) $\lim_{x\to 2} h(x)$ and $h(2)$ (if it exists)

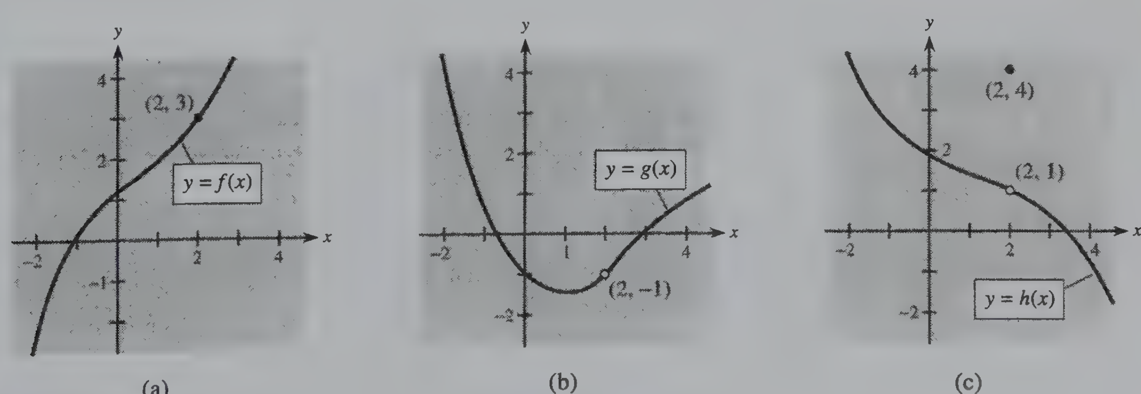


Figure 9.2

Solution

(a) From the graph in Figure 9.2(a), we see that as x approaches 2 from both the left and the right, the graph approaches the point (2, 3). Thus $f(x)$ approaches the single value 3. That is,

$$\lim_{x \to 2} f(x) = 3$$

The value of $f(2)$ is the y-coordinate of the point on the graph at $x = 2$. Thus $f(2) = 3$.

(b) Figure 9.2(b) shows that as x approaches 2 from both the left and the right, the graph approaches the open circle at $(2, -1)$. Thus

$$\lim_{x \to 2} g(x) = -1$$

The figure also shows that at $x = 2$ there is no point on the graph. Thus $g(2)$ is undefined.

(c) Figure 9.2(c) shows that

$$\lim_{x \to 2} h(x) = 1$$

The figure also shows that at $x = 2$ there is a point on the graph at (2, 4). Thus $h(2) = 4$, and we see that $\lim\limits_{x \to 2} h(x) \neq h(2)$.

As Example 1 shows, the limit of the function as x approaches c may or may not be the same as the value of the function at $x = c$. This leads to our intuitive definition of *limit*.

Limit Let $f(x)$ be a function defined on an open interval containing c, except perhaps at c. Then

$$\lim_{x \to c} f(x) = L$$

is read "the limit of $f(x)$ as x approaches c equals L." The number L exists if we can make values of $f(x)$ as close to L as we desire by choosing values of x sufficiently close to c. When the values of $f(x)$ do not approach a single finite value L as x approaches c, we say the limit does not exist.

As the definition states, a limit as $x \to c$ can exist only if the function approaches a single finite value as x approaches c from both the left and right of c. In the next example, we consider some cases where a limit does not exist.

EXAMPLE 2

Using the functions graphed in Figure 9.3, determine why the limit as $x \to 2$ does not exist for

(a) $f(x)$ (b) $g(x)$ (c) $h(x)$

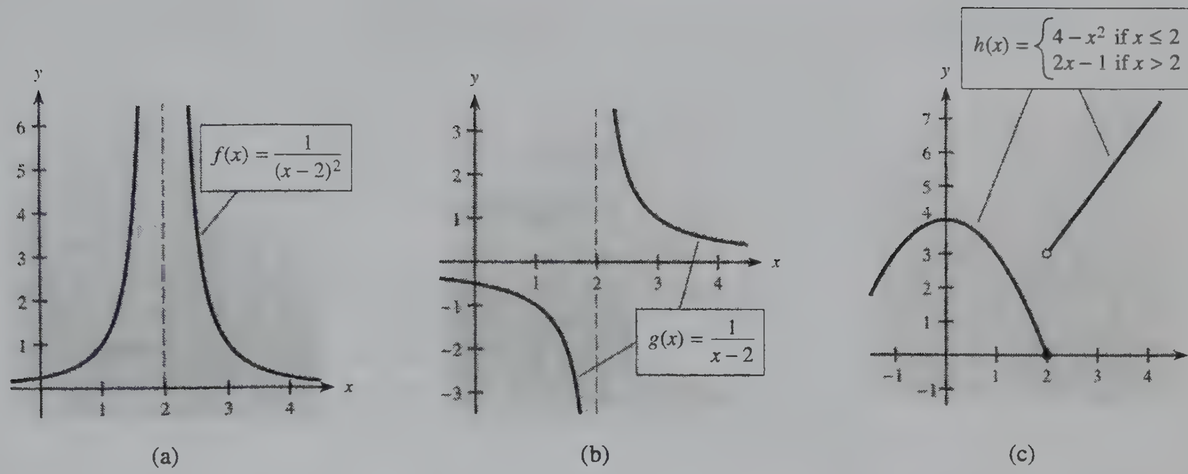


Figure 9.3

Solution

(a) As $x \to 2$ from the left side and the right side of $x = 2$, $f(x)$ increases without bound, which we denote by saying that $f(x)$ approaches $+\infty$ as $x \to 2$. In this case, $\lim_{x\to 2} f(x)$ does not exist (briefly noted by DNE) because $f(x)$ does not approach a finite value as $x \to 2$. The graph has a vertical asymptote at $x = 2$. We say that this is an **infinite limit.**

(b) As $x \to 2$ from the left, $g(x)$ approaches $-\infty$, and as $x \to 2$ from the right, $g(x)$ approaches $+\infty$, so $g(x)$ does not approach a finite value as $x \to 2$. Therefore, the limit does not exist. The graph of $y = g(x)$ has a vertical asymptote at $x = 2$.

(c) As $x \to 2$ from the left (while $x < 2$, denoted by $x \to 2^-$), the graph approaches the point at (2, 0), so $h(x)$ approaches the value 0, but as $x \to 2$ from the right (while $x > 2$, denoted by $x \to 2^+$), the graph approaches the open circle at (2, 3), so $h(x)$ approaches the value 3. Because $h(x)$ approaches two different numbers as x approaches 2, the limit does not exist.

Examples 1 and 2 illustrate the following two important facts regarding limits.

1. The limit of a function as x approaches c is independent of the value of the function at c. When $\lim_{x\to c} f(x)$ exists, the value of the function at c may be (i) the same as the limit, (ii) undefined, or (iii) defined but different from the limit (see Figure 9.2 and Example 1).
2. The limit is said to exist only if the following conditions are satisfied:
   (a) The limit L is a finite value (real number).
   (b) The limit as x approaches c from the left equals the limit as x approaches c from the right. That is, we must have

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$$

Figure 9.3 and Example 2 illustrate cases where $\lim_{x\to c} f(x)$ does not exist.

CHECKPOINT

1. Can $\lim_{x \to c} f(x)$ exist if $f(c)$ is undefined?
2. Does $\lim_{x \to c} f(x)$ exist if $f(c) = 0$?
3. Does $f(c) = 1$ if $\lim_{x \to c} f(x) = 1$?
4. If $\lim_{x \to c} f(x) = 0$, does $\lim_{x \to c} f(x)$ exist?

Fact 1 regarding limits tells us that the value of the limit of a function as $x \to c$ will not always be the same as the value of the function at $x = c$. However, there are many functions for which the limit and the functional value agree [see Figure 9.2(a)], and for these functions we can easily evaluate limits. The following properties of limits allow us to identify certain classes or types of functions for which $\lim_{x \to c} f(x)$ equals $f(c)$.

Properties of Limits

If k is a constant, $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then the following are true.

I. $\lim_{x \to c} k = k$

II. $\lim_{x \to c} x = c$

III. $\lim_{x \to c} [f(x) \pm g(x)] = L \pm M$

IV. $\lim_{x \to c} [f(x) \cdot g(x)] = LM$

V. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ if $M \neq 0$

VI. $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} = \sqrt[n]{L}$

If f is a polynomial function, then Properties I–IV imply that $\lim_{x \to c} f(x)$ can be found by evaluating $f(c)$. Moreover, if h is a rational function whose denominator is not zero at $x = c$, then Property V implies that $\lim_{x \to c} h(x)$ can be found by evaluating $h(c)$. The following summarizes these observations and recalls the definitions of polynomial and rational functions.

Function	Definition	Limit
Polynomial function	The function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where n is a positive integer, is called a **polynomial function.**	$\lim_{x \to c} f(x) = f(c)$ for all values c (by Properties I–IV)
Rational function	The function $h(x) = \dfrac{f(x)}{g(x)}$ where both $f(x)$ and $g(x)$ are polynomial functions, is called a **rational function.**	$\lim_{x \to c} h(x) = \lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{f(c)}{g(c)}$ when $g(c) \neq 0$ (by Property V)

EXAMPLE 3

Find the following limits, if they exist.

(a) $\lim_{x \to -1} (x^3 - 2x)$ (b) $\lim_{x \to 4} \dfrac{x^2 - 4x}{x - 2}$

Solution

(a) Note that $f(x) = x^3 - 2x$ is a polynomial, so

$$\lim_{x \to -1} f(x) = f(-1) = (-1)^3 - 2(-1) = 1$$

Figure 9.4(a) shows the graph of $f(x) = x^3 - 2x$.

(b) Note that this limit has the form

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are polynomials and $g(c) \neq 0$. Therefore, we have

$$\lim_{x \to 4} \frac{x^2 - 4x}{x - 2} = \frac{4^2 - 4(4)}{4 - 2} = \frac{0}{2} = 0$$

Figure 9.4(b) shows the graph of $g(x) = \dfrac{x^2 - 4x}{x - 2}$.

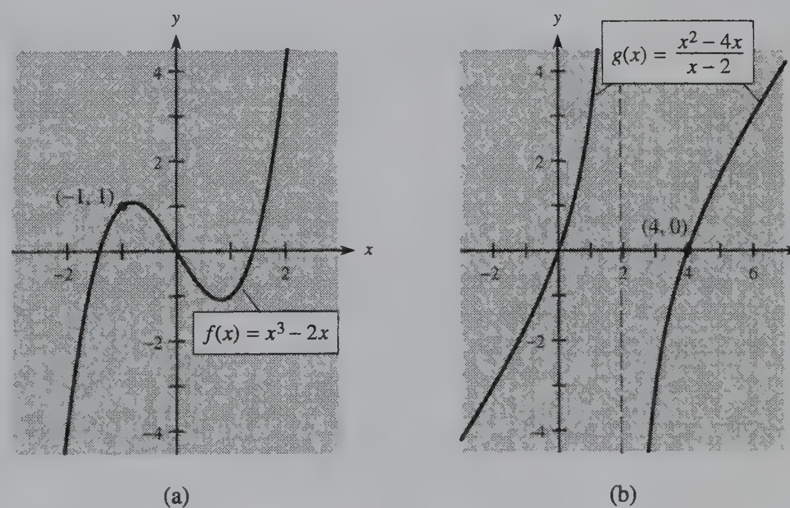


Figure 9.4

We have seen that we can use Property V to find the limit of a rational function $f(x)/g(x)$ as long as the denominator is *not* zero. If the limit of the denominator of $f(x)/g(x)$ *is* zero, then there are two possible cases.

I. Both $\lim_{x \to c} g(x) = 0$ and $\lim_{x \to c} f(x) = 0$, or

II. $\lim_{x \to c} g(x) = 0$ and $\lim_{x \to c} f(x) \neq 0$.

In case I we say that $f(x)/g(x)$ has the form 0/0 at $x = c$. We call this the **0/0 indeterminate form;** the limit cannot be evaluated until $x - c$ is factored from both $f(x)$ and $g(x)$ and the fraction is reduced. Example 4 will illustrate this case.

In case II, the limit has the form $a/0$, where a is a constant, $a \neq 0$. This expression is undefined, and the limit does not exist. Example 5 will illustrate this case.

EXAMPLE 4

Evaluate the following limits, if they exist.

(a) $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

(b) $\lim\limits_{x \to 1} \dfrac{x^2 - 3x + 2}{x^2 - 1}$

Solution

(a) We cannot find the limit by using Property V because the denominator is zero at $x = 2$. The numerator is also zero at $x = 2$, so the expression

$$\frac{x^2 - 4}{x - 2}$$

has the 0/0 indeterminate form at $x = 2$. Thus we can factor $x - 2$ from both the numerator and the denominator and reduce the fraction. (We can divide by $x - 2$ because $x - 2 \neq 0$ while $x \to 2$.)

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

Figure 9.5(a) shows the graph of $f(x) = (x^2 - 4)/(x - 2)$. Note the open circle at (2, 4).

(b) By substituting 1 for x in $(x^2 - 3x + 2)/(x^2 - 1)$, we see that the expression has the 0/0 indeterminate form at $x = 1$, so $x - 1$ is a factor of both the numerator and the denominator. (We can then reduce the fraction because $x - 1 \neq 0$ while $x \to 1$.)

$$\begin{aligned}\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^2 - 1} &= \lim_{x \to 1} \frac{(x - 1)(x - 2)}{(x - 1)(x + 1)} \\ &= \lim_{x \to 1} \frac{x - 2}{x + 1} \\ &= \frac{1 - 2}{1 + 1} = \frac{-1}{2} \quad \text{(by Property V)}\end{aligned}$$

Figure 9.5(b) shows the graph of $g(x) = (x^2 - 3x + 2)/(x^2 - 1)$. Note the open circle at $\left(1, -\frac{1}{2}\right)$.

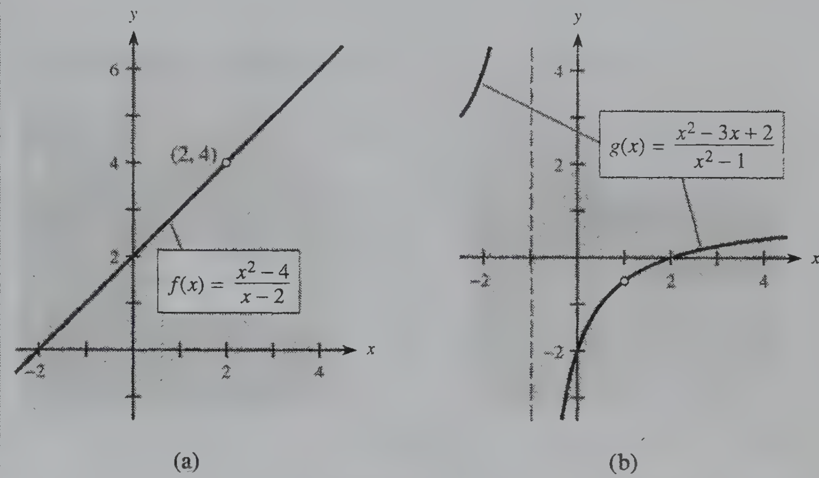


Figure 9.5 (a) (b)

Note that although both problems in Example 4 had the 0/0 indeterminate form, they had different answers.

EXAMPLE 5

Find $\lim_{x \to 1} \dfrac{x^2 + 3x + 2}{x - 1}$, if it exists.

Solution

Substituting 1 for x in the function results in 6/0, so this limit has the form $a/0$, with $a \neq 0$, and is like case II discussed previously. Hence the limit does not exist. Because the numerator is not zero when $x = 1$, we know that $x - 1$ is *not* a factor of the numerator, and we cannot divide numerator and denominator as we did in Example 4. Table 9.2 confirms that this limit does not exist, because the values of the expression are unbounded near $x = 1$.

TABLE 9.2

Left of $x = 1$		*Right of* $x = 1$	
x	$\dfrac{x^2 + 3x + 2}{x - 1}$	x	$\dfrac{x^2 + 3x + 2}{x - 1}$
0	−2	2	12
0.5	−7.5	1.5	17.5
0.7	−15.3	1.2	35.2
0.9	−55.1	1.1	65.1
0.99	−595.01	1.01	605.01
0.999	−5,995.001	1.001	6,005.001
0.9999	−59,999.0001	1.001	60,005.0001

$\lim_{x \to 1^-} \dfrac{x^2 + 3x + 2}{x - 1} = -\infty$ $\qquad$ $\lim_{x \to 1^+} \dfrac{x^2 + 3x + 2}{x - 1} = +\infty$

The left-hand and right-hand limits are not finite, so they do not exist. Thus $\lim_{x \to 1} \frac{x^2 + 3x + 2}{x - 1}$ does not exist.

The results of Examples 4 and 5 can be summarized as follows:

Rational Functions: Evaluating Limits of the Form $\lim_{x \to c} \frac{f(x)}{g(x)}$ ***where*** $\lim_{x \to c} g(x) = 0$

Type I. If $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$, then the fractional expression has the **0/0 indeterminate form** at $x = c$. We can factor $x - c$ from $f(x)$ and $g(x)$, reduce the fraction, and then find the limit of the resulting expression, if it exists.

Type II. If $\lim_{x \to c} f(x) \neq 0$ and $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} \frac{f(x)}{g(x)}$ does not exist. In this case, the values of $f(x)/g(x)$ are unbounded near $x = c$.

In Example 5, even though the left-hand and right-hand limits do not exist (see Table 9.2), knowledge that the functional values are unbounded (that is, that they become infinite) is helpful in graphing. The graph is shown in Figure 9.6. We see that $x = 1$ is a vertical asymptote.

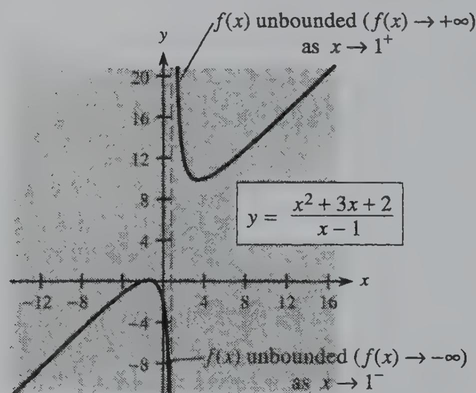


Figure 9.6

EXAMPLE 6

As mentioned in the Application Preview, USA Steel has shown that the cost C of removing p percent of the particulate pollution from the smokestack emissions at one of its plants is

$$C = C(p) = \frac{7300p}{100 - p}$$

To investigate the cost of removing as much of the pollution as possible, find:

(a) the cost of removing 50% of the pollution.
(b) the cost of removing 90% of the pollution.
(c) the cost of removing 99% of the pollution.
(d) the cost of removing 100% of the pollution.

Solution

(a) The cost of removing 50% of the pollution is \$7300 because

$$C(50) = \frac{7300(50)}{100 - 50} = \frac{365{,}000}{50} = 7300$$

(b) The cost of removing 90% of the pollution is \$65,700 because

$$C(90) = \frac{7300(90)}{100 - 90} = \frac{657{,}000}{10} = 65{,}700$$

(c) The cost of removing 99% of the pollution is \$722,700 because

$$C(99) = \frac{7300(99)}{100 - 99} = \frac{722{,}700}{1} = 722{,}700$$

(d) The cost of removing 100% of the pollution is undefined because the denominator of the function is 0 when $p = 100$. To see what the cost approaches as p approaches 100 from values smaller than 100, we evaluate $\lim\limits_{x \to 100^-} \dfrac{7300p}{100 - p}$. This limit has the Type II form for rational functions. Thus $\lim\limits_{x \to 100^-} \dfrac{7300}{100 - p} = +\infty$, which means that the cost of removing 100% of the pollution approaches infinity. (That is, it is impossible to remove 100% of the pollution.)

CHECKPOINT

5. Evaluate the following limits (if they exist).

(a) $\lim\limits_{x \to -3} \dfrac{2x^2 + 5x - 3}{x^2 - 9}$ (b) $\lim\limits_{x \to 5} \dfrac{x^2 - 3x - 3}{x^2 - 8x + 1}$ (c) $\lim\limits_{x \to -3/4} \dfrac{4x}{4x + 3}$

Assume that f, g, and h are polynomials.

6. Does $\lim\limits_{x \to c} f(x) = f(c)$? 7. Does $\lim\limits_{x \to c} \dfrac{g(x)}{h(x)} = \dfrac{g(c)}{h(c)}$?

8. If $g(c) = 0$ and $h(c) = 0$, can we be certain that

(a) $\lim\limits_{x \to c} \dfrac{g(x)}{h(x)} = 0$? (b) $\lim\limits_{x \to c} \dfrac{g(x)}{h(x)}$ exists?

9. If $g(c) \neq 0$ and $h(c) = 0$, what can be said about $\lim\limits_{x \to c} \dfrac{g(x)}{h(x)}$ and $\lim\limits_{x \to c} \dfrac{h(x)}{g(x)}$?

As we noted in Section 2.4, "Special Functions and Their Graphs," many applications are modeled by piecewise defined functions. To see how we evaluate a limit involving a piecewise defined function, consider the following example.

EXAMPLE 7

Find $\lim\limits_{x \to 1^-} f(x)$, $\lim\limits_{x \to 1^+} f(x)$, and $\lim\limits_{x \to 1} f(x)$, if they exist for

$$f(x) = \begin{cases} x^2 + 1 \text{ for } x \leq 1 \\ x + 2 \text{ for } x > 1 \end{cases}$$

Solution

We select values of x that approach 1 from the left and right and evaluate f at these values (see Table 9.3). For values of x less than 1, we use $f(x) = x^2 + 1$ to find the value of the function (see the left-hand side of the table). For values of x that are greater than 1, we use $f(x) = x + 2$ to find the value of f (see the right-hand side of the table).

TABLE 9.3

Left of 1		*Right of* 1	
x	$f(x) = x^2 + 1$	x	$f(x) = x + 2$
0.1	1.01	1.2	3.2
0.9	1.81	1.01	3.01
0.99	1.98	1.001	3.001
0.999	1.998	1.0001	3.0001
0.9999	1.9998	1.00001	3.00001

In this case, we observe that $f(x)$ appears to be approaching 2 as x approaches 1 from the left, whereas $f(x)$ appears to be approaching 3 as x approaches 1 from the right. Figure 9.7 shows the graph of $f(x)$, with $\lim_{x\to 1^-} f(x) = 2$ and $\lim_{x\to 1^+} f(x) = 3$, as we determined from Table 9.3. We show this result algebraically as follows. Because $f(x)$ is defined by $x^2 + 1$ when $x < 1$,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2 + 1) = 2.$$

Because $f(x)$ is defined by $x + 2$ when $x > 1$,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x + 2) = 3$$

And because

$$2 = \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x) = 3$$

$\lim_{x\to 1} f(x)$ does not exist.

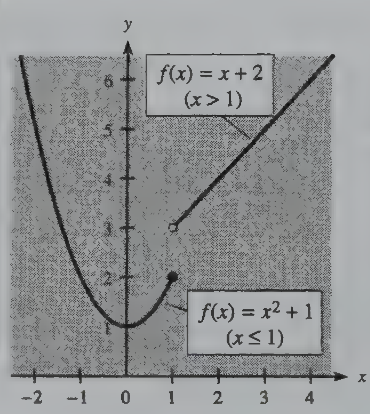


Figure 9.7

Graphing Utilities We have used graphical, numerical, and algebraic methods to understand and evaluate limits. Graphing utilities can be especially effective when we are exploring limits graphically or numerically.

EXAMPLE 8

Consider the following limits.

(a) $\lim_{x\to 5} \dfrac{x^2 + 2x - 35}{x^2 - 6x + 5}$ (b) $\lim_{x\to -1} \dfrac{2x}{x + 1}$

Investigate each limit by using the following methods.

(i) Graphically: Graph the function with a graphing utility and trace near the limiting x-value.
(ii) Numerically: Use the table feature of a graphing utility to evaluate the function very close to the limiting x-value.
(iii) Algebraically: Use properties of limits and algebraic techniques.

Solution

(a) $\lim_{x\to 5} \dfrac{x^2 + 2x - 35}{x^2 - 6x + 5}$

(i) Figure 9.8(a) on the next page shows the graph of $y = (x^2 + 2x - 35)/(x^2 - 6x + 5)$. Tracing near $x = 5$ shows y-values getting close to 3.
(ii) Figure 9.8(b) on the next page shows a table from a graphing utility with $y_1 = (x^2 + 2x - 35)/(x^2 - 6x + 5)$ and x-values approaching 5 from both sides (note that the function is undefined at $x = 5$). Again, the y-values approach 3.

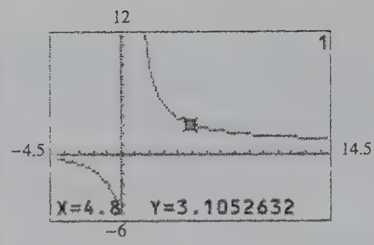
Both (i) and (ii) strongly suggest $\lim_{x\to 5} \dfrac{x^2 + 2x - 35}{x^2 - 6x + 5} = 3$.

(iii) Algebraic evaluation of this limit confirms what the graph and the table suggest.

$$\lim_{x\to 5} \frac{x^2 + 2x - 35}{x^2 - 6x + 5} = \lim_{x\to 5} \frac{(x+7)(x-5)}{(x-1)(x-5)} = \lim_{x\to 5} \frac{x+7}{x-1} = \frac{12}{4} = 3$$

(b) $\lim_{x\to -1} \dfrac{2x}{x + 1}$

(i) Figure 9.9(a) on the next page shows the graph of $y = 2x/(x + 1)$; it indicates that the graph is broken near $x = -1$. Tracing confirms that the break occurs at $x = -1$ and also suggests that the function becomes unbounded near $x = -1$. In addition, we can see that as x approaches -1 from either side, the function is headed in different directions. All this suggests that the limit does not exist.
(ii) Figure 9.9(b) shows a graphing utility table of values for $y_1 = 2x/(x + 1)$ and with x-values approaching $x = -1$. The table reinforces our preliminary conclusions from the graph that the limit does not exist, because the function is unbounded near $x = -1$.
(iii) Algebraically we see that this limit has the form $-2/0$. Thus $\lim_{x\to -1} \dfrac{2x}{x + 1}$ DNE.



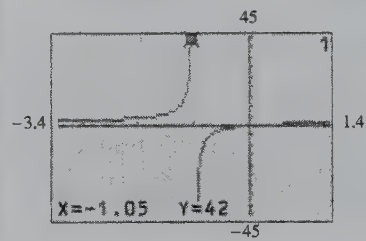
(a)

X	Y1
4.9	3.0513
4.99	3.005
4.999	3.0005
5	ERROR
5.001	2.9995
5.01	2.995
5.1	2.9512

X=4.9

(b)

Figure 9.8



(a)

X	Y1
-1.1	22
-1.01	202
-1.001	2002
-1	ERROR
-.999	-1998
-.99	-198
-.9	-18

X=-.9

(b)

Figure 9.9

We could also use the graphing and table features of spreadsheets to explore limits.

CHECKPOINT SOLUTIONS

1. Yes. For example, Figure 9.1 on page 625 and Table 9.1 show that this is possible for $g(x) = \dfrac{x^2 - x - 6}{x + 2}$. Remember that $\lim\limits_{x \to c} f(x)$ does not depend on $f(c)$.
2. Not necessarily. Figure 9.3(c) on page 627 shows the graph of $y = h(x)$ with $h(2) = 0$, but $\lim\limits_{x \to 2} h(x)$ does not exist.
3. Not necessarily. Figure 9.2(c) on page 625 shows the graph of $y = h(x)$ with $\lim\limits_{x \to 2} f(x) = 1$ but $h(2) = 4$.
4. Not necessarily. For example, Figure 9.3(c) on page 627 shows the graph of $y = h(x)$ with $\lim\limits_{x \to 2^-} h(x) = 0$, but with $\lim\limits_{x \to 2^+} h(x) = 2$, so the limit doesn't exist. Recall that if $\lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c^+} f(x) = L$, then $\lim\limits_{x \to c} f(x) = L$.
5. (a) $\lim\limits_{x \to -3} \dfrac{2x^2 + 5x - 3}{x^2 - 9} = \lim\limits_{x \to -3} \dfrac{(2x - 1)(x + 3)}{(x + 3)(x - 3)} = \lim\limits_{x \to -3} \dfrac{2x - 1}{x - 3} = \dfrac{-7}{-6} = \dfrac{7}{6}$

   (b) $\lim\limits_{x \to 5} \dfrac{x^2 - 3x - 3}{x^2 - 8x + 1} = \dfrac{7}{-14} = -\dfrac{1}{2}$

   (c) Substituting $x = -3/4$ gives $-3/0$, so $\lim\limits_{x \to -3/4} \dfrac{4x}{4x + 3}$ does not exist.
6. Yes, Properties I–IV yield this result.
7. Not necessarily. If $h(c) \neq 0$, then this is true. Otherwise, it is not true.

8. For both (a) and (b), $g(x)/h(x)$ has the 0/0 indeterminate form at $x = c$. In this case we can make no general conclusion about the limit. It is possible for the limit to exist (and be zero or nonzero) or not to exist. Consider the following 0/0 indeterminate forms.

(i) $\lim\limits_{x\to 0} \dfrac{x^2}{x} = \lim\limits_{x\to 0} x = 0$ (ii) $\lim\limits_{x\to 0} \dfrac{x(x+1)}{x} = \lim\limits_{x\to 0} (x+1) = 1$

(iii) $\lim\limits_{x\to 0} \dfrac{x}{x^2} = \lim\limits_{x\to 0} \dfrac{1}{x}$, which does not exist

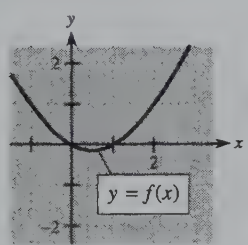
9. $\lim\limits_{x\to c} \dfrac{g(x)}{h(x)}$ does not exist and $\lim\limits_{x\to c} \dfrac{h(x)}{g(x)} = 0$

EXERCISE 9.1

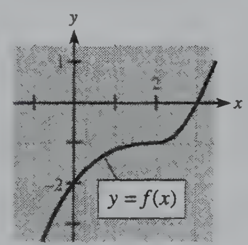
In Problems 1–8, a graph of $y = f(x)$ is shown and a c-value is given. For each problem, use the graph to find the following, whenever they exist.

(a) $f(c)$ and (b) $\lim\limits_{x\to c} f(x)$

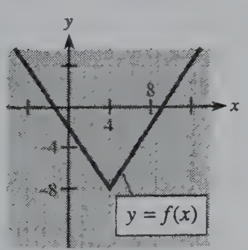
1. $c = 2$



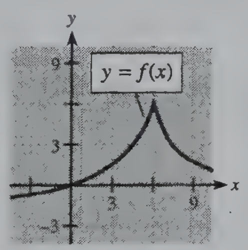
2. $c = 2$



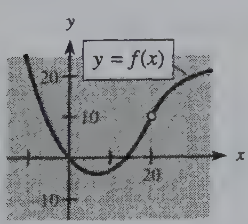
3. $c = 4$



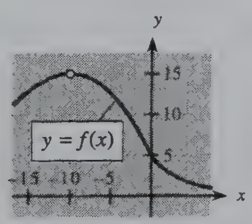
4. $c = 6$



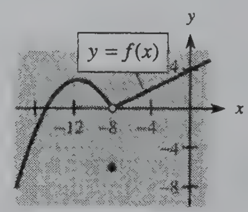
5. $c = 20$



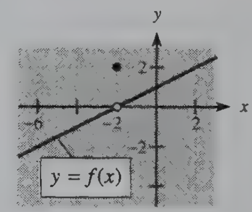
6. $c = -10$



7. $c = -8$



8. $c = -2$

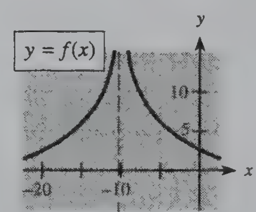


In Problems 9–12, use the graph of $y = f(x)$ and the given c-value to find the following, whenever they exist.

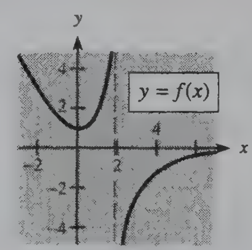
(a) $\lim\limits_{x\to c^-} f(x)$ (b) $\lim\limits_{x\to c^+} f(x)$

(c) $\lim\limits_{x\to c} f(x)$ (d) $f(c)$

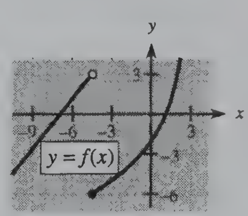
9. $c = -10$



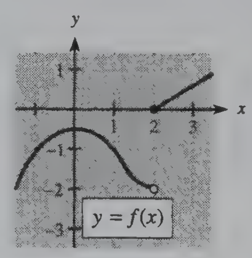
10. $c = 2$



11. $c = -4\frac{1}{2}$



12. $c = 2$



In Problems 13–18, complete each table and predict the limit, if it exists.

13. $f(x) = \dfrac{x^2 + 4x - 12}{x^2 - 2x}$

$\lim_{x\to 2} f(x) = ?$

x	$f(x)$
1.9	
1.99	
1.999	
↓	↓
2	?
↑	↑
2.001	
2.01	
2.1	

14. $f(x) = \dfrac{2x - 10}{x^2 - 25}$

$\lim_{x\to 5} f(x) = ?$

x	$f(x)$
4.9	
4.99	
4.999	
↓	↓
5	?
↑	↑
5.001	
5.01	
5.1	

15. $f(x) = \dfrac{2 - x - x^2}{x - 1}$

$\lim_{x\to 1} f(x) = ?$

x	$f(x)$
0.9	
0.99	
0.999	
↓	↓
1	?
↑	↑
1.001	
1.01	
1.1	

16. $f(x) = \dfrac{2x + 1}{\frac{1}{4} - x^2}$

$\lim_{x\to -0.5} f(x) = ?$

x	$f(x)$
−0.51	
−0.501	
−0.5001	
↓	↓
−0.5	?
↑	↑
−0.4999	
−0.499	
−0.49	

17. $f(x) = \begin{cases} 5x - 1 & \text{for } x < 1 \\ 8 - 2x - x^2 & \text{for } x \ge 1 \end{cases}$

$\lim_{x\to 1} f(x) = ?$

x	$f(x)$
0.9	
0.99	
0.999	
↓	↓
1	?
↑	↑
1.001	
1.01	
1.1	

18. $f(x) = \begin{cases} 4 - x^2 & \text{for } x \le -2 \\ x^2 + 2x & \text{for } x > -2 \end{cases}$

$\lim_{x\to -2} f(x) = ?$

x	$f(x)$
−2.1	
−2.01	
−2.001	
↓	↓
−2	?
↑	↑
−1.999	
−1.99	

In Problems 19–40, use properties of limits and algebraic methods to find the limits, if they exist.

19. $\lim_{x\to -35} (34 + x)$

20. $\lim_{x\to 80} (82 - x)$

21. $\lim_{x\to -1} (4x^3 - 2x^2 + 2)$

22. $\lim_{x\to 3} (2x^3 - 12x^2 + 5x + 3)$

23. $\lim_{x\to -1/2} \dfrac{4x - 2}{4x^2 + 1}$

24. $\lim_{x\to -1/3} \dfrac{1 - 3x}{9x^2 + 1}$

25. $\lim_{x\to 3} \dfrac{x^2 - 9}{x - 3}$

26. $\lim_{x\to -4} \dfrac{x^2 - 16}{x + 4}$

27. $\lim_{x\to 7} \dfrac{x^2 - 8x + 7}{x^2 - 6x - 7}$

28. $\lim_{x\to -5} \dfrac{x^2 + 8x + 15}{x^2 + 5x}$

29. $\lim_{x\to -2} \dfrac{x^2 + 4x + 4}{x^2 + 3x + 2}$

30. $\lim_{x\to 10} \dfrac{x^2 - 8x - 20}{x^2 - 11x + 10}$

31. $\lim_{x\to 3} f(x)$, where $f(x) = \begin{cases} 10 - 2x & \text{for } x < 3 \\ x^2 - x & \text{for } x \ge 3 \end{cases}$

32. $\lim_{x\to 5} f(x)$, where $f(x) = \begin{cases} 7x - 10 & \text{for } x < 5 \\ 25 & \text{for } x \ge 5 \end{cases}$

33. $\lim_{x\to -1} f(x)$, where $f(x) = \begin{cases} x^2 + \dfrac{4}{x} & \text{for } x \le -1 \\ 3x^3 - x - 1 & \text{for } x > -1 \end{cases}$

34. $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} \dfrac{x^3 - 4}{x - 3} & \text{for } x \le 2 \\ \dfrac{3 - x^2}{x} & \text{for } x > 2 \end{cases}$

35. $\lim_{x\to 2} \dfrac{x^2 + 6x + 9}{x - 2}$

36. $\lim_{x\to 5} \dfrac{x^2 - 6x + 8}{x - 5}$

37. $\lim_{x\to -1} \dfrac{x^2 + 5x + 6}{x + 1}$

38. $\lim_{x\to 3} \dfrac{x^2 + 2x - 3}{x - 3}$

39. $\lim_{h\to 0} \dfrac{(x + h)^3 - x^3}{h}$

40. $\lim_{h\to 0} \dfrac{2(x + h)^2 - 2x^2}{h}$

In Problems 41–46, graph each function with a graphing utility and use TRACE to predict the limit. Check your work either by using the table feature of the graphing utility or by finding the limit algebraically.

41. $\lim_{x\to 2} \dfrac{x^3 - 4x}{2x^2 - x^3}$

42. $\lim_{x\to 1/2} \dfrac{x^2 - \frac{1}{4}}{2x - 1}$

43. $\lim_{x\to 10} \dfrac{x^2 - 19x + 90}{3x^2 - 30x}$

44. $\lim_{x\to -3} \dfrac{x^4 + 3x^3}{2x^4 - 18x^2}$

45. $\lim_{x\to -1} \dfrac{x^3 - x}{x^2 + 2x + 1}$

46. $\lim_{x\to 5} \dfrac{x^2 - 7x + 10}{x^2 - 10x + 25}$

In Problems 47–52, use the table feature of a graphing utility to predict each limit. Check your work by using either a graphical or an algebraic approach.

47. $\lim_{x\to 6} \dfrac{x^2 - 2x - 24}{x^2 + 2x - 48}$

48. $\lim_{x\to 9} \dfrac{x^3 - 6x^2 - 27x}{x^2 - 12x - 27}$

49. $\lim_{x\to -2} \dfrac{x^4 - 4x^2}{x^2 + 8x + 12}$

50. $\lim_{x\to -4} \dfrac{x^3 + 4x^2}{2x^2 + 7x - 4}$

51. $\lim_{x\to 4} f(x)$, where $f(x) = \begin{cases} 12 - \frac{3}{4}x & \text{for } x \le 4 \\ x^2 - 7 & \text{for } x > 4 \end{cases}$

52. $\lim_{x\to 7} f(x)$,
where $f(x) = \begin{cases} 2 + x - x^2 & \text{for } x \le 7 \\ 23 - 9x & \text{for } x > 7 \end{cases}$

53. Use values 0.1, 0.01, 0.001, 0.0001, and 0.00001 with your calculator to approximate

$$\lim_{a\to 0} (1 + a)^{1/a}$$

to three decimal places. This limit equals the special number e that is discussed in Section 5.1, "Exponential Functions," and Section 6.2, "Compound Interest; Geometric Sequences."

54. If $\lim_{x\to 2} [f(x) + g(x)] = 5$ and $\lim_{x\to 2} g(x) = 11$, find
(a) $\lim_{x\to 2} f(x)$
(b) $\lim_{x\to 2} \{[f(x)]^2 - [g(x)]^2\}$
(c) $\lim_{x\to 2} \dfrac{3g(x)}{f(x) - g(x)}$

55. If $\lim_{x\to 3} f(x) = 4$ and $\lim_{x\to 3} g(x) = -2$, find
(a) $\lim_{x\to 3} [f(x) + g(x)]$ (b) $\lim_{x\to 3} [f(x) - g(x)]$
(c) $\lim_{x\to 3} [f(x) \cdot g(x)]$ (d) $\lim_{x\to 3} \dfrac{g(x)}{f(x)}$

56. (a) If $\lim_{x\to 2^+} f(x) = 5$, $\lim_{x\to 2^-} f(x) = 5$, and $f(2) = 0$, find $\lim_{x\to 2} f(x)$, if it exists. Explain your conclusions.
(b) If $\lim_{x\to 0^+} f(x) = 3$, $\lim_{x\to 0^-} f(x) = 0$, and $f(0) = 0$, find $\lim_{x\to 0} f(x)$, if it exists. Explain your conclusions.

Applications

57. ***Revenue*** The total revenue for a product is given by

$$R(x) = 1600x - x^2$$

where x is the number of units sold. What is $\lim_{x\to 100} R(x)$?

58. ***Profit*** If the profit function for a product is given by

$$P(x) = 92x - x^2 - 1760$$

find $\lim_{x\to 40} P(x)$.

59. ***Sales and training*** The average monthly sales volume (in thousands of dollars) for a firm depends on the number of hours x of training of its sales staff, according to

$$S(x) = \frac{4}{x} + 30 + \frac{x}{4}, \quad 4 \le x \le 100$$

(a) Find $\lim_{x\to 4^+} S(x)$. (b) Find $\lim_{x\to 100^-} S(x)$.

60. ***Sales and training*** During the first 4 months of employment, the monthly sales S (in thousands of dollars) for a new salesperson depends on the number of hours x of training, as follows:

$$S = S(x) = \frac{9}{x} + 10 + \frac{x}{4}, \quad x \ge 4$$

(a) Find $\lim_{x\to 4^+} S(x)$. (b) Find $\lim_{x\to 10} S(x)$.

61. ***Advertising and sales*** Suppose that the daily sales S (in dollars) t days after the end of an advertising campaign is

$$S = S(t) = 400 + \frac{2400}{t + 1}$$

(a) Find $S(0)$. (b) Find $\lim_{t\to 7} S(t)$.
(c) Find $\lim_{t\to 14} S(t)$.

62. ***Advertising and sales*** Sales y (in thousands of dollars) are related to advertising expenses x (in thousands of dollars) according to

$$y = y(x) = \frac{200x}{x + 10}, \quad x \ge 0$$

(a) Find $\lim_{x\to 10} y(x)$. (b) Find $\lim_{x\to 0^+} y(x)$.

63. ***Productivity*** During an 8-hour shift, the rate of change of productivity (in units per hour) of children's phonographs assembled after t hours on the job is

$$r(t) = \frac{128(t^2 + 6t)}{(t^2 + 6t + 18)^2}, \quad 0 \le t \le 8$$

(a) Find $\lim_{x \to 4} r(t)$. (b) Find $\lim_{x \to 8^-} r(t)$.

(c) Is the rate of productivity higher near the lunch break (at $t = 4$) or near quitting time (at $t = 8$)?

64. ***Revenue*** If the revenue for a product is $R(x) = 100x - 0.1x^2$, and the average revenue per unit is

$$\overline{R}(x) = \frac{R(x)}{x}, \quad x > 0$$

find (a) $\lim_{x \to 100} \frac{R(x)}{x}$ and (b) $\lim_{x \to 0^+} \frac{R(x)}{x}$.

65. ***Cost-benefit*** Suppose that the cost C of obtaining water that contains p percent impurities is given by

$$C(p) = \frac{120{,}000}{p} - 1200$$

(a) Find $\lim_{p \to 100^-} C(p)$, if it exists. Interpret this result.

(b) Find $\lim_{p \to 0^+} C(p)$, if it exists.

(c) Is complete purity possible? Explain.

66. ***Cost-benefit*** Suppose that the cost C of removing p percent of the particulate pollution from the smokestacks of an industrial plant is given by

$$C(p) = \frac{730{,}000}{100 - p} - 7300$$

(a) Find $\lim_{p \to 80} C(p)$.

(b) Find $\lim_{p \to 100^-} C(p)$, if it exists.

(c) Can 100% of the particulate pollution be removed? Explain.

67. ***Federal income tax*** Use the following tax rate schedule for single taxpayers, and create a table of values that could be used to find the following limits, if they exist. Let x represent the amount on Form 1040, line 37, and let $T(x)$ represent the tax due (entered on Form 1040, line 38).

(a) $\lim_{x \to 24{,}650^-} T(x)$ (b) $\lim_{x \to 24{,}650^+} T(x)$

(c) $\lim_{x \to 24{,}650} T(x)$

Schedule X–Use if your filing status is Single

If the amount on Form 1040, line 37, is: over—	*But not over—*	*Enter on Form 1040, line 38*	*of the amount over—*
$0	$24,650	—15%	$0
24,650	59,750	$3,697.50 + 28%	24,650
59,750	124,650	13,525.50 + 31%	59,750
124,650	271,050	33,644.50 + 36%	124,650
271,050	—	86,348.50 + 39.6%	271,050

Source: Internal Revenue Service, 1997 Form 1040 Instructions

68. ***Parking costs*** The Ace Parking Garage charges \$2.00 for parking for 2 hours or less, and 50 cents for each extra hour or part of an hour after the 2-hour minimum. The parking charges for the first 5 hours could be written as a function of the time as follows:

$$f(t) = \begin{cases} \$2.00 & \text{if } 0 < t \le 2 \\ \$2.50 & \text{if } 2 < t \le 3 \\ \$3.00 & \text{if } 3 < t \le 4 \\ \$3.50 & \text{if } 4 < t \le 5 \end{cases}$$

(a) Find $\lim_{t \to 1} f(t)$, if it exists.

(b) Find $\lim_{t \to 2} f(t)$, if it exists.

69. ***Municipal water rates*** The Beaver, Pennsylvania, Borough Municipal Authority has the following rates per 1000 gallons of water used.

Usage (x)	*Cost per 1000 Gallons* ($C(x)$)
First 100,000 gallons	1.557
Next 900,000 gallons	1.040
Over 1,000,000 gallons	0.689

Write a function $C = C(x)$ that models the charges, and find $\lim_{x \to 1000} C(x)$ (that is, as usage approaches 1,000,000 gallons).

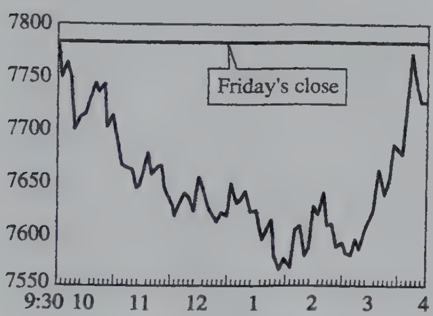
70. ***Telephone charges*** A direct-dial call from Savannah, Georgia, to Atlanta, Georgia, costs \$0.28 for the first minute and \$0.24 for each additional minute or part of a minute. If $C = C(t)$ is the charge for a call lasting t minutes, create a table of charges for calls lasting close to 1 minute and use it to find the following limits, if they exist.

(a) $\lim_{t \to 1^-} C(t)$ (b) $\lim_{t \to 1^+} C(t)$ (c) $\lim_{t \to 1} C(t)$

Dow Jones average The graph in the figure shows the Dow Jones Industrial Average (DJIA) at 5-minute intervals for Monday, October 5, 1998. Use the graph for Problems 71 and 72, with t as the time of day and $D(t)$ as the DJIA at time t.

71. Estimate $\lim_{t \to 9:30\text{AM}^+} D(t)$, if it exists. Explain what this limit corresponds to.
72. Estimate $\lim_{t \to 4:00\text{PM}^-} D(t)$, if it exists. Explain what this limit corresponds to.

DJIA at 5-minute intervals yesterday



SOURCES: *Telerate, WSJ Statistics*
SOURCE: *Wall Street Journal*, October 6, 1998

Farm workers The percentage of U.S. workers in farm occupations during certain years is shown in the table.

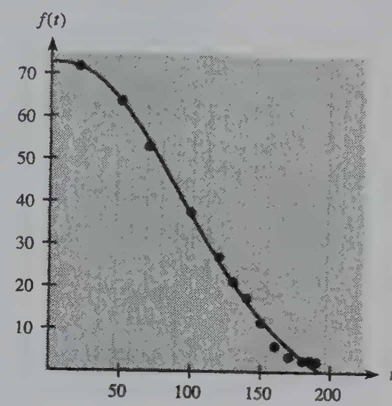
Year	*Percent*	*Year*	*Percent*
1820	71.8	1950	11.6
1850	63.7	1960	6.1
1870	53	1970	3.6
1900	37.5	1980	2.7
1920	27	1985	2.8
1930	21.2	1990	2.4
1940	17.4		

Source: *The World Almanac and Book of Facts*, 1993

Assume that the percentage of U.S. workers in farm occupations can be modeled with the function

$$f(t) = 1000 \cdot \frac{-8.0912t + 1558.9}{1.09816t^2 - 122.183t + 21472.6}$$

where t is the number of years past 1800. (A graph of $f(t)$ along with the data in the table is shown in the figure.) Use the table and equation in Problems 73 and 74.



73. (a) Find $\lim_{t \to 200} f(t)$, if it exists.
    (b) What does this limit predict?
    (c) Is the equation accurate as $t \to 200$? Explain.
74. (a) Find $\lim_{t \to 100} f(t)$, if it exists.
    (b) What does this limit predict?
    (c) Is the equation accurate as $t \to 100$? Explain.

9.2 Continuous Functions; Limits at Infinity

OBJECTIVES

- To determine whether a function is continuous or discontinuous
- To determine where a function is discontinuous
- To find limits at infinity

APPLICATION PREVIEW

Suppose that a friend of yours and her husband have a taxable income of \$99,600, and she tells you that she doesn't want to make any more money because that would put them in a higher tax bracket. She makes this statement because the tax rate schedule for married taxpayers filing a joint return (shown in the table) appears to have a jump in taxes for taxable income at \$99,600.

Schedule Y-1—Use if your filing status is Married filing jointly or Qualifying widow(er)

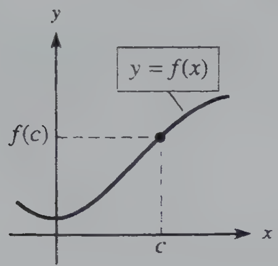
If the amount on form 1040, line 37, is over—	*But not over—*	*Enter on Form 1040, line 38*	*of the amount over—*
\$0	\$41,200	15%	\$0
41,200	99,600	6,180 + 28%	41,200
99,600	151,750	22,532 + 31%	99,600
151,750	271,050	38,698.50 + 36%	151,750
271,050	—	81,646.50 + 39.6%	271,050

Source: Internal Revenue Service, 1997 Form 1040 Instructions

To see whether the couple's taxes would jump to some higher level, we will write the function that gives income tax for married taxpayers as a function of taxable income and show that the function is **continuous.** That is, we will see that the tax paid does not jump at \$99,600 even though the tax on income above \$99,600 is collected at a higher rate. In this section, we will show how to determine whether a function is continuous, and we will investigate some different types of discontinuous functions.

We have found that $f(c)$ is the same as the limit as $x \to c$ for any polynomial function $f(x)$ and any real number c. Any function for which this special property holds is called a **continuous function.** The graphs of such functions can be drawn without lifting the pencil from the paper, and graphs of others may have holes, vertical asymptotes, or jumps that make it impossible to draw them without lifting the pencil. Even though a function may not be continuous everywhere, it is likely to have some points where the limit of the function as $x \to c$ is the same as $f(c)$. In general, we define continuity of a function at the value $x = c$ as follows:

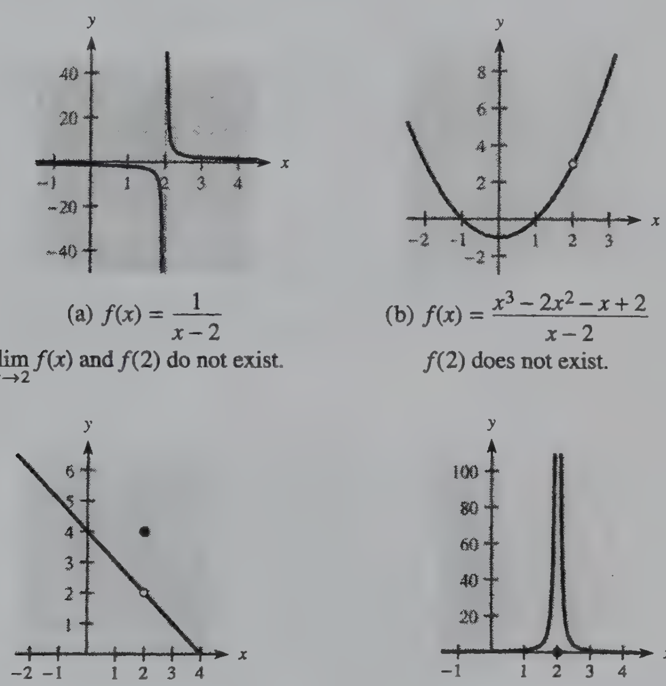
Continuity at a Point



The function f is **continuous at $x = c$** if *all* of the following conditions are satisfied.

1. $f(c)$ exists 2. $\lim_{x \to c} f(x)$ exists 3. $\lim_{x \to c} f(x) = f(c)$

If one or more of the conditions above do not hold, we say the function is **discontinuous at $x = c$.** Figure 9.10 shows graphs of some functions that are discontinuous at $x = 2$.



(a) $f(x) = \dfrac{1}{x-2}$
$\lim_{x \to 2} f(x)$ and $f(2)$ do not exist.

(b) $f(x) = \dfrac{x^3 - 2x^2 - x + 2}{x - 2}$
$f(2)$ does not exist.

(c) $f(x) = \begin{cases} 4 - x & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$
$\lim_{x \to 2} f(x) = 2 \neq 4 = f(2)$

(d) $f(x) = \begin{cases} 1/(x-2)^2 & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}$
$\lim_{x \to 2} f(x)$ does not exist.

Figure 9.10

In the previous section, we saw that if f is a polynomial function, then $\lim_{x \to c} f(x) = f(c)$ for every real number c, and also that $\lim_{x \to c} h(x) = h(c)$ if $h(x) = \dfrac{f(x)}{g(x)}$ is a rational function and $g(c) \neq 0$. Thus, by definition, we have the following.

Every polynomial function is continuous for all real numbers.

Every rational function is continuous at all values of x except those that make the denominator 0.

EXAMPLE 1

For what values of x, if any, is the function $h(x) = \dfrac{3x+2}{4x-6}$ continuous?

Solution

This is a rational function, so it is continuous for all values of x except for those that make the denominator, $4x - 6$, equal to 0. Because $4x - 6 = 0$ at $x = 3/2$, $h(x)$ is continuous for all real numbers except $x = 3/2$.

EXAMPLE 2

For what values of x, if any, is the function discontinuous if

$$f(x) = \frac{x^2 - x - 2}{x^2 - 4}$$

Solution

This is a rational function, so it is continuous everywhere except where the denominator is 0. To find the zeros of the denominator, we factor $x^2 - 4$.

$$f(x) = \frac{x^2 - x - 2}{x^2 - 4} = \frac{x^2 - x - 2}{(x-2)(x+2)}$$

Because the denominator is 0 for $x = 2$ and for $x = -2$, $f(2)$ and $f(-2)$ do not exist (recall that division by 0 is undefined). Thus the function is discontinuous at $x = 2$ and $x = -2$. The graph of this function (see Figure 9.11) shows a hole at $x = 2$ and a vertical asymptote at $x = -2$.

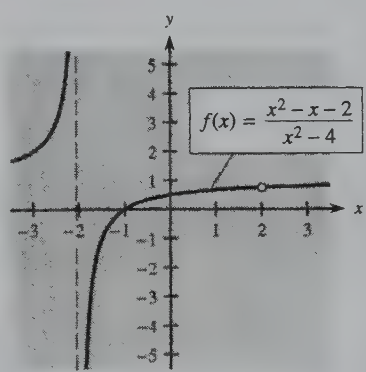


Figure 9.11

CHECKPOINT

1. Find any x-values where the following functions are discontinuous.

   (a) $f(x) = x^3 - 3x + 1$ (b) $g(x) = \dfrac{x^3 - 1}{(x-1)(x+2)}$

If the pieces of a piecewise defined function are polynomials, the only values of x where the function might be discontinuous are those at which the definition of the function changes.

EXAMPLE 3

Determine the values of x, if any, for which the following functions are discontinuous.

(a) $g(x) = \begin{cases} (x+2)^3 + 1 & \text{if } x \le -1 \\ 3 & \text{if } x > -1 \end{cases}$ (b) $f(x) = \begin{cases} 4 - x^2 & \text{if } x < 2 \\ x - 2 & \text{if } x \ge 2 \end{cases}$

Solution

(a) $g(x)$ is a piecewise defined function in which each part is a polynomial. Thus, to see whether a discontinuity exists, we need only check the value of x for which the definition of the function changes—that is, at $x = -1$. Because $x = -1$ satisfies $x \le -1$, $g(-1) = (-1 + 2)^3 + 1 = 2$. Evaluating the left- and right-hand limits gives

$$\lim_{x\to-1^-} g(x) = \lim_{x\to-1^-} [(x+2)^3 + 1] = (-1+2)^3 + 1 = 2$$

and

$$\lim_{x\to-1^+} g(x) = \lim_{x\to-1^+} 3 = 3$$

Because the left- and right-hand limits differ, $\lim_{x\to-1} g(x)$ does not exist, so $g(x)$ is discontinuous at $x = -1$. This result is confirmed by examining the graph of g, shown in Figure 9.12.

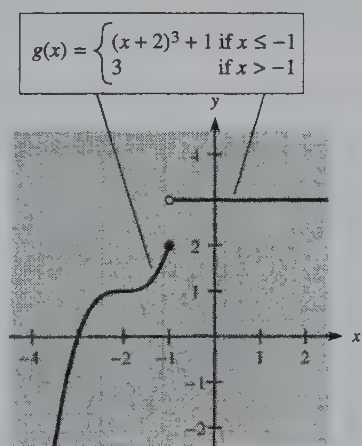


Figure 9.12

(b) As with $g(x)$, $f(x)$ is continuous everywhere except perhaps at $x = 2$, where the definition of $f(x)$ changes. Because $x = 2$ satisfies $x \ge 2$, $f(2) = 2 - 2 = 0$. The left- and right-hand limits are

$$\lim_{x\to2^-} f(x) = \lim_{x\to2^-} (4 - x^2) = 4 - 2^2 = 0$$

and

$$\lim_{x\to2^+} f(x) = \lim_{x\to2^+} (x - 2) = 2 - 2 = 0$$

Because the right- and left-hand limits are equal, we conclude that $\lim_{x\to2} f(x) = 0$. The limit is equal to the functional value

$$\lim_{x\to2} f(x) = f(2)$$

so we conclude that f is continuous at $x = 2$ and thus f is continuous for all values of x. This result is confirmed by the graph of f, shown in Figure 9.13.

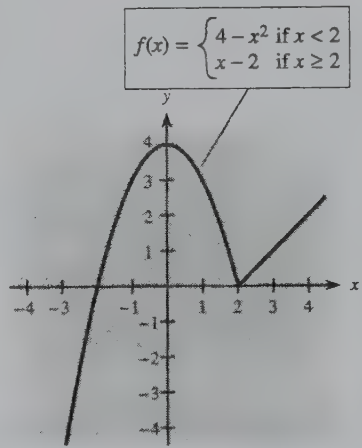


Figure 9.13

We now consider the problem posed in the Application Preview.

EXAMPLE 4

The tax rate schedule for married taxpayers filing a joint return (shown in the table) appears to have a jump in taxes for taxable income at $99,600.

Schedule Y-1—Use if your filing status is Married filing jointly or Qualifying widow(er)

If the amount on Form 1040, line 37, is: over—	*But not over—*	*Enter on Form 1040, line 38*	*of the amount over—*
$0	$41,200	15%	$0
41,200	99,600	$6180.00 + 28%	41,200
99,600	151,750	22,532.00 + 31%	99,600
151,750	271,050	38,698.50 + 36%	151,750
271,050	—	81,646.50 + 39.6%	271,050

Source: Internal Revenue Service, 1997 Form 1040 Instructions

(a) Use the table and write the function that gives income tax for married taxpayers as a function of taxable income, and graph the function.
(b) Is the function in (a) continuous at $x = 99{,}600$?
(c) A married friend of yours and her husband have a taxable income of $99,600, and she tells you that she doesn't want to make any more money because doing so would put her in a higher tax bracket. What would you tell her to do if she is offered a raise?

Solution

(a) The function that gives the tax due for married taxpayers is

$$T(x) = \begin{cases} 0.15x & \text{if } 0 \le x \le 41{,}200 \\ 6180 + 0.28(x - 41{,}200) & \text{if } 41{,}200 < x \le 99{,}600 \\ 22{,}532 + 0.31(x - 99{,}600) & \text{if } 99{,}600 < x \le 151{,}750 \\ 38{,}698.50 + 0.36(x - 151{,}750) & \text{if } 151{,}750 < x \le 271{,}050 \\ 81{,}646.50 + 0.396(x - 271{,}050) & \text{if } x > 271{,}050 \end{cases}$$

(b) This function is continuous at $x = 99{,}600$, because

(i) $T(99{,}600) = 22{,}532$, so $T(99{,}600)$ exists.

(ii) Because the function is piecewise defined near 99,600, we evaluate $\lim_{x\to 99{,}600} T(x)$ by evaluating $\lim_{x\to 99{,}600^-} T(x)$ and $\lim_{x\to 99{,}600^+} T(x)$.

$$\lim_{x\to 99{,}600^-} T(x) = \lim_{x\to 99{,}600^-} [6180 + 0.28(x - 41{,}200)] = 22{,}532$$
$$\lim_{x\to 99{,}600^+} T(x) = \lim_{x\to 99{,}600^+} [22{,}532 + 0.31(x - 99{,}600)] = 22{,}532$$

Because these limits are the same, $\lim_{x\to 99{,}600} T(x) = 22{,}532$, and so the limit exists.

(iii) Because $T(99{,}600) = \lim_{x\to 99{,}600} T(x) = 22{,}532$, the function is continuous at 99,600.

(c) If your friend earned more than \$99,600, she and her husband would pay taxes at a higher rate on the money earned *above* the \$99,600, but it would not increase the tax rate on any income *up to* \$99,600. Thus she should take any raise that's offered.

CHECKPOINT

2. If $f(x)$ and $g(x)$ are polynomials, $h(x) = \begin{cases} f(x) & \text{if } x \le a \\ g(x) & \text{if } x > a \end{cases}$ is continuous everywhere except perhaps at ________.

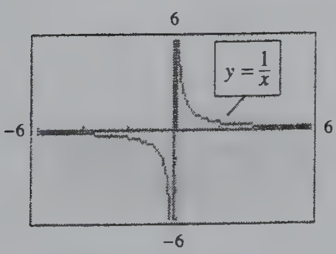
Graphing Utilities

We noted earlier (in Chapter 2, "Special Functions") that the graph of $y = 1/x$ has a vertical asymptote at $x = 0$ [shown in Figure 9.14(a)]. By graphing $y = 1/x$ with a large x-range or by using TRACE to let x get very large, we can see that $y = 1/x$ never becomes negative for positive x-values regardless of how large the x-value is. Although no value of x makes $1/x$ equal to 0, it is easy to see that $1/x$ approaches 0 as x gets very large. This is denoted by

$$\lim_{x\to+\infty} \frac{1}{x} = 0$$

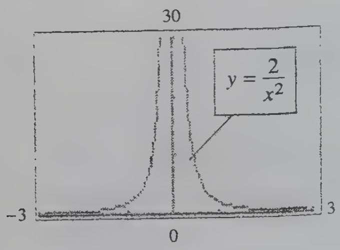
We say that $y = 0$ (the x-axis) is a horizontal asymptote for $y = 1/x$. We also see that $y = 1/x$ approaches 0 as x decreases without bound, and we denote this by

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$



(a)

Figure 9.14



(b)

Figure 9.14 (continued)

These limits can also be established with numerical tables.

x	$f(x) = 1/x$	x	$f(x) = 1/x$
100	0.01	−100	−0.01
100,000	0.00001	−100,000	−0.00001
100,000,000	0.00000001	−100,000,000	−0.00000001
↓	↓	↓	↓
$+\infty$	0	$-\infty$	0
	$\lim_{x\to+\infty} \frac{1}{x} = 0$		$\lim_{x\to-\infty} \frac{1}{x} = 0$

From Figure 9.14(b), we also see that

$$\lim_{x\to+\infty} \frac{2}{x^2} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{2}{x^2} = 0$$

By using graphs and/or tables of values, we can generalize the results for the functions shown in Figure 9.14 and conclude the following.

Limits at Infinity If c is any constant, then

1. $\lim_{x\to+\infty} c = c$ and $\lim_{x\to-\infty} c = c$.
2. $\lim_{x\to+\infty} \frac{c}{x^p} = 0$, where $p > 0$.
3. $\lim_{x\to-\infty} \frac{c}{x^n} = 0$, where $n > 0$ is any integer.

In order to use these properties for finding the limits of rational functions as x approaches $+\infty$ or $-\infty$, we first divide each term of the numerator and denominator by the highest power of x present and then determine the limit of the resulting expression.

EXAMPLE 5

Find each of the following limits, if they exist.

(a) $\lim_{x\to+\infty} \frac{2x-1}{x+2}$ (b) $\lim_{x\to-\infty} \frac{x^2+3}{1-x}$

Solution

(a) The highest power of x present is x^1, so we divide each term in the numerator and denominator by x and then use the properties for limits at infinity.

$$\lim_{x\to+\infty} \frac{2x-1}{x+2} = \lim_{x\to+\infty} \frac{\frac{2x}{x} - \frac{1}{x}}{\frac{x}{x} + \frac{2}{x}} = \lim_{x\to+\infty} \frac{2 - \frac{1}{x}}{1 + \frac{2}{x}}$$

$$= \frac{2-0}{1+0} = 2 \quad \text{(by Properties 1 and 2)}$$

Figure 9.15(a) shows the graph of this function with the y-coordinates of the graph approaching 2 as x approaches $+\infty$ and as x approaches $-\infty$. That is, $y = 2$ is a horizontal asymptote. Note also that there is a discontinuity (vertical asymptote) where $x = -2$.

(b) We divide each term in the numerator and denominator by x^2 and then use the properties.

$$\lim_{x\to-\infty} \frac{x^2+3}{1-x} = \lim_{x\to-\infty} \frac{\frac{x^2}{x^2}+\frac{3}{x^2}}{\frac{1}{x^2}-\frac{x}{x^2}} = \lim_{x\to-\infty} \frac{1+\frac{3}{x^2}}{\frac{1}{x^2}-\frac{1}{x}} = +\infty$$

This limit is $+\infty$ because the numerator approaches 1 and the denominator approaches 0 through positive values. Thus

$$\lim_{x\to-\infty} \frac{x^2+3}{1-x} \quad \text{does not exist}$$

The graph of this function, shown in Figure 9.15(b), has y-coordinates that increase without bound as x approaches $-\infty$ and that decrease without bound as x approaches $+\infty$. (There is no horizontal asymptote.) Note also that there is a vertical asymptote at $x = 1$.

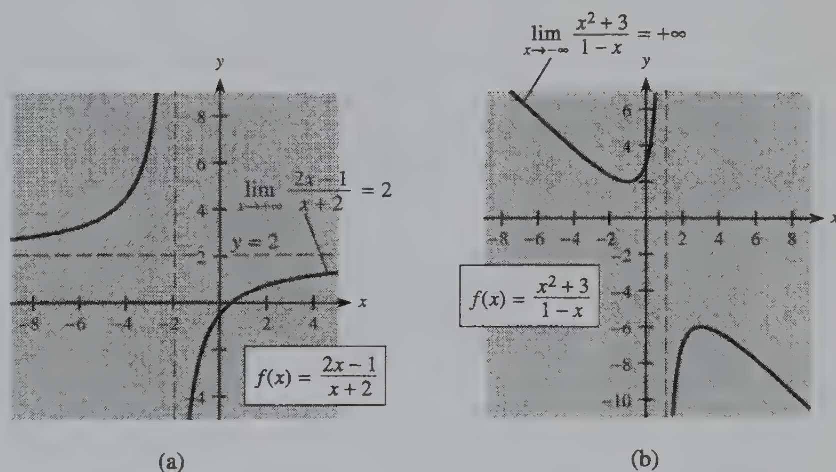


Figure 9.15

CHECKPOINT

3. Evaluate $\lim_{x\to+\infty} \dfrac{x^2-4}{2x^2-7}$.

Graphing Utilities We can use the graphing and table features of a graphing utility to help locate and investigate discontinuities. The utility can be used to focus our attention on a possible discontinuity and to support or suggest appropriate algebraic calculations.

EXAMPLE 6

Use a graphing utility to investigate the continuity of the following functions.

(a) $f(x) = \dfrac{x^2 + 1}{x + 1}$ (b) $g(x) = \dfrac{x^2 - 2x - 3}{x^2 - 1}$

(c) $h(x) = \dfrac{|x + 1|}{x + 1}$ (d) $k(x) = \begin{cases} \dfrac{-x^2}{2} - 2x & \text{if } x \le -1 \\ \dfrac{x}{2} + 2 & \text{if } x > -1 \end{cases}$

Solution

(a) Figure 9.16(a) shows that $f(x)$ has a discontinuity (vertical asymptote) near $x = -1$. Because $f(-1)$ DNE, we know that $f(x)$ is not continuous at $x = -1$.

(b) Figure 9.16(b) shows that $g(x)$ is discontinuous (vertical asymptote) near $x = 1$, and this looks like the only discontinuity. However, the denominator of $g(x)$ is zero at $x = 1$ and $x = -1$, so $g(x)$ must have discontinuities at both of these x-values. Tracing, evaluating, or using the table feature confirms that $x = -1$ is a discontinuity (a hole, or missing point).

(c) Figure 9.16(c) shows a discontinuity (jump) at $x = -1$. We also see that $h(-1)$ DNE, which confirms the observations from the graph.

(d) The graph in Figure 9.16(d) appears to be continuous. The only "suspicious" x-value is $x = -1$, where the formula for $k(x)$ changes. Evaluating $k(-1)$ and tracing or examining a table near $x = -1$ indicates that $k(x)$ is continuous there. Algebraic evaluation of the two one-sided limits confirms this.

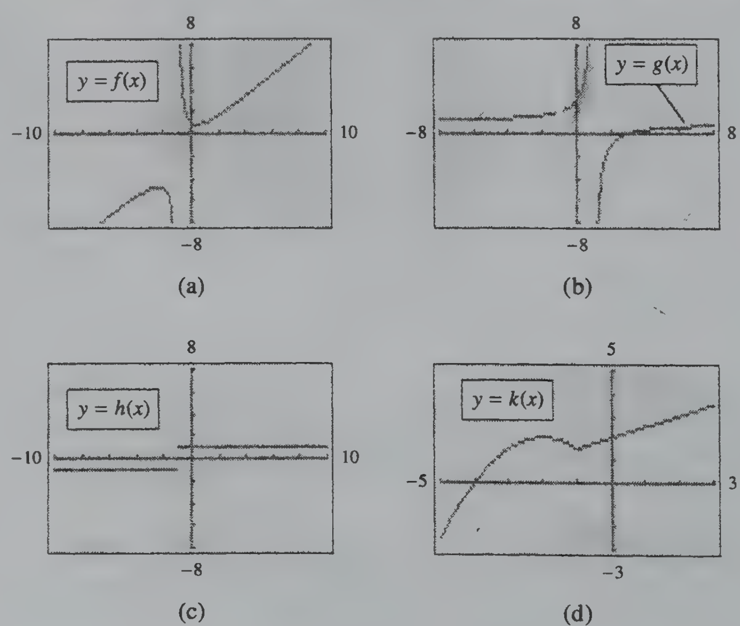


Figure 9.16

Summary

The following information is useful in discussing continuity of functions.

A. A polynomial function is continuous everywhere.

B. A rational function is a function of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials.

1. If $g(x) \neq 0$ at any value of x, the function is continuous everywhere.
2. If $g(c) = 0$, the function is discontinuous at $x = c$.
   (a) If $g(c) = 0$ and $f(c) \neq 0$, then there is a vertical asymptote at $x = c$.
   (b) If $g(c) = 0$ and $\lim_{x \to c} \frac{f(x)}{g(x)} = L$, then the graph has an open circle at $x = c$.

C. A piecewise defined function *may* have a discontinuity at any x-value where the function changes its formula. One-sided limits must be used to see whether the limit exists.

The following steps are useful when we are evaluating limits at infinity for a rational function $f(x) = p(x)/q(x)$.

1. Divide both $p(x)$ and $q(x)$ by the highest power of x found in either polynomial.
2. Use the properties of limits at infinity to complete the evaluation.

CHECKPOINT SOLUTIONS

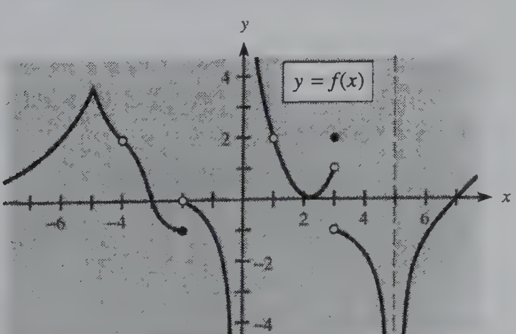
1. (a) This is a polynomial function, so it is continuous at all values of x (discontinuous at none).
   (b) This is a rational function. It is discontinuous at $x = 1$ and $x = -2$ because these values make its denominator 0.

2. $x = a$.

3. $$\lim_{x\to+\infty} \frac{x^2-4}{2x^2-7} = \lim_{x\to+\infty} \frac{1-\frac{4}{x^2}}{2-\frac{7}{x^2}} = \frac{1-0}{2-0} = \frac{1}{2}$$

EXERCISE 9.2

Problems 1 and 2 refer to the figure below. For each given x-value, use the figure to determine whether the function is continuous or discontinuous at that x-value. If the function is discontinuous, state which of the three conditions that define continuity is not satisfied.



1. (a) $x = -5$ (b) $x = 1$ (c) $x = 3$ (d) $x = 0$
2. (a) $x = 2$ (b) $x = -4$ (c) $x = -2$ (d) $x = 5$

In Problems 3–14, determine whether each function is continuous or discontinuous at the given x-value. Examine the three conditions in the definition of continuity.

3. $f(x) = x^2 - 5x,\quad x = 0$
4. $f(x) = 3x - 5x^3,\quad x = 2$
5. $f(x) = \frac{x^2-4}{x-2},\quad x = -2$
6. $y = \frac{x^2-9}{x+3},\quad x = 3$
7. $y = \frac{x^2-9}{x+3},\quad x = -3$

8. $f(x) = \dfrac{x^2 - 4}{x - 2}, x = 2$

9. $y = \dfrac{x^2 + 5x - 6}{x + 1}, x = -1$

10. $y = \dfrac{x^2 - 2x - 3}{x - 1}, x = 1$

11. $f(x) = \begin{cases} 2 & \text{if } x \le 0 \\ x + 2 & \text{if } x > 0 \end{cases} \quad x = 0$

12. $f(x) = \begin{cases} x - 3 & \text{if } x \le 2 \\ 4x - 7 & \text{if } x > 2 \end{cases} \quad x = 2$

13. $f(x) = \begin{cases} x^2 + 1 & \text{if } x \le 1 \\ 2x^2 - 1 & \text{if } x > 1 \end{cases} \quad x = 1$

14. $f(x) = \begin{cases} x^2 - x & \text{if } x \le 2 \\ 8 - 3x & \text{if } x > 2 \end{cases} \quad x = 2$

In Problems 15–22, determine whether the given function is continuous. If it is not, identify where it is discontinuous and which condition fails to hold. You can verify your conclusions by graphing each function with a graphing utility, if one is available.

15. $f(x) = 4x^2 - 1$

16. $y = 5x^2 - 2x$

17. $g(x) = \dfrac{4x^2 + 3x + 2}{x + 2}$

18. $y = \dfrac{4x^2 + 4x + 1}{x + 1/2}$

19. $y = \dfrac{x}{x^2 + 1}$

20. $y = \dfrac{2x - 1}{x^2 + 3}$

21. $f(x) = \begin{cases} 3 & \text{if } x \le 1 \\ x^2 + 2 & \text{if } x > 1 \end{cases}$

22. $f(x) = \begin{cases} x^3 + 1 & \text{if } x \le 1 \\ 2 & \text{if } x > 1 \end{cases}$

In Problems 23–26, use the trace and table features of a graphing utility to investigate whether each of the following functions has any discontinuities.

23. $y = \dfrac{x^2 - 5x - 6}{x + 1}$

24. $y = \dfrac{x^2 - 5x + 4}{x - 4}$

25. $f(x) = \begin{cases} x - 4 & \text{if } x \le 3 \\ x^2 - 8 & \text{if } x > 3 \end{cases}$

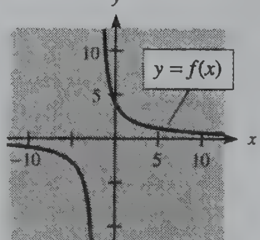
26. $f(x) = \begin{cases} x^2 + 4 & \text{if } x \ne 1 \\ 5 & \text{if } x = 1 \end{cases}$

Each of Problems 27–30 contains a function and its graph. For each problem, answer (a) and (b).

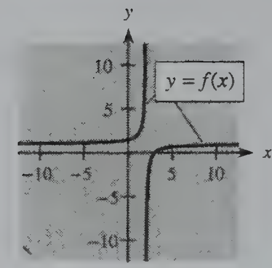
(a) Use the graph to determine, as well as you can, (i) vertical asymptotes, (ii) $\lim\limits_{x \to +\infty} f(x)$, (iii) $\lim\limits_{x \to -\infty} f(x)$

(b) Check your conclusions in (a) by using the functions to determine items (i)–(iii) analytically.

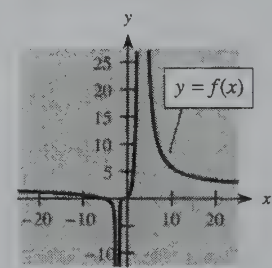
27. $f(x) = \dfrac{8}{x + 2}$



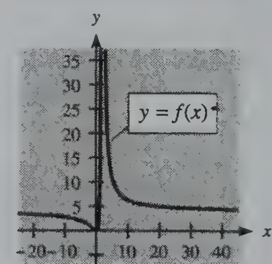
28. $f(x) = \dfrac{x - 3}{x - 2}$



29. $f(x) = \dfrac{2(x + 1)^3(x + 5)}{(x - 3)^2(x + 2)^2}$



30. $f(x) = \dfrac{4x^2}{x^2 - 4x + 4}$



Use analytic methods to evaluate the limits in Problems 31–38. You can verify your conclusions by graphing the functions with a graphing utility, if one is available.

31. $\lim\limits_{x \to +\infty} \dfrac{3}{x + 1}$

32. $\lim\limits_{x \to -\infty} \dfrac{4}{x^2 - 2x}$

33. $\lim\limits_{x \to +\infty} \dfrac{x^3 - 1}{x^3 + 4}$

34. $\lim\limits_{x \to -\infty} \dfrac{3x^2 + 2}{x^2 - 4}$

35. $\lim\limits_{x \to -\infty} \dfrac{5x^3 - 4x}{3x^3 - 2}$

36. $\lim\limits_{x \to +\infty} \dfrac{4x^2 + 5x}{x^2 - 4x}$

37. $\lim\limits_{x \to +\infty} \dfrac{3x^2 + 5x}{6x + 1}$

38. $\lim\limits_{x \to -\infty} \dfrac{5x^3 - 8}{4x^2 + 5x}$

In Problems 39 and 40, use a graphing utility to complete (a) and (b).

(a) Graph each function in the window $0 \le x \le 300$ and $-2 \le y \le 2$. What does the graph indicate about $\lim\limits_{x \to +\infty} f(x)$?

(b) Use the table feature with x-values larger than 10,000 to investigate $\lim\limits_{x \to +\infty} f(x)$. Does the table support your conclusions in (a)?

39. $f(x) = \dfrac{x^2 - 4}{3 + 2x^2}$

40. $f(x) = \dfrac{5x^3 - 7x}{1 - 3x^3}$

In Problems 41 and 42, complete (a)–(c). Use analytic methods to locate (a) any points of discontinuity and (b) any horizontal asymptotes. (c) Then explain why, for these functions, a graphing utility is better as a support tool for the analytic methods than as the primary tool for investigation.

41. $f(x) = \dfrac{1000x - 1000}{x + 1000}$ 42. $f(x) = \dfrac{3000x}{4350 - 2x}$

Applications

43. ***Sales volume*** Suppose that the weekly sales volume (in thousands of units) for a product is given by

$$y = \frac{32}{(p+8)^{2/5}}$$

where p is the price in dollars per unit. Is this function continuous
(a) for all values of p?
(b) at $p = 24$?
(c) for all $p \geq 0$?
(d) What is the domain for this application?

44. ***Worker productivity*** Suppose that the average number of minutes M that it takes a new employee to assemble one unit of a product is given by

$$M = \frac{40 + 30t}{2t + 1}$$

where t is the number of days on the job. Is this function continuous
(a) for all values of t?
(b) at $t = 14$?
(c) for all $t \geq 0$?
(d) What is the domain for this application?

45. ***Demand*** Suppose that the demand for a product is defined by the equation

$$p = \frac{200{,}000}{(q+1)^2}$$

where p is the price and q is the quantity demanded.
(a) Is this function discontinuous at any value of q? What value?
(b) Because q represents quantity, we know that $q \geq 0$. Is this function continuous for $q \geq 0$?

46. ***Advertising and sales*** The sales volume y (in thousands of dollars) is related to advertising expenditures x (in thousands of dollars) according to

$$y = \frac{200x}{x + 10}$$

(a) Is this function discontinuous at any points?
(b) Advertising expenditures x must be nonnegative. Is this function continuous for these values of x?

47. ***Annuities*** If an annuity makes an infinite series of equal payments at the end of the interest periods, it is called a **perpetuity.** If a lump sum investment of A_n is needed to result in n periodic payments of R when the interest rate per period is i, then

$$A_n = R\left[\frac{1 - (1+i)^{-n}}{i}\right]$$

(a) Evaluate $\lim_{n\to\infty} A_n$ to find a formula for the lump sum payment for a perpetuity.
(b) Find the lump sum investment needed to make payments of \$100 per month in perpetuity if interest is 12%, compounded monthly.

48. ***Response to adrenalin*** Experimental evidence suggests that the response y of the body to the concentration x of injected adrenalin is given by

$$y = \frac{x}{a + bx}$$

where a and b are experimental constants.
(a) Is this function continuous for all x?
(b) On the basis of your conclusion in (a) and the fact that in reality $x \geq 0$ and $y \geq 0$, must a and b be both positive, be both negative, or have opposite signs?

49. ***Cost-benefit*** Suppose that the cost C of removing p percent of the impurities from the waste water in a manufacturing process is given by

$$C(p) = \frac{9800p}{101 - p}$$

Is this function continuous for all those p-values for which the problem makes sense?

50. ***Cost-benefit*** Suppose that the cost C of removing p percent of the particulate pollution from the exhaust gases at an industrial site is given by

$$C(p) = \frac{8100p}{100 - p}$$

Describe any discontinuities for $C(p)$. Explain what each discontinuity means.

51. ***Cost-benefit*** The percentage p of particulate pollution that can be removed from the smokestacks of an industrial plant by spending C dollars is given by

$$p = \frac{100C}{7300 + C}$$

Find the percentage of the pollution that could be removed if spending C were allowed to increase without bound. Can 100% of the pollution be removed? Explain.

52. ***Cost-benefit*** The percentage p of impurities that can be removed from the waste water of a manufacturing process at a cost of C dollars is given by

$$p = \frac{100C}{8100 + C}$$

Find the percentage of the impurities that could be removed if cost were no object (that is, if cost were allowed to increase without bound). Can 100% of the impurities be removed? Explain.

53. ***Federal income tax*** The tax owed by a married couple filing jointly and their tax rates can be found in the following tax rate schedule.

Schedule Y-1—Use if your filing status is Married filing jointly or Qualifying widow(er)

If the amount on Form 1040, line 37, is: over—	*But not over—*	*Enter on Form 1040, line 38*	*of the amount over—*
\$0	\$41,200	15%	\$0
41,200	99,600	\$6180 + 28%	41,200
99,600	151,750	22,532 + 31%	99,600
151,750	271,050	38,698.50 + 36%	151,750
271,050	—	81,646.50 + 39.6%	271,050

Source: Internal Revenue Service, 1997 Form 1040 Instructions

From this schedule, the tax rate $R(x)$ is a function of income x (the amount on Form 1040, line 37) as follows.

$$R(x) = \begin{cases} 0.15 & \text{if } \quad 0 \le x \le 41{,}200 \\ 0.28 & \text{if } \quad 41{,}200 < x \le 99{,}600 \\ 0.31 & \text{if } \quad 99{,}600 < x \le 151{,}750 \\ 0.36 & \text{if } \quad 151{,}750 < x \le 271{,}050 \\ 0.396 & \text{if } \quad 271{,}050 < x \end{cases}$$

Identify any discontinuities in $R(x)$.

54. ***Calories and temperature*** Suppose that the number of calories of heat required to raise 1 gram of water (or ice) from $-40°C$ to $x°C$ is given by

$$f(x) = \begin{cases} \frac{1}{2}x + 20 & \text{if } -40 \le x < 0 \\ x + 100 & \text{if } 0 \le x \end{cases}$$

(a) What can be said about the continuity of the function $f(x)$?

(b) What accounts for the behavior of the function at 0°C?

55. ***Electrical usage costs*** The monthly charge in dollars for x kilowatt hours (kWh) of electricity used by a residential consumer of Excelsior Electric Membership Corporation from November through June is given by the function

$$C(x) = \begin{cases} 10 + .094x & \text{if } \quad 0 \le x \le 100 \\ 19.40 + .075(x - 100) & \text{if } \quad 100 < x \le 500 \\ 49.40 + .05(x - 500) & \text{if } \quad x > 500 \end{cases}$$

(a) What is the monthly charge if 1100 kWh of electricity is consumed in a month?

(b) Find $\lim_{x\to 100} C(x)$ and $\lim_{x\to 500} C(x)$, if the limits exist.

(c) Is C continuous at $x = 100$ and at $x = 500$?

56. ***Postage costs*** First-class postage is 33 cents for the first ounce or part of an ounce that a letter weighs and is an additional 22 cents for each additional ounce or part of an ounce above 1 ounce. Use the table or graph of the postage function, $f(x)$, to determine the following.

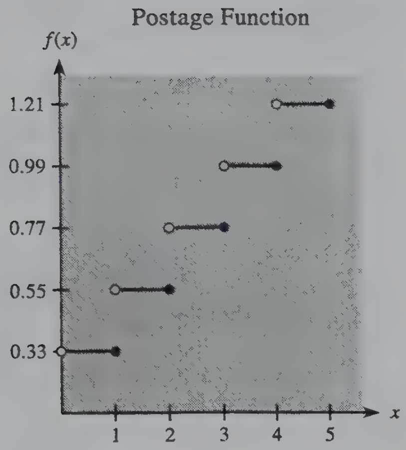
(a) $\lim_{x\to 2.5} f(x)$ (b) $f(2.5)$

(c) Is $f(x)$ continuous at 2.5?

(d) $\lim_{x\to 4} f(x)$ (e) $f(4)$

(f) Is $f(x)$ continuous at 4?

Weight x	*Postage f(x)*
$0 < x \le 1$	\$0.33
$1 < x \le 2$	0.55
$2 < x \le 3$	0.77
$3 < x \le 4$	0.99
$4 < x \le 5$	1.21



57. ***Public debt of the United States*** The interest paid on the public debt of the United States of America as a percentage of federal expenditures for selected years is shown in the following table.

Year	*Interest Paid as a Percentage of Federal Expenditures*	*Point Coordinates if t = 0 in 1900*
1930	0	(30, 0)
1940	10.5	(40, 10.5)
1950	13.4	(50, 13.4)
1955	9.4	(55, 9.4)
1960	10.0	(60, 10.0)
1965	9.6	(65, 9.6)
1970	9.9	(70, 9.9)
1975	9.8	(75, 9.8)
1980	12.7	(80, 12.7)
1985	18.9	(85, 18.9)
1990	21.1	(90, 21.1)
1995	22.0	(95, 22.0)

Source: Bureau of Public Debt, Department of the Treasury

If t is the number of years past 1900, use the table to complete the following.

(a) Use the data in the table to find a cubic and a fourth-degree function that model the percentage of federal expenditures devoted to payment of interest on the public debt. Let $d(t)$ be the one that better fits the data.

(b) Use $d(t)$ to predict the percentage of federal expenditures devoted to payment of interest in 2005.

(c) Calculate $\lim_{t\to+\infty} d(t)$.

(d) Can $d(t)$ be used to predict the percentage of federal expenditures devoted to payment of interest on the public debt for large values of t? Explain.

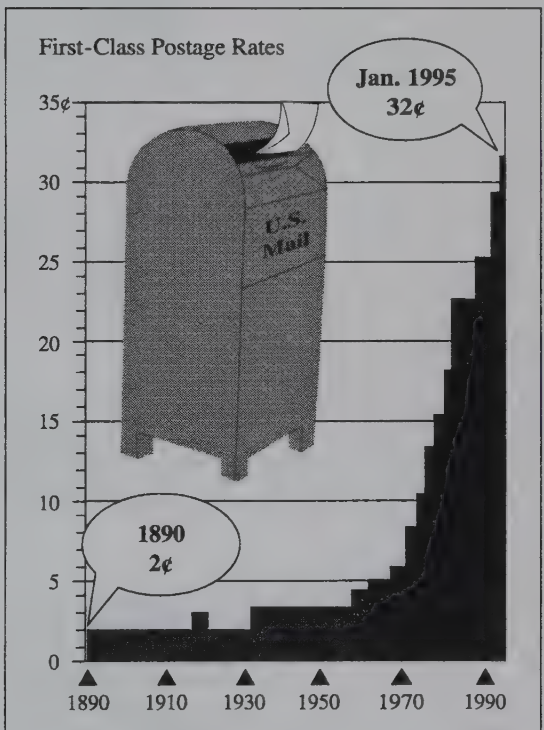
(e) For what years can you guarantee that $d(t)$ cannot be used to predict the percentage of federal expenditures devoted to payment of interest on the public debt? Explain.

58. ***Postal rates*** The following graphic shows the history of postal rates from 1890 to 1995. Use the figure to answer the following.

(a) If $P(t)$ represents first-class postage in year t, is $P(t)$ continuous?

(b) Identify the longest period of years after World War I when $P(t)$ was continuous.

(c) If $P(t)$ were modeled by a continuous curve, do you think the best model would be linear, exponential, or logarithmic? Explain.



Source: *Oil City Derrick*, Oil City, PA, December 1, 1994.

9.3 The Derivative: Rates of Change; Tangent to a Curve

OBJECTIVES

- To define the derivative as a rate of change
- To use the definition of derivative to find derivatives of functions
- To use derivatives to find slopes of tangents to curves

APPLICATION PREVIEW

Suppose an oil company's revenue (in thousand of dollars) is given by

$$R = 100x - x^2, \quad x \geq 0$$

where x is the number of thousands of barrels of oil sold per day. Then we can use the **derivative** of this function to find the marginal revenue when 20,000 barrels are sold.

In Chapter 1, "Linear Equations and Functions," we studied linear revenue functions and defined the marginal revenue for a product as the rate of change of the revenue function. For linear revenue functions, this rate is also the slope of the line that is the graph of the revenue function. In this section, we will define **marginal revenue** as the rate of change of the revenue function, even when the revenue function is not linear. We will discuss the relationship between the marginal revenue at a given point and the slope of the line tangent to the revenue function at that point. We will see how the derivative of the revenue function can be used to find both the slope of this tangent line and the marginal revenue.

We will begin our study of rates of change (that is, derivatives) by investigating a common rate of change, velocity.

Suppose a ball is thrown straight upward at 64 feet per second from a spot 96 feet above ground level. The equation that describes the height y of the ball after x seconds is

$$y = f(x) = 96 + 64x - 16x^2$$

Figure 9.17 shows the graph of this function for $0 \leq x \leq 5$. The average velocity of the ball over a given time interval is the change in the height divided by the length of time that has passed. Table 9.4 shows some average velocities over time intervals beginning at $x = 1$.

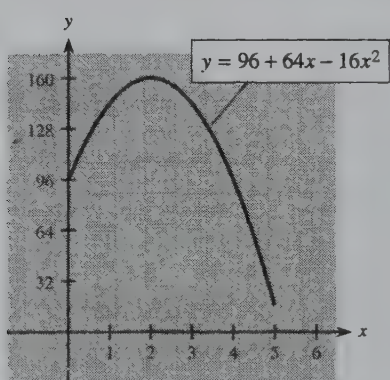


Figure 9.17

TABLE 9.4 Average Velocities

Time			*Height*			
Beginning	*Ending*	*Change* (Δx)	*Beginning*	*Ending*	*Change* (Δy)	*Average Velocity* ($\Delta y/\Delta x$)
1	2	1	144	160	16	16/1 = 16
1	1.5	0.5	144	156	12	12/0.5 = 24
1	1.1	0.1	144	147.04	3.04	3.04/01 = 30.4
1	1.01	0.01	144	144.3184	0.3184	0.3184/0.01 = 31.84

In Table 9.4, the smaller the time interval, the more closely the average velocity approximates the instantaneous velocity at $x = 1$. Thus the instantaneous velocity at $x = 1$ is closer to 31.84 ft/s than to 30.4 ft/s.

If we represent the change in time by h, then the average velocity from $x = 1$ to $x = 1 + h$ approaches the instantaneous velocity at $x = 1$ as h approaches 0. This is illustrated in the following example.

EXAMPLE 1

Suppose a ball is thrown straight upward so that its height $f(x)$ (in feet) is given by the equation

$$f(x) = 96 + 64x - 16x^2$$

where x is time (in seconds).

(a) Find the average velocity from $x = 1$ to $x = 1 + h$.
(b) Find the instantaneous velocity at $x = 1$.

Solution

(a) Let h represent the change in x (time) from 1 to $1 + h$. Then the corresponding change in $f(x)$ (height) is

$$\begin{aligned} f(1 + h) - f(1) &= [96 + 64(1 + h) - 16(1 + h)^2] - [96 + 64 - 16] \\ &= 96 + 64 + 64h - 16(1 + 2h + h^2) - 144 \\ &= 16 + 64h - 16 - 32h - 16h^2 \\ &= 32h - 16h^2 \end{aligned}$$

The average velocity V_{av} is the change in height divided by the change in time.

$$\begin{aligned} V_{av} &= \frac{f(1 + h) - f(1)}{h} \\ &= \frac{32h - 16h^2}{h} \\ &= 32 - 16h \end{aligned}$$

(b) The instantaneous velocity V is the limit of the average velocity as h approaches 0.

$$\begin{aligned} V = \lim_{h \to 0} V_{av} &= \lim_{h \to 0} (32 - 16h) \\ &= 32 \text{ ft/s} \end{aligned}$$

Note that average velocity is found over a time interval. Instantaneous velocity is usually called **velocity,** and it can be found at any time x, as follows:

Velocity Suppose that an object moving in a straight line has its position y at time x given by $y = f(x)$. Then the **velocity** of the object at time x is

$$V = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that this limit exists.

The instantaneous rate of change of any function (commonly called *rate of change*) can be found in the same way we find velocity. The function that gives this instantaneous rate of change of a function f is called the **derivative** of f.

Derivative If f is a function defined by $y = f(x)$, then the **derivative** of $f(x)$ at any value x, denoted $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if this limit exists. If $f'(c)$ exists, we say that f is **differentiable** at c.

The following procedure illustrates how to find the derivative of a function $y = f(x)$ at any value x.

Derivative Using the Definition

Procedure	**Example**
To find the derivative of $y = f(x)$ at any value x:	Find the derivative of $f(x) = 4x^2$.
1. Let h represent the change in x from x to $x + h$.	1. The change in x from x to $x + h$ is h.
2. The corresponding change in $y = f(x)$ is $f(x+h) - f(x)$	2. The change in $f(x)$ is $f(x+h) - f(x) = 4(x+h)^2 - 4x^2$ $= 4(x^2 + 2xh + h^2) - 4x^2$ $= 4x^2 + 8xh + 4h^2 - 4x^2$ $= 8xh + 4h^2$
3. Form the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ and simplify.	3. $\dfrac{f(x+h) - f(x)}{h} = \dfrac{8xh + 4h^2}{h}$ $= 8x + 4h$
4. Find $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to determine $f'(x)$, the derivative of $f(x)$.	4. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ $f'(x) = \lim\limits_{h \to 0} (8x + 4h) = 8x$

Note that in the previous example, we could have found the derivative of the function $f(x) = 4x^2$ at a particular value of x, say $x = 3$, by evaluating the derivative formula at that value:

$$f'(x) = 8x \quad \text{so} \quad f'(3) = 8(3) = 24$$

In addition to $f'(x)$, the derivative at any point x may be denoted by

$$\frac{dy}{dx}, \quad y', \quad \frac{d}{dx}f(x), \quad D_x y, \quad \text{or} \quad D_x f(x)$$

We can, of course, use variables other than x and y to represent functions and their derivatives. For example, we can represent the derivative of the function defined by $p = 2q^2 - 1$ by dp/dq.

CHECKPOINT

1. For the function $y = f(x) = x^2 - x + 1$, find
   (a) $f(x + h) - f(x)$
   (b) $\dfrac{f(x+h) - f(x)}{h}$
   (c) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$
   (d) $f'(2)$

In Section 1.6, "Applications of Functions in Business and Economics," we defined the **marginal revenue** for a product as the rate of change of the total revenue function for the product. If the total revenue function for a product is not linear, we define the marginal revenue for the product as the instantaneous rate of change, or the derivative, of the revenue function.

Marginal Revenue

Suppose that the total revenue function for a product is given by $R = R(x)$, where x is the number of units sold. Then the **marginal revenue** at x units is

$$\overline{MR} = R'(x) = \lim_{h \to 0} \frac{R(x+h) - R(x)}{h}$$

provided that the limit exists.

Note that the marginal revenue (derivative of the revenue function) can be found by using the steps in the Procedure/Example above. These steps can also be combined, as they are in Example 2, which is the Application Preview problem.

EXAMPLE 2

Suppose that an oil company's revenue (in thousands of dollars) is given by the equation

$$R = R(x) = 100x - x^2, \quad x \geq 0$$

where x is the number of thousands of barrels of oil sold each day.

(a) Find the function that gives the marginal revenue at any value of x.
(b) Find the marginal revenue when 20,000 barrels are sold (that is, at $x = 20$).

Solution

(a) The marginal revenue function is found by evaluating

$$R'(x) = \lim_{h\to 0} \frac{R(x+h) - R(x)}{h}$$

$$= \lim_{h\to 0} \frac{[100(x+h) - (x+h)^2] - (100x - x^2)}{h}$$

$$= \lim_{h\to 0} \frac{100x + 100h - (x^2 + 2xh + h^2) - 100x + x^2}{h}$$

$$= \lim_{h\to 0} \frac{100h - 2xh - h^2}{h} = \lim_{h\to 0} (100 - 2x - h) = 100 - 2x$$

(b) The marginal revenue function found in (a) gives the marginal revenue at *any* value of x. To find the marginal revenue when 20 units are sold, we evaluate $R'(20)$.

$$R'(20) = 100 - 2(20) = 60$$

Hence the marginal revenue at $x = 20$ is \$60,000. Because the marginal revenue is used to approximate the revenue from the sale of one additional unit, we interpret $R'(20) = 60$ to mean that the expected revenue from the sale of the next thousand barrels (after 20,000) will be approximately \$60,000. [*Note:* The actual revenue from this sale is $R(21) - R(20) = 1659 - 1600 = 59$ (thousand dollars).]

As mentioned earlier, the rate of change of revenue (the marginal revenue) for a linear revenue function is given by the slope of the line. In fact, the slope of the revenue curve gives us the marginal revenue even if the revenue function is not linear. We will show that the slope of the graph of a function at any point is the same as the derivative at that point. In order to show this, we must define the slope of a curve at a point on the curve. We will define the slope of a curve at a point as the slope of the line tangent to the curve at the point.

In geometry, a **tangent** to a circle is defined as a line that has one point in common with the circle. [See Figure 9.18(a).] This definition does not apply to all curves, as Figure 9.18(b) shows. Many lines can be drawn through the point A that touch the curve only at A. One of the lines, line l, looks like it is tangent to the curve.

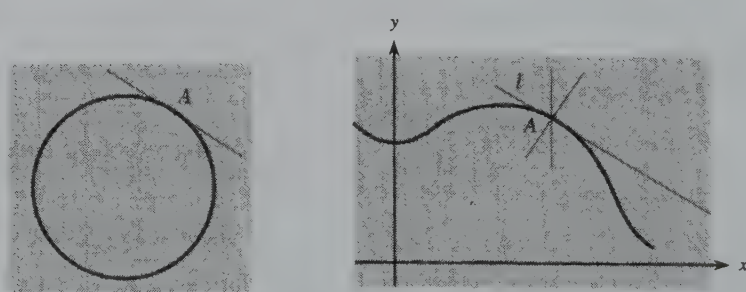


Figure 9.18 (a) (b)

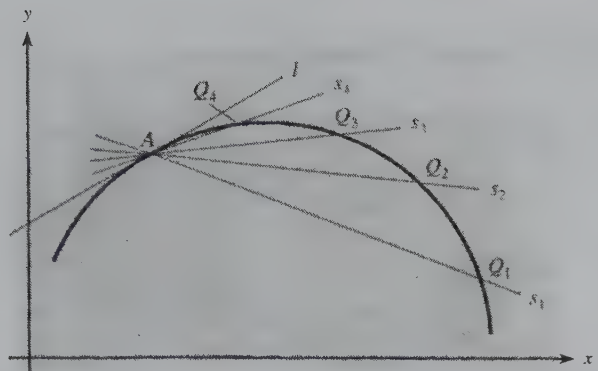


Figure 9.19

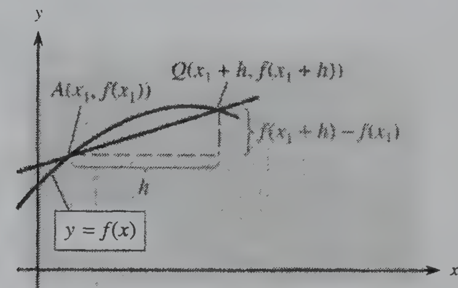


Figure 9.20

We can use **secant lines** (lines that intersect the curve at two points) to determine the tangent to a curve at a point. In Figure 9.19, we have a set of secant lines s_1, s_2, s_3, and s_4 that pass through a point A on the curve and points Q_1, Q_2, Q_3, and Q_4 on the curve near A. The line l represents the tangent line to the curve at point A. We can get a secant line as close as we wish to the tangent line l by choosing a "second point" Q sufficiently close to point A.

As we choose points on the curve closer and closer to A, the limiting position of the secant lines that pass through A is the **tangent line** to the curve at point A, and the slopes of those secant lines approach the slope of the tangent line at A. Thus we can find the slope of the tangent line by finding the slope of a secant line and taking the limit of this slope as the "second point" Q approaches A. To find the slope of the tangent to the graph of $y = f(x)$ at A $(x_1, f(x_1))$, we first draw a secant line from point A to a second point Q $(x_1 + h, f(x_1 + h))$ on the curve (see Figure 9.20).

The slope of this secant line is

$$m_{AQ} = \frac{f(x_1 + h) - f(x_1)}{h}$$

As Q approaches A, we see that the difference between the x-coordinates of these two points decreases, so h approaches 0. Thus the slope of the tangent is given by the following.

Slope of the Tangent The **slope of the tangent** to the graph of $y = f(x)$ at point $A(x_1, f(x_1))$ is

$$m = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

if this limit exists. That is, $m = f'(x_1)$.

EXAMPLE 3

Find the slope of $y = f(x) = x^2$ at the point A (2, 4).

Solution

The formula for the slope of the tangent to $y = f(x)$ at (2, 4) is

$$m = f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

Thus for $f(x) = x^2$, we have

$$m = f'(2) = \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h}$$

Taking the limit immediately would result in both the numerator and the denominator approaching 0. To avoid this, we simplify the fraction before taking the limit.

$$m = \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0} (4 + h) = 4$$

Thus the slope of the tangent to $y = x^2$ at (2, 4) is 4 (see Figure 9.21).

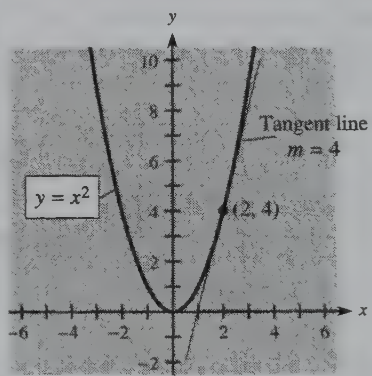


Figure 9.21

The statement "the slope of the tangent to the curve at (2, 4) is 4" is frequently simplified to the statement "the slope of the curve at (2, 4) is 4." Knowledge that the slope is a positive number on an interval tells us that the function is increasing on that interval, which means that a point moving along the graph of the function rises as it moves to the right on that interval. If the derivative (and thus the slope) is negative on an interval, the curve is decreasing on the interval; that is, a point moving along the graph falls as it moves to the right on that interval.

EXAMPLE 4

Given $y = f(x) = 3x^2 + 2x$, find

(a) the derivative of $f(x)$ at any point $(x, f(x))$.
(b) the slope of the curve at (1, 5).
(c) the equation of the line tangent to $y = 3x^2 + 2x$ at (1, 5).

Solution

(a) The derivative of $f(x)$ at any value x is denoted by $f'(x)$ and is

$$\begin{aligned} y' = f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{[3(x+h)^2 + 2(x+h)] - (3x^2+2x)}{h} \\ &= \lim_{h\to 0} \frac{3(x^2+2xh+h^2)+2x+2h-3x^2-2x}{h} \\ &= \lim_{h\to 0} \frac{6xh+3h^2+2h}{h} \\ &= \lim_{h\to 0}(6x+3h+2) \\ &= 6x+2 \end{aligned}$$

(b) The derivative is $f'(x) = 6x + 2$, so the slope of the tangent to the curve at $(1, 5)$ is $f'(1) = 6(1) + 2 = 8$.

(c) The equation of the tangent line uses the given point $(1, 5)$ and the slope $m = 8$. It is $y - 5 = 8(x - 1)$, or $y = 8x - 3$.

Technology Note

Note in Figure 9.21 that near the point of tangency at (2, 4), the tangent line and the function look coincident. In fact, if we graphed both with a graphing utility and repeatedly zoomed in near the point (2, 4), the two graphs would eventually appear as one. Thus the derivative of $f(x)$ at the point where $x = a$ can be approximated by finding the slope between $(a, f(a))$ and a second point that is nearby.

In addition, we know that the slope of the tangent to $f(x)$ at $x = a$ is defined by

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Hence we could also estimate $f'(a)$—that is, the slope of the tangent at $x = a$—by evaluating

$$\frac{f(a+h)-f(a)}{h} \quad \text{when } h \approx 0$$

EXAMPLE 5

(a) Let $f(x) = 3x^2 + 2x$. Use $\dfrac{f(a+h)-f(a)}{h}$ and two values of h to make estimates of the slope of the tangent to $f(x)$ at $x = 3$ on opposite sides of $x = 3$.

(b) Use the following table of values of x and $g(x)$ to estimate $g'(3)$.

x	1	1.9	2.7	2.9	2.999	3	3.002	3.1	4	5
$g(x)$	1.6	4.3	11.4	10.8	10.513	10.5	10.474	10.18	6	−5

Solution

The table feature of a graphing utility can facilitate the following calculations.

(a) We can use $h = 0.0001$ and $h = -0.0001$ as follows:

$$\text{With } h = 0.0001: \quad f'(3) \approx \frac{f(3+0.0001) - f(3)}{0.0001} = \frac{f(3.0001) - f(3)}{0.0001} = 20.0003 \approx 20$$

$$\text{With } h = -0.0001: \quad f'(3) \approx \frac{f(3+(-0.0001)) - f(3)}{-0.0001} = \frac{f(2.9999) - f(3)}{-0.0001} = 19.9997 \approx 20$$

(b) We use the given table and measure the slope between (3, 10.5) and another point that is nearby (the closer, the better). Using (2.999, 10.513), we obtain

$$g'(3) \approx \frac{y_2 - y_1}{x_2 - x_1} = \frac{10.5 - 10.513}{3 - 2.999} = \frac{-0.013}{0.001} = -13$$

Most graphing utilities have a feature called the **numerical derivative** (usually denoted by nDer or nDeriv) that can approximate the derivative of a function at a point. On most utilities this feature uses a calculation similar to our method in Example 5(a). The numerical derivative of $f(x) = 3x^2 + 2x$ with respect to x at $x = 3$ can be found as follows on many graphing utilities:

$$\text{nDeriv}(3x^2 + 2x, x, 3) = 20$$

The discussion in this section indicates that the derivative of a function can be used to accomplish the following.

1. Find the velocity of an object moving in a straight line.
2. Find the instantaneous rate of change of a function.
3. Find the marginal revenue function for a given revenue function.
4. Find the slope of the tangent to the graph of a function.

So far we have talked about how the derivative is defined, what it represents, and how to find it. However, there are functions for which derivatives do not exist at every value of x. Figure 9.22 shows some common cases where $f'(c)$ does not exist but where $f'(x)$ exists for all other values of x. These cases occur where there is a discontinuity, a corner, or a vertical tangent line.

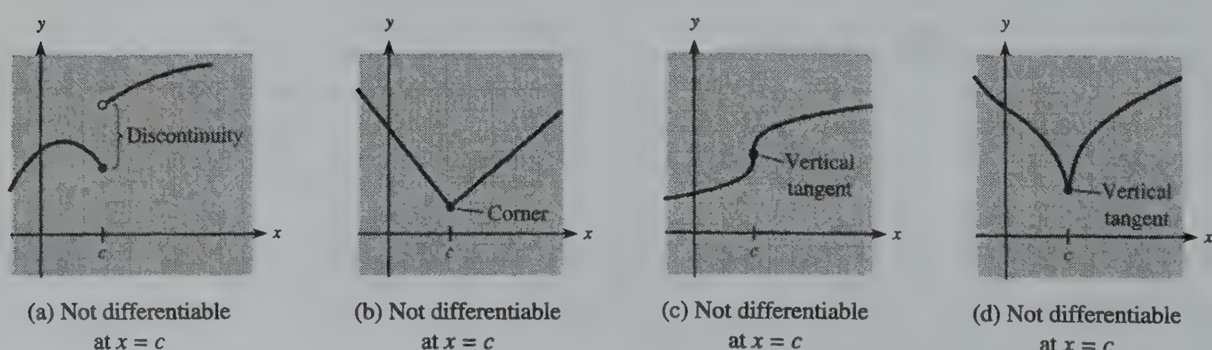


Figure 9.22

From Figure 9.22 we see that a function may be continuous at $x = c$ even though $f'(c)$ does not exist. Thus continuity does not imply differentiability at a point. However, differentiability does imply continuity.

Differentiability Implies Continuity

If a function f is differentiable at $x = c$, then f is continuous at $x = c$.

EXAMPLE 6

The monthly charge for water in a small town is given by

$$y = f(x) = \begin{cases} 18 & \text{if } 0 \le x \le 20 \\ 0.1x + 16 & \text{if } x > 20 \end{cases}$$

(a) Is this function continuous at $x = 20$?
(b) Is this function differentiable at $x = 20$?

Solution

(a) We must check the three properties for continuity.

1. $f(x) = 18$ for $x \le 20$, so $f(20) = 18$
2. $$\left.\begin{aligned} \lim_{x\to 20^-} f(x) &= \lim_{x\to 20^-} 18 = 18 \\ \lim_{x\to 20^+} f(x) &= \lim_{x\to 20^+} (0.1x + 16) = 18 \end{aligned}\right\} \Rightarrow \lim_{x\to 20} f(x) = 18$$
3. $\lim_{x\to 20} f(x) = f(20)$

Thus $f(x)$ is continuous at $x = 20$.

(b) Because the function is defined differently on either side of $x = 20$, we need to test to see whether $f'(20)$ exists by evaluating both

$$\text{(i)} \lim_{h\to 0^-} \frac{f(20+h) - f(20)}{h} \quad \text{and} \quad \text{(ii)} \lim_{h\to 0^+} \frac{f(20+h) - f(20)}{h}$$

and determining whether they are equal.

$$\begin{aligned} \text{(i)} \lim_{h\to 0^-} \frac{f(20+h) - f(20)}{h} &= \lim_{h\to 0^-} \frac{18 - 18}{h} \\ &= \lim_{h\to 0^-} 0 = 0 \end{aligned}$$

$$\begin{aligned} \text{(ii)} \lim_{h\to 0^+} \frac{f(20+h) - f(20)}{h} &= \lim_{h\to 0^+} \frac{[0.1(20+h) + 16] - 18}{h} \\ &= \lim_{h\to 0^+} \frac{0.1h}{h} \\ &= \lim_{h\to 0^+} 0.1 = 0.1 \end{aligned}$$

Because these limits are not equal, the derivative does not exist.

CHECKPOINT

2. Which of the following are given by $f'(c)$?
   (a) The slope of the tangent when $x = c$
   (b) The y-coordinate of the point where $x = c$
   (c) The instantaneous rate of change of $f(x)$ at $x = c$
   (d) The marginal revenue at $x = c$, if $f(x)$ is the revenue function
3. Must a graph that has no discontinuity, corner, or cusp at $x = c$ be differentiable at $x = c$?

EXAMPLE 7

If the point (a, b) lies on the graph of $y = x^2$, then the equation of the secant line to $y = x^2$ from (1, 1) to (a, b) has the equation

$$y - 1 = \frac{b-1}{a-1}(x-1), \quad \text{or} \quad y = \frac{b-1}{a-1}(x-1) + 1$$

(a) Write the equation of the secant line from (1, 1) to (5, 25) and graph $y = x^2$ and this secant line.

(b) Write the equation of the secant line from (1, 1) to (3, 9) and graph $y = x^2$ and this secant line.

(c) Write the equation of the secant line from (1, 1) to (1.01, 1.0201) and graph $y = x^2$ and this secant line.

(d) Which secant line appears as if it might be closest to the tangent line at (1, 1)?

(e) Express the slope of the secant line from (1, 1) to (a, b) in terms of a and find the limit of this slope as $a \to 1$. Is this limit the slope of the tangent line to $y = x^2$ at (1, 1)?

Solution

(a) The equation of the secant line from (1, 1) to $a = 5$, $b = 25$ is

$$y = \frac{25-1}{5-1}(x-1) + 1, \quad \text{or} \quad y = 6x - 5$$

The graph of $y = x^2$ and the secant line are shown in Figure 9.23(a).

(b) The equation of the secant line from (1, 1) to $a = 3$, $b = 9$ is

$$y = \frac{9-1}{3-1}(x-1) + 1, \quad \text{or} \quad y = 4x - 3$$

The graph of $y = x^2$ and the secant line are shown in Figure 9.23(b).

(c) The equation of the secant line from (1, 1) to $a = 1.01$, $b = 1.0201$ is

$$y = \frac{1.0201-1}{1.01-1}(x-1) + 1, \quad \text{or} \quad y = 2.01x - 1.01$$

The graph of $y = x^2$ and the secant line are shown in Figure 9.23(c).

(d) The secant line from (1, 1) to (1.01, 1.0201) is closest to the tangent line at (1, 1).

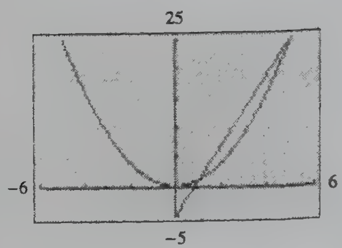
(e) The slope of the secant line from (1, 1) to (a, b) is

$$\frac{b-1}{a-1} = \frac{a^2-1}{a-1}$$

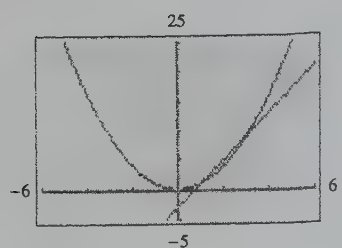
The limit of this slope as a approaches 1, the x-value of the point (1, 1), is

$$\lim_{a\to 1} \frac{b-1}{a-1} = \lim_{a\to 1} \frac{a^2-1}{a-1} = \lim_{a\to 1}(a+1) = 2$$

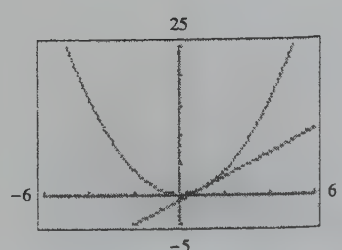
This limit, 2, is the slope of the tangent line at (1, 1). That is, the derivative of $y = x^2$ at (1, 1) is 2. [Note that a graphing utility's calculation of the numerical derivative of $f(x) = x^2$ with respect to x at $x = 1$ gives $f'(1) = 2$.]



(a)



(b)



(c)

Figure 9.23

CHECKPOINT SOLUTIONS

1. (a) $f(x+h) - f(x) = [(x+h)^2 - (x+h) + 1] - (x^2 - x + 1)$
$= x^2 + 2xh + h^2 - x - h + 1 - x^2 + x - 1$
$= 2xh + h^2 - h$

(b) $\dfrac{f(x+h) - f(x)}{h} = \dfrac{2xh + h^2 - h}{h}$
$= 2x + h - 1$

(c) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} (2x + h - 1)$
$= 2x - 1$

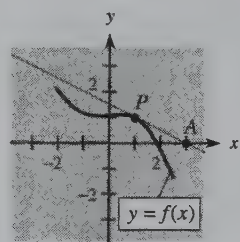
(d) $f'(x) = 2x - 1$, so $f'(2) = 3$.

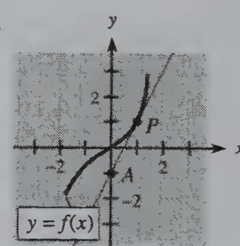
2. Parts (a), (c), and (d) are given by $f'(c)$. The y-coordinate where $x = c$ is given by $f(c)$.
3. No. Figure 9.22(c) shows such an example.

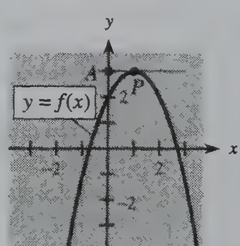
EXERCISE 9.3

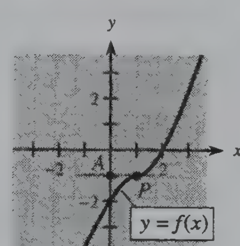
1. In the Procedure/Example table in this section we were given $f(x) = 4x^2$ and found $f'(x) = 8x$. Find
   (a) the instantaneous rate of change of $f(x)$ at $x = 4$.
   (b) the slope of the tangent to the graph of $y = f(x)$ at $x = 4$.
   (c) the point on the graph of $y = f(x)$ at $x = 4$.
2. In Example 4 of this section we were given $f(x) = 3x^2 + 2x$ and found $f'(x) = 6x + 2$. Find
   (a) the instantaneous rate of change of $f(x)$ at $x = 6$.
   (b) the slope of the tangent to the graph of $y = f(x)$ at $x = 6$.
   (c) the point on the graph of $y = f(x)$ at $x = 6$.
3. Let $f(x) = 2x^2 - x$.
   (a) Use the Procedure/Example in this section to verify that $f'(x) = 4x - 1$.
   (b) Find the instantaneous rate of change of $f(x)$ at $x = -1$.
   (c) Find the slope of the tangent to the graph of $y = f(x)$ at $x = -1$.
   (d) Find the point on the graph of $y = f(x)$ at $x = -1$.
4. Let $f(x) = 9 - \frac{1}{2}x^2$.
   (a) Use the Procedure/Example in this section to verify that $f'(x) = -x$.
   (b) Find the instantaneous rate of change of $f(x)$ at $x = 2$.
   (c) Find the slope of the tangent to the graph of $y = f(x)$ at $x = 2$.
   (d) Find the point on the graph of $y = f(x)$ at $x = 2$.

In Problems 5–8, the tangent line to the graph of $f(x)$ at $x = 1$ is shown. On the tangent line, P is the point of tangency and A is another point on the line.
(a) Find the coordinates of the points P and A.
(b) Use the coordinates of P and A to find the slope of the tangent line.
(c) Find $f'(1)$.
(d) Find the instantaneous rate of change of $f(x)$ at P.

5.


6.


7.


8.


For each function in Problems 9–14, find
(a) the derivative, by using the definition.
(b) the instantaneous rate of change of the function at any value and at the given value.
(c) the slope of the tangent at the given value.

9. $f(x) = 1 - 6x;\quad x = 20$
10. $f(x) = 4 - 5x;\quad x = -8$
11. $f(x) = 4x^2 - 2x + 1;\quad x = -3$
12. $f(x) = 16x^2 - 4x + 2;\quad x = 1$
13. $p(q) = q^2 + 4q + 1;\quad q = 5$
14. $p(q) = 2q^2 - 4q + 5;\quad q = 2$

For each function in Problems 15–18, approximate $f'(a)$ in the following ways.

(a) Use the numerical derivative feature of a graphing utility.

(b) Use $\dfrac{f(a+h) - f(a)}{h}$ with $h = 0.0001$.

(c) Graph the function on a graphing utility. Then zoom in near the point until the graph appears straight, pick two points, and find the slope of the line you see.

15. $f'(2)$ for $f(x) = 3x^4 - 7x - 5$
16. $f'(-1)$ for $f(x) = 2x^3 - 11x + 9$
17. $f'(4)$ for $f(x) = (2x - 1)^3$
18. $f'(3)$ for $f(x) = \dfrac{3x + 1}{2x - 5}$

In Problems 19 and 20, use the given tables to approximate $f'(a)$ as accurately as you can.

19. $a = 13$

x	12.0	12.99	13	13.1
$f(x)$	1.41	17.42	17.11	22.84

20. $a = -7.5$

x	−7.4	−7.50	−7.51	−7
$f(x)$	22.12	22.351	22.38	24.12

In Problems 21 and 22, a point (a, b) on the graph of $y = f(x)$ is given, and the equation of the line tangent to the graph of $f(x)$ at (a, b) is given. In each case, find $f'(a)$ and $f(a)$.

21. $(-3, -9);\quad 5x - 2y = 3$
22. $(-1, 6);\quad x + 10y = 59$
23. If the instantaneous rate of change of $f(x)$ at $(1, -1)$ is 3, write the equation of the line tangent to the graph of $f(x)$ at $x = 1$.
24. If the instantaneous rate of change of $g(x)$ at $(-1, -2)$ is 1/2, write the equation of the line tangent to the graph of $g(x)$ at $x = -1$.

Because the derivative of a function represents both the slope of the tangent to the curve and the instantaneous rate of change of the function, it is possible to use information about one to gain information about the other. In Problems 25 and 26, use the graph of the function $y = f(x)$ given in Figure 9.24.

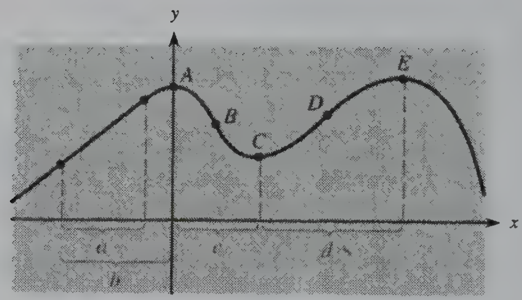


Figure 9.24

25. (a) Over what interval(s) (*a*) through (*d*) is the rate of change of $f(x)$ positive?
    (b) Over what interval(s) (*a*) through (*d*) is the rate of change of $f(x)$ negative?
    (c) At what point(s) A through E is the rate of change of $f(x)$ equal to zero?
26. (a) At what point(s) A through E does the rate of change of $f(x)$ change from positive to negative?
    (b) At what point(s) A through E does the rate of change of $f(x)$ change from negative to positive?
27. Given the graph of $y = f(x)$ in Figure 9.25, determine for which x-values A, B, C, D, or E the function is
    (a) continuous.
    (b) differentiable.
28. Given the graph of $y = f(x)$ in Figure 9.25, determine for which x-values F, G, H, I, or J the function is
    (a) continuous.
    (b) differentiable.

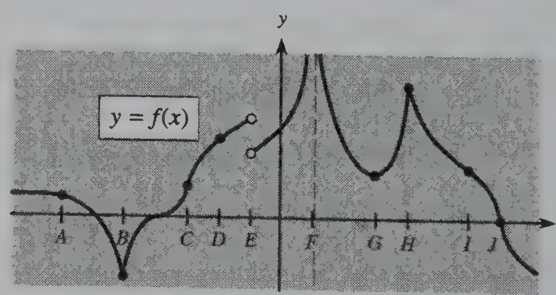


Figure 9.25

In Problems 29–32, (a) find the slope of the tangent to the graph of $f(x)$ at any point, (b) find the slope of the tangent at the given x-value, (c) write the equation of the line tangent to the graph of $f(x)$ at the given point, and (d) graph both $f(x)$ and its tangent line (use a graphing utility if one is available).

29. (a) $f(x) = x^2 + x$
(b) $x = 2$
(c) $(2, 6)$

30. (a) $f(x) = x^2 + 3x$
(b) $x = -1$
(c) $(-1, -2)$

31. (a) $f(x) = x^3 + 3$
(b) $x = 1$
(c) $(1, 4)$

32. (a) $f(x) = 5x^3 + 2$
(b) $x = -1$
(c) $(-1, -3)$

Applications

33. ***Average velocity*** When a ball is dropped from a height of 256 feet, its position (height above the ground) after x seconds is given by

$$S(x) = 256 - 16x^2$$

(a) What is the average velocity in the first 2 seconds of the fall?
(b) What does the negative average velocity in (a) mean?

34. ***Average velocity*** If an object is thrown upward at 64 ft/s from a height of 20 feet, its height S after x seconds is given by

$$S(x) = 20 + 64x - 16x^2$$

What is the average velocity in the
(a) first 2 seconds after it is thrown?
(b) next 2 seconds?

35. ***Demand*** If the demand for a product is given by

$$D(p) = \frac{1000}{\sqrt{p}} - 1$$

what is the average rate of change of demand when p increases from
(a) 1 to 25?
(b) 25 to 100?

36. ***Revenue*** If the total revenue function for a blender is

$$R(x) = 36x - 0.01x^2$$

where x is the number of units sold, what is the average rate of change in revenue $R(x)$ as x increases from 10 to 20 units?

37. ***Speed limit enforcement*** If 0.05 second elapses while a car on a road travels over two sensors that are 5 feet apart, what is the average velocity (in miles per hour) of the car as it travels between the sensors (88 feet per second is equivalent to 60 miles per hour)?

38. ***Speed limit enforcement*** One speed-check system used by local police measures the elasped time between two marks 0.1 mile apart. If 5.85 seconds elapse while a car on the highway travels between the two marks, find the average velocity of the car (in miles per hour) as it travels between the two marks.

39. ***Marginal revenue*** Say the revenue function for a stereo system is

$$R(x) = 300x - x^2$$

where x denotes the number of units sold.
(a) What is the function that gives marginal revenue?
(b) What is the marginal revenue if 50 units are sold and what does it mean?
(c) What is the marginal revenue if 200 units are sold and what does it mean?
(d) What is the marginal revenue if 150 units are sold and what does it mean?
(e) As the number of units sold passes through 150, what happens to revenue?

40. ***Marginal revenue*** Suppose the total revenue function for a blender is

$$R(x) = 36x - 0.01x^2$$

where x is the number of units sold.
(a) What function gives the marginal revenue?
(b) What is the marginal revenue when 600 units are sold and what does it mean?
(c) What is the marginal revenue when 2000 units are sold and what does it mean?
(d) What is the marginal revenue when 1800 units are sold and what does it mean?

41. ***Labor force and output*** The monthly output at the Olek Carpet Mill is

$$Q(x) = 15{,}000 + 2x^2 \text{ units}, \quad (40 \leq x \leq 60)$$

where x is the number of workers employed at the mill. If there are currently 50 workers, find the instantaneous rate of change of monthly output with respect to the number of workers. That is, find $Q'(50)$.

42. ***Consumer expenditure*** Suppose that the demand x for a product is

$$x = 10{,}000 - 100p$$

where p dollars is the price per unit. Then the consumer expenditure for the product is

$$E(p) = px = p(10{,}000 - 100p) = 10{,}000p - 100p^2$$

What is the instantaneous rate of change of consumer expenditure with respect to price at
(a) any price p? (b) $p = 5$? (c) $p = 20$?

In Problems 43–46, find derivatives with the numerical derivative feature of a graphing utility.

43. ***Profit*** Suppose that the profit function for the monthly sales of a car by a dealership is

$$P(x) = 500x - x^2 - 100$$

where x is the number of cars sold. What is the instantaneous rate of change of profit when

(a) 200 cars are sold? Explain its meaning.

(b) 300 cars are sold? Explain its meaning.

44. ***Profit*** If the total revenue function for a toy is

$$R(x) = 2x$$

and the total cost function is

$$C(x) = 100 + 0.2x^2 + x$$

what is the instantaneous rate of change of profit if 10 units are produced and sold? Explain its meaning.

45. ***Heat index*** The highest recorded temperature in the state of Alaska was 100°F and occurred on June 27, 1915, at Fort Yukon. The *heat index* is the apparent temperature of the air at a given temperature and humidity level. If x denotes the relative humidity (in percent), then the heat index (in degrees Fahrenheit) for an air temperature of 100°F can be approximated with the function

$$f(x) = 0.009x^2 + 0.139x + 91.875$$

(a) At what rate is the heat index changing when the humidity is 50%?

(b) Write a sentence that explains the meaning of your answer in (a).

46. ***Receptivity*** In learning theory, receptivity is defined as the ability of students to understand a complex concept. Receptivity is highest when the topic is introduced and tends to decrease as time passes in a lecture. Suppose that the receptivity of a group of students in a mathematics class is given by

$$g(t) = -0.2t^2 + 3.1t + 32$$

where t is minutes after the lecture begins.

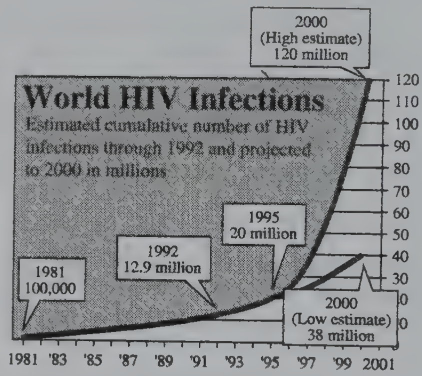
(a) At what rate is receptivity changing 10 minutes after the lecture begins?

(b) Write a sentence that explains the meaning of your answer in (a).

47. ***HIV infections*** The figure shows the number of world HIV infections since AIDS was discovered in 1981. The data are accurate through 1992, but are estimated beyond that time.

(a) Find the average rate of change in world HIV infections from 1981 to 1992.

(b) Find the average rates of change in world HIV infections from 1995 to 2000, using first the low estimate and then the high estimate.

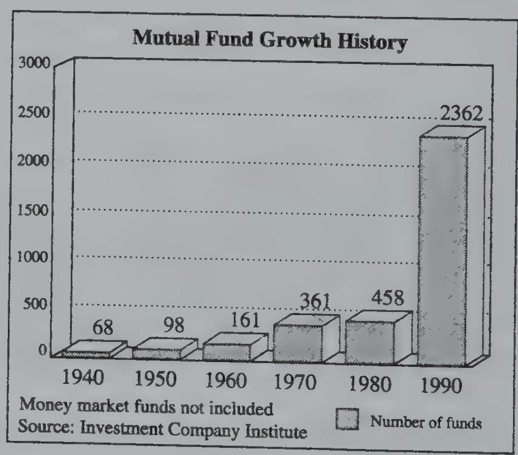


SOURCE: World Health Organization and the Global AIDS Policy Coalition, as reported in the *El Paso Times*, December 6, 1992.

48. ***Mutual funds*** The figure shows the history of the growth of mutual funds.

(a) Find the average annual rate of change in the number of funds from 1950 to 1960.

(b) Find the average rate of change in the number of funds from 1980 to 1990.



Published in *Investment Digest* of Valic Co., Vol. 5, No. 2, Summer, 1992.

9.4 Derivative Formulas

OBJECTIVES

- *To find derivatives of powers of x*
- *To find derivatives of constant functions*
- *To find derivatives of functions involving constant coefficients*
- *To find derivatives of sums and differences of functions*

APPLICATION PREVIEW

The killer bees bred in South America have entered the United States in spite of efforts to halt their spread. The first bees were recorded entering California in 1985 and were quickly destroyed. However, the first American was killed by the bees on July 15, 1993. Suppose that the bees enter a county in Texas and that the bee population in that county grows over a 6-week period, with the number of bees given by the equation

$$P(t) = 2t^2 + 10t + 1$$

where t is the number of weeks since the first bee is discovered. We can find the rate of growth of the bee population 2 weeks after the first bee is discovered by using the derivative $P'(t)$ of the growth function.

As we discussed in the previous section, the derivative of a function can be used to find the rate of change of the function. In this section we will develop formulas that will make it easier to find certain derivatives.

We can use the definition of derivative to show the following:

$$\text{If } f(x) = x^2, \text{ then } f'(x) = 2x.$$
$$\text{If } f(x) = x^3, \text{ then } f'(x) = 3x^2.$$
$$\text{If } f(x) = x^4, \text{ then } f'(x) = 4x^3.$$
$$\text{If } f(x) = x^5, \text{ then } f'(x) = 5x^4.$$

Do you recognize a pattern that could be used to find the derivative of $f(x) = x^6$? What is the derivative of $f(x) = x^n$? If you said that the derivative of $f(x) = x^6$ is $f'(x) = 6x^5$ and the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$, you're right. We can use the definition of derivative to show this. If n is a positive integer, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \end{aligned}$$

Because we are assuming that n is a positive integer, we can use the binomial formula to expand $(x + h)^n$. You may recall from Section 8.3 that this formula is stated as follows:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \cdots + b^n$$

Thus replacing a with x and b with h gives

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{1\cdot 2}x^{n-2}h^2 + \cdots + h^n\right] - x^n}{h} \\ &= \lim_{h\to 0} \left[nx^{n-1} + \frac{n(n-1)}{1\cdot 2}x^{n-2}h + \cdots + h^{n-1}\right] \end{aligned}$$

Now, each term after nx^{n-1} contains h as a factor, so all terms except nx^{n-1} will approach 0 as $h \to 0$. Thus

$$f'(x) = nx^{n-1}$$

Even though we proved this derivative rule only for the case when n is a positive integer, the rule applies for any real number n.

Powers of x Rule If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$.

EXAMPLE 1

Find the derivatives of the following functions.

(a) $g(x) = x^6$ (b) $f(x) = x^{-2}$
(c) $y = x^4$ (d) $y = x^{1/3}$

Solution

(a) If $g(x) = x^6$, then $g'(x) = 6x^{6-1} = 6x^5$.
(b) The Powers of x Rule applies for all real values. Thus for $f(x) = x^{-2}$, we have

$$f'(x) = -2x^{-2-1} = -2x^{-3} = \frac{-2}{x^3}$$

(c) If $y = x^4$, then $dy/dx = 4x^{4-1} = 4x^3$.
(d) The Powers of x Rule applies to $y = x^{1/3}$.

$$\frac{dy}{dx} = \frac{1}{3}x^{1/3-1} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

In Example 1 we took the derivative with respect to x of *both sides* of each equation. We denote the operation "take the derivative with respect to x" by $\frac{d}{dx}$. Thus for $y = x^4$, in (c),

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^4) \quad \text{gives} \quad \frac{dy}{dx} = 4x^3$$

Similarly, for $f(x) = x^{-2}$, in (b),

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(x^{-2}) \quad \text{gives} \quad f'(x) = -2x^{-3}$$

The differentiation rules are stated and proved for the independent variable x, but they also apply to other independent variables. The following examples illustrate differentiation with variables other than x.

EXAMPLE 2

Find the derivatives of the following functions.

(a) $u(s) = s^8$ (b) $p = q^{2/3}$ (c) $C(t) = \sqrt{t}$ (d) $s = \frac{1}{\sqrt{t}}$

Solution

(a) If $u(s) = s^8$, then $u'(s) = 8s^{8-1} = 8s^7$.

(b) If $p = q^{2/3}$, then

$$\frac{dp}{dq} = \frac{2}{3}q^{2/3-1} = \frac{2}{3}q^{-1/3} = \frac{2}{3q^{1/3}}$$

(c) Writing $\sqrt{t}$ in its equivalent form, $t^{1/2}$, permits us to use the derivative formula.

$$C'(t) = \frac{1}{2}t^{1/2-1} = \frac{1}{2}t^{-1/2}$$

Writing the derivative in radical form gives

$$C'(t) = \frac{1}{2} \cdot \frac{1}{t^{1/2}} = \frac{1}{2\sqrt{t}}$$

(d) Writing $1/\sqrt{t}$ as a power of t gives

$$s = \frac{1}{t^{1/2}} = t^{-1/2}, \quad \text{so} \quad \frac{ds}{dt} = -\frac{1}{2}t^{-1/2-1} = -\frac{1}{2}t^{-3/2}$$

Writing the derivative in a form similar to that of the original function gives

$$\frac{ds}{dt} = -\frac{1}{2} \cdot \frac{1}{t^{3/2}} = -\frac{1}{2\sqrt{t^3}}$$

EXAMPLE 3

Find the slope of the tangent to the curve $y = x^3$ at $x = 1$.

Solution

Finding the slope of the tangent line to $y = x^3$ at $x = 1$ involves two steps.

1. Find the derivative of $y = x^3$.

$$y' = 3x^2$$

2. Evaluate the derivative at $x = 1$.

$$m_{\tan} = y'\big|_{x=1} = y'(1) = 3(1)^2 = 3$$

The graph of $y = x^3$ and the tangent line to the graph at $x = 1$, $y = 1^3 = 1$ are shown in Figure 9.26.

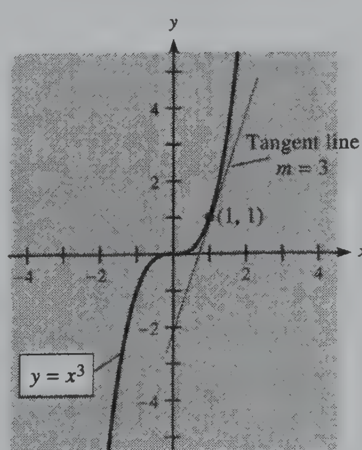


Figure 9.26

A function of the form $y = f(x) = c$, where c is a constant, is called a **constant function.** We can show that the derivative of a constant function is 0, as follows:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0$$

We can state this rule formally.

Constant Function Rule If $f(x) = c$, where c is a constant, then $f'(x) = 0$.

EXAMPLE 4

Find the derivative of the function defined by $y = 4$.

Solution

Because 4 is a constant, $\frac{dy}{dx} = 0$.

Recall that the function defined by $y = 4$ has a horizontal line as its graph. Thus the slope of the line (and the derivative of the function) is 0.

We now can take derivatives of constant functions and powers of x. But we do not yet have a rule for taking derivatives of functions of the form $f(x) = 4x^5$ or $g(t) = \frac{1}{2}t^2$. The following rule provides a method for handling functions of this type.

Coefficient Rule If $f(x) = c \cdot u(x)$, where c is a constant and $u(x)$ is a differentiable function of x, then $f'(x) = c \cdot u'(x)$.

The above formula says that the derivative of a constant times a function is the constant times the derivative of the function.

We can use the fact that

$$\lim_{h\to 0} c \cdot g(h) = c \cdot \lim_{h\to 0} g(h)$$

which was discussed in the section "Limits," to verify the coefficient rule. If $f(x) = c \cdot u(x)$, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c \cdot u(x+h) - c \cdot u(x)}{h} \\ &= \lim_{h\to 0} c \cdot \left[\frac{u(x+h) - u(x)}{h}\right] = c \cdot \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \end{aligned}$$

$$\text{so } f'(x) = c \cdot u'(x)$$

EXAMPLE 5

Find the derivatives of the following functions.

(a) $f(x) = 4x^5$ (b) $f(t) = \frac{1}{2}t^2$ (c) $p = \dfrac{5}{\sqrt{q}}$

Solution

(a) $f'(x) = 4(5x^4) = 20x^4$ (b) $g'(t) = \frac{1}{2}(2t) = t$

(c) $p = \dfrac{5}{\sqrt{q}} = 5q^{-1/2}$, so

$$\frac{dp}{dq} = 5\left(-\frac{1}{2}q^{-3/2}\right) = -\frac{5}{2\sqrt{q^3}}$$

In Example 4 of Section 9.3, "The Derivative: Rates of Change; Tangent to a Curve," we found the derivative of $f(x) = 3x^2 + 2x$ to be $f'(x) = 6x + 2$. This result, along with the results of several of the derivatives calculated in the exercise for that section, suggest that we can find the derivative of a function by finding the derivatives of its terms and combining them. The following rules state this formally.

Sum Rule If $f(x) = u(x) + v(x)$, where u and v are differentiable functions of x, then $f'(x) = u'(x) + v'(x)$.

We can prove this rule as follows. If $f(x) = u(x) + v(x)$, then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h} \\
&= \lim_{h\to 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h}\right] \\
&= \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} \\
&= u'(x) + v'(x)
\end{aligned}$$

Difference Rule If $f(x) = u(x) - v(x)$, where u and v are differentiable functions of x, then $f'(x) = u'(x) - v'(x)$.

EXAMPLE 6

Find the derivatives of the following functions.

(a) $y = x^2 + 3$ (b) $p = q^2 - 4q$ (c) $y = 3x + 5$

Solution

(a) $y' = 2 \cdot x + 0 = 2x$
(b) $dp/dq = 2 \cdot q - 4 \cdot 1 = 2q - 4$
(c) $y' = 3 \cdot 1 + 0 = 3$

In Example 6(c) we saw that the derivative of $y = 3x + 5$ is 3. Because the slope of a line is the same at all points on the line, it is reasonable that the derivative of a linear equation is a constant. In particular, the slope of the graph of the equation $y = mx + b$ is m at all points on its graph because the derivative of $y = mx + b$ is $y' = f'(x) = m$.

The rules regarding the derivatives of sums and differences of two functions also apply if more than two functions are involved. For example, the derivative of $f(x) = 4x^3 - 2x^2 + 5x - 3$ is $f'(x) = 12x^2 - 4x + 5$. We may think of the functions that are added and subtracted as terms of the function f. Then it would be correct to say that we may take the derivative of a function term by term.

EXAMPLE 7

Find the derivatives of the following functions.

(a) $y = 3x^3 - 4x^2$
(b) $p = \frac{1}{3}q^3 + 2q^2 - 3$
(c) $u(x) = 5x^4 + x^{1/3}$
(d) $y = 4x^3 + \sqrt{x}$
(e) $s = 5t^6 - \dfrac{1}{t^2}$

Solution

(a) $y' = 3(3x^2) - 4(2x) = 9x^2 - 8x$

(b) $\dfrac{dp}{dq} = \frac{1}{3}(3q^2) + 2(2q) - 0 = q^2 + 4q$

(c) $u'(x) = 5(4x^3) + \frac{1}{3}x^{-2/3} = 20x^3 + \dfrac{1}{3x^{2/3}}$

(d) We may write the function as

$$y = 4x^3 + x^{1/2}$$

so

$$\begin{aligned} y' &= 4(3x^2) + \frac{1}{2}x^{-1/2} = 12x^2 + \frac{1}{2x^{1/2}} \\ &= 12x^2 + \frac{1}{2\sqrt{x}} \end{aligned}$$

(e) We may write $s = 5t^6 - 1/t^2$ as

$$s = 5t^6 - t^{-2}$$

so

$$\frac{ds}{dt} = 5(6t^5) - (-2t^{-3}) = 30t^5 + 2t^{-3}$$
$$= 30t^5 + \frac{2}{t^3}$$

EXAMPLE 8

Find the slope of the tangent to $f(x) = \frac{1}{2}x^2 + 5x$ at each of the following.

(a) $x = 2$ (b) $x = -5$

Solution

The derivative of $f(x) = \frac{1}{2}x^2 + 5x$ is $f'(x) = x + 5$.

(a) At $x = 2$, the slope is $f'(2) = 2 + 5 = 7$.
(b) At $x = -5$, the slope is $f'(-5) = -5 + 5 = 0$. Thus when $x = -5$, the tangent to the curve is a horizontal line.

CHECKPOINT

1. True or false: The derivative of a constant times a function is equal to the constant times the derivative of the function.
2. True or false: The derivative of the sum of two functions is equal to the sum of the derivatives of the two functions.
3. True or false: The derivative of the difference of two functions is equal to the difference of the derivatives of the two functions.
4. Does the Coefficient Rule apply to $f(x) = x^n/c$, where c is a constant? Explain.
5. Find the derivative of each of the following functions.
   (a) $f(x) = x^{10} - 10x + 5$ (b) $s = \frac{1}{t^5} - 10^7 + 1$
6. Find the slope of the line tangent to $f(x) = x^3 - 4x^2 + 1$ at $x = -1$.

EXAMPLE 9

Find all points on the graph of $f(x) = x^3 + 3x^2 - 45x + 4$ where the tangent line is horizontal.

Solution

A horizontal line has slope equal to 0. Thus, to find the desired points, we solve $f'(x) = 0$.

$$f'(x) = 3x^2 + 6x - 45$$

We solve $3x^2 + 6x - 45 = 0$ as follows:

$$3x^2 + 6x - 45 = 0$$
$$3(x^2 + 2x - 15) = 0$$
$$3(x + 5)(x - 3) = 0$$

Solving $3(x + 5)(x - 3) = 0$ gives $x = -5$ and $x = 3$. The y-coordinates for these x-values come from $f(x)$. The desired points are $(-5, f(-5)) = (-5, 179)$ and $(3, f(3)) = (3, -77)$. Figure 9.27 shows the graph of $y = f(x)$ with these points and the tangent lines at them indicated.

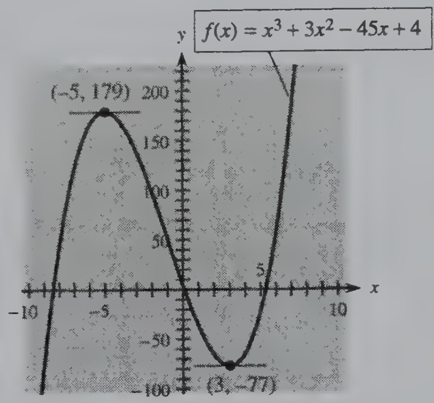


Figure 9.27

The marginal revenue $R'(x)$ is used to estimate the change in revenue caused by the sale of one additional unit.

EXAMPLE 10

Suppose that a manufacturer of a product knows that because of the demand for this product, his revenue is given by

$$R(x) = 1500x - 0.02x^2, \qquad 0 \leq x \leq 1000$$

where x is the number of units sold and $R(x)$ is in dollars.

(a) Find the marginal revenue at $x = 500$.
(b) Find the change in revenue caused by the increase in sales from 500 to 501 units.
(c) Find the difference between the marginal revenue found in (a) and the change in revenue found in (b).

Solution

(a) The marginal revenue for any value of x is

$$R'(x) = 1500 - 0.04x$$

The marginal revenue at $x = 500$ is

$$R'(500) = 1500 - 20 = 1480 \text{ (dollars)}$$

We can interpret this to mean that the approximate revenue from the sale of the 501st unit will be \$1480.

(b) The revenue at $x = 500$ is $R(500) = 745{,}000$, and the revenue at $x = 501$ is $R(501) = 746{,}479.98$, so the change in revenue is

$$R(501) - R(500) = 746{,}479.98 - 745{,}000 = 1479.98 \text{ (dollars)}$$

(c) The difference is $1480 - 1479.98 = 0.02$. Thus we see that the marginal revenue at $x = 500$ is a good estimate of the revenue from the 501st unit.

EXAMPLE 11

Suppose that the killer bees mentioned in the Application Preview enter a county in Texas and that the bee population in that county grows over a 6-week period, with the number of bees given by the equation

$$P(t) = 2t^2 + 10t + 1$$

where t is the number of weeks since the first bee is discovered. Find the rate of growth of the bee population 2 weeks after the first bee is discovered.

Solution

The rate of growth of the bees is given by $P'(t) = 4t + 10$, so the rate of growth 2 weeks after the first bee is discovered is $P'(2) = 18$ bees per week.

Graphing Utilities

We have mentioned that graphing utilities have a numerical derivative feature that can be used to estimate the derivative of a function at a specific value of x. This feature can also be used to check the derivative of a function that has been computed with a formula. We graph both the derivative calculated with a formula and the numerical derivative. If the two graphs lie on top of one another, the computed derivative agrees with the numerical derivative. Figure 9.28 illustrates this idea for the derivative of $f(x) = \frac{1}{3}x^3 - 2x^2 + 4$. Figure 9.28(a) shows $f'(x) = x^2 - 4x$ as y_1 and the calculator's numerical derivative of $f(x)$ as y_2. Figure 9.28(b) shows the graphs of both y_1 and y_2 (the graphs are coincident).

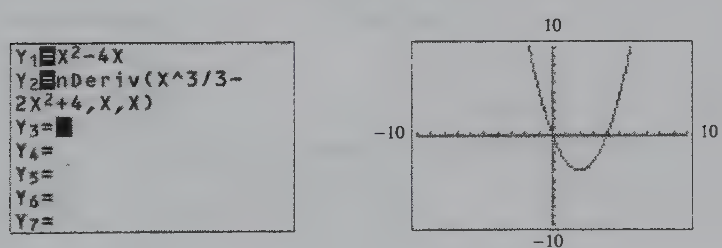


Figure 9.28 (a) (b)

EXAMPLE 12

(a) Graph $f(x) = x^3 - 3x + 3$ and its derivative $f'(x)$ on the same set of axes so that all values of x that make $f'(x) = 0$ are in the x-range.

(b) Investigate the graph of $y = f(x)$ near values of x where $f'(x) = 0$. Does the graph of $y = f(x)$ appear to turn at values where $f'(x) = 0$?

(c) Compare the interval of x values where $f'(x) < 0$ with the interval where the graph of $y = f(x)$ is decreasing from left to right.
(d) What is the relationship between the intervals where $f'(x) > 0$ and where the graph of $y = f(x)$ is increasing from left to right?

Solution

(a) The graphs of $f(x) = x^3 - 3x + 3$ and $f'(x) = 3x^2 - 3$ are shown in Figure 9.29.
(b) The values where $f'(x) = 0$ are the x-intercepts, $x = -1$ and $x = 1$. The graph of $y = x^3 - 3x + 3$ appears to turn at both these values.
(c) $f'(x) < 0$ where the graph of $y = f'(x)$ is below the x-axis, for $-1 < x < 1$. The graph of $y = f(x)$ appears to be decreasing on this interval.
(d) They appear to be the same intervals.

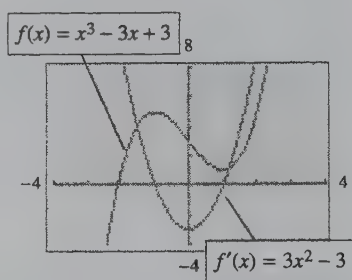


Figure 9.29

CHECKPOINT SOLUTIONS

1. True, by the Coefficient Rule.
2. True, by the Sum Rule.
3. True, by the Difference Rule.
4. Yes, $f(x) = x^n/c = (1/c)x^n$, so the coefficient is $(1/c)$.
5. (a) $f'(x) = 10x^9 - 10$
   (b) Note that $s = t^{-5} - 10^7 + 1$ and that 10^7 and 1 are constants.
$$\frac{ds}{dt} = -5t^{-6} = \frac{-5}{t^6}$$
6. The slope of the tangent at $x = -1$ is $f'(-1)$.
$$f'(x) = 3x^2 - 8x \qquad f'(-1) = 3(-1)^2 - 8(-1) = 11$$

EXERCISE 9.4

Find the derivatives of the functions in Problems 1–10.

1. $y = 4$
2. $f(s) = 6$
3. $y = x$
4. $s = t^2$
5. $f(x) = 2x^3 - x^5$
6. $f(x) = 3x^4 - x^9$
7. $y = 6x^4 - 5x^2 + x - 2$
8. $y = 3x^5 - 5x^3 - 8x + 8$
9. $g(x) = 10x^9 - 5x^5 + 7x^3 + 5x - 6$
10. $h(x) = 12x^{20} + 8x^{10} - 2x^7 + 17x - 9$

In Problems 11–14, at the indicated points, find
(a) the slope of the tangent to the curve, and
(b) the instantaneous rate of change of the function.

11. $y = 4x^2 + 3x, \quad x = 2$
12. $C(x) = 3x^2 - 5, \quad (3, 22)$
13. $P(x) = x^2 - 4x, \quad (2, -4)$
14. $R(x) = 16x + x^2, \quad x = 1$

In Problems 15–22, find the derivative of each function.

15. $y = x^{-5} + x^{-8} - 3$
16. $y = x^{-1} - x^{-2} + 13$
17. $y = 3x^{11/3} - 2x^{7/4} - x^{1/2} + 8$
18. $y = 5x^{8/5} - 3x^{5/6} + x^{1/3} + 5$
19. $f(x) = 5x^{-4/5} + 2x^{-4/3}$
20. $f(x) = 6x^{-8/3} - x^{-2/3}$
21. $g(x) = \dfrac{3}{x^4} + \dfrac{2}{x^5} + 5\sqrt[3]{x}$
22. $h(x) = \dfrac{7}{x^7} - \dfrac{3}{x^3} + 8\sqrt{x}$

In Problems 23–26, write the equations of the tangent lines to the curves at the indicated points.

23. $y = x^3 - 3x^2 + 5 \quad \text{at } x = 1$
24. $y = x^4 - 4x^3 - 2 \quad \text{at } x = 2$
25. $f(x) = 4x^2 - \dfrac{1}{x} \quad \text{at } x = -\dfrac{1}{2}$
26. $f(x) = \dfrac{x^3}{3} - \dfrac{3}{x^3} \quad \text{at } x = -1$

In Problems 27–30, find the coordinates of points where the graph of $f(x)$ has horizontal tangents.

27. $f(x) = -x^3 + 9x^2 - 15x + 6$
28. $f(x) = \dfrac{1}{3}x^3 - 3x^2 - 16x + 8$
29. $f(x) = x^4 - 4x^3 + 9$
30. $f(x) = 3x^5 - 5x^3 + 2$

In Problems 31 and 32, find each derivative at the given x-value (a) by finding the derivative with the appropriate rule and (b) with the numerical derivative feature of a graphing utility.

31. $y = 5 - 2\sqrt{x} \quad \text{at } x = 4$
32. $y = 1 + 3x^{2/3} \quad \text{at } x = -8$

In Problems 33–36, complete the following.

(a) Calculate the derivative of each function with the appropriate formula.
(b) Check your result from (a) by graphing your calculated derivative and the numerical derivative of the given function with respect to x evaluated at x.

33. $f(x) = 2x^3 + 5x - \pi^4 + 8$
34. $f(x) = 3x^2 - 8x + 2^5 - 20$
35. $h(x) = \dfrac{10}{x^3} - \dfrac{10}{\sqrt[5]{x^2}} + x^2 + 1$
36. $g(x) = \dfrac{5}{x^{10}} + \dfrac{4}{\sqrt[4]{x^3}} + x^5 - 4$

The tangent line to a curve at a point closely approximates the curve near the point. In fact, for x-values close enough to the point of tangency, the function and its tangent line are virtually indistinguishable. Problems 37 and 38 explore this relationship. Use each given function and the indicated point to complete the following.

(a) Write the equation of the tangent line to the curve at the indicated point.
(b) Use a graphing utility to graph both the function and its tangent line. Be sure your graph shows the point of tangency.
(c) Repeatedly zoom on the point of tangency until the function and the tangent line cannot be distinguished. Identify the x- and y-ranges in this window.

37. $f(x) = 3x^2 + 2x \quad \text{at } x = 1$
38. $f(x) = 4x - x^2 \quad \text{at } x = 5$

For each function in Problems 39–42, do the following.

(a) Find $f'(x)$.
(b) Graph both $f(x)$ and $f'(x)$ with a graphing utility.
(c) Use the graph of $f'(x)$ to identify x-values where $f'(x) = 0, f'(x) > 0$, and $f'(x) < 0$.
(d) Use the graph of $f(x)$ to identify x-values where $f(x)$ has a maximum or minimum point, where the graph of $f(x)$ is rising, and where the graph of $f(x)$ is falling.

39. $f(x) = 8 - 2x - x^2$
40. $f(x) = x^2 + 4x - 12$
41. $f(x) = x^3 - 12x - 5$
42. $f(x) = 7 - 3x^2 - \dfrac{x^3}{3}$

Applications

43. ***Revenue*** Suppose that a wholesaler expects that his monthly revenue for small television sets will be

$$R(x) = 100x - 0.1x^2, \quad 0 \le x \le 800$$

where x is the number of units sold. Find his marginal revenue and interpret it when the quantity sold is
(a) $x = 300$ (b) $x = 600$

44. ***Revenue*** The total revenue for a commodity is described by the function

$$R = 300x - 0.02x^2$$

(a) What is the marginal revenue when 40 units are sold?
(b) Interpret your answer to (a).

45. ***Workers and output*** The weekly output of a certain product is

$$Q(x) = 200x + 6x^2$$

where x is the number of workers on the assembly line. There are presently 60 workers on the line.
(a) Find $Q'(x)$ and estimate the change in the weekly output caused by the addition of one worker.
(b) Calculate $Q(61) - Q(60)$ to see the actual change in the weekly output.

46. ***Capital investment and output*** The monthly output of a certain product is

$$Q(x) = 800x^{5/2}$$

where x is the capital investment in millions of dollars. Find dQ/dx, which can be used to estimate the effect on the output if an additional capital investment of \$1 million is made.

47. ***Demand*** The demand q for a product depends on the price p (in dollars) according to

$$q = \frac{1000}{\sqrt{p}} - 1, \qquad \text{for } p > 0$$

Find and explain the meaning of the instantaneous rate of change of demand with respect to price when the price is
(a) \$25 (b) \$100

48. ***Demand*** Suppose that the demand for a product depends on the price p according to

$$D(p) = \frac{50{,}000}{p^2} - \frac{1}{2}, \qquad p > 0$$

where p is in dollars. Find and explain the meaning of the instantaneous rate of change of demand with respect to price when
(a) $p = 50$ (b) $p = 100$

49. ***Cost and average cost*** Suppose that the total cost function for the production of x units of a product is given by

$$C(x) = 4000 + 55x + 0.1x^2$$

Then the average cost of producing x items is

$$\overline{C(x)} = \frac{\text{total cost}}{x} = \frac{4000}{x} + 55 + 0.1x$$

(a) Find the instantaneous rate of change of average cost with respect to the number of units produced, at any level of production.
(b) Find the level of production at which this rate of change equals zero.

50. ***Cost and average cost*** Suppose that the total cost function for a certain commodity is given by

$$C(x) = 40{,}500 + 190x + 0.2x^2$$

where x is the number of units produced.
(a) Find the instantaneous rate of change of the average cost

$$\overline{C} = \frac{40{,}500}{x} + 190 + 0.2x$$

for any level of production.
(b) Find the level of production where this rate of change equals zero.

51. ***Cost-benefit*** Suppose that for a certain city the cost C of obtaining drinking water that contains p percent impurities (by volume) is given by

$$C = \frac{120{,}000}{p} - 1200$$

(a) Find the rate of change of cost with respect to p when impurities account for 1% (by volume).
(b) Write a sentence that explains the meaning of your answer in (a).

52. ***Cost-benefit*** Suppose that the cost C of processing the exhaust gases at an industrial site to ensure that only p percent of the particulate pollution escapes is given by

$$C(p) = \frac{8100(100 - p)}{p}$$

(a) Find the rate of change of cost C with respect to the percentage of particulate pollution that escapes when $p = 2$ (percent).
(b) Write a sentence interpreting your answer to (a).

53. ***Wind chill*** One form of the formula that meteorologists use to calculate wind chill temperature (WC) is

$$WC = 48.064 + 0.474t - 0.020ts - 1.85s + 0.304t\sqrt{s} - 27.74\sqrt{s}$$

where s is the wind speed in mph and t is the actual air temperature in degrees Fahrenheit. Suppose temperature is constant at 15°.
(a) Express wind chill WC as a function of wind speed s.
(b) Find the rate of change of wind chill with respect to wind speed when the wind speed is 25 mph.
(c) Interpret your answer to (b).

54. ***Allometric relationships—crabs*** For fiddler crabs, data gathered by Thompson* show that the allometric relationship between the weight C of the claw and the weight W of the body is given by

$$C = 0.11W^{1.54}$$

Find the function that gives the rate of change of claw weight with respect to body weight.

55. ***Union participation*** The following table shows the percentage of U.S. workers who belonged to unions for selected years from 1930 to 1996.

Year	*Percentage*
1930	11.6
1940	26.9
1950	31.5
1960	31.4
1970	27.3
1975	25.5
1980	21.9
1985	18
1990	16.1
1993	15.8
1994	15.5
1995	14.9
1996	14.5

Source: *World Almanac,* 1998

(a) Model these data with a cubic function, $u(x)$, where x is the number of years past 1900.
(b) For the period from 1980 to 1985, use the data points to find the average rate of change of the percentage of U.S. workers who belonged to unions.
(c) Find the instantaneous rate of change of the modeling function $u(x)$ for the year 1980.

56. ***Population below poverty level*** The table below shows the number of millions of people in the United States who lived below the poverty level for selected years between 1960 and 1996.

Year	*Persons Living Below the Poverty Level (millions)*
1960	39.9
1965	33.2
1970	25.4
1975	25.9
1980	29.3
1986	32.4
1990	33.6
1993	39.3
1994	38.1
1995	36.4
1996	36.5

Source: *The World Almanac and Book of Facts,* 1998

(a) Model these data with a cubic function $p = p(t)$, where p is the number of millions of people and t represents the number of years since 1900.
(b) Find the rate of growth in the number of persons living below the poverty level in 1980.
(c) Interpret your answer to (b).

57. ***Inflation rate*** The annual change in the consumer price index (CPI) for a 10-year period is shown in the table below. Assume that the annual change in the CPI can be modeled with a cubic function, $f(t)$, where t is the number of years past 1900.

Year	*Annual Percent Change in the CPI*
1987	3.6
1988	4.1
1989	4.8
1990	5.4
1991	4.2
1992	3.0
1993	3.0
1994	2.6
1995	2.8
1996	3.0

Source: *The World Almanac and Book of Facts,* 1998

(a) Find the function, $f(t)$, that models the data.
(b) Find the function that models the instantaneous rate of change of the CPI.
(c) Use the model found in (b) to find the instantaneous rate of change in 1989 and 1994.
(d) Interpret the two rates of change in (c).

*d'Arcy Thompson, *On Growth and Form* (Cambridge, England: Cambridge University Press, 1961).

58. ***Consumer debt*** The percentage of disposable income spent on consumer debt during certain years is shown in the table below.

Year	Consumer Debt as a Percentage of Disposable Income	Year	Consumer Debt as a Percentage of Disposable Income
1980	18.2	1990	20.0
1982	16.9	1991	18.9
1983	17.7	1994	19.7
1985	20.8	1995	21.3
1986	21.4	1996	23.7
1988	21.0		

Source: Federal Reserve System

Assume that consumer debt as a percentage of disposable income can be modeled with a fourth-degree function, $d(t)$, where t is the number of years past 1900.

(a) Find the function $d(t)$ that models the data.
(b) Find the function that models the instantaneous rate of change of consumer debt as a percentage of disposable income.
(c) Use the model found in (b) to find the instantaneous rate of change in 1985 and in 1995.
(d) Interpret the two rates of change found in (c).

9.5 Product and Quotient Rules

OBJECTIVES

- *To use the Product Rule to find the derivative of certain functions*
- *To use the Quotient Rule to find the derivative of certain functions*

APPLICATION PREVIEW

When medicine is administered, reaction (measured in change of blood pressure or temperature) can be modeled by

$$R = m^2\left(\frac{c}{2} - \frac{m}{3}\right)$$

where c is a positive constant and m is the amount of medicine absorbed into the blood.* The rate of change of R with respect to m is the sensitivity of the body to medicine. To find an expression for sensitivity as a function of m, we calculate dR/dm. We can find this derivative with the **Product Rule** for derivatives.

We have simple formulas for finding the derivatives of the sums and differences of functions. But we are not so lucky with products. The derivative of a product is *not* the product of the derivatives. To see this, we consider the function $f(x) = x \cdot x$. Because this function is $f(x) = x^2$, its derivative is $f'(x) = 2x$. But the product of the derivatives of x and x would give $1 \cdot 1 = 1 \neq 2x$. Thus we need a different formula to find the derivative of a product. This formula is given by the Product Rule.

Product Rule If $f(x) = u(x) \cdot v(x)$, where u and v are differentiable functions of x, then

$$f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)$$

*Source: Thrall, R. M., *et al.*, *Some Mathematical Models in Biology,* U.S. Department of Commerce, 1967.

Thus the derivative of a product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.

We can prove the Product Rule as follows. If $f(x) = u(x) \cdot v(x)$, then

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{u(x+h)\cdot v(x+h) - u(x)\cdot v(x)}{h}$$

Subtracting and adding $u(x + h) \cdot v(x)$ in the numerator gives

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{u(x+h)\cdot v(x+h) - u(x+h)\cdot v(x) + u(x+h)\cdot v(x) - u(x)\cdot v(x)}{h} \\ &= \lim_{h\to 0}\left(u(x+h)\left[\frac{v(x+h)-v(x)}{h}\right] + v(x)\left[\frac{u(x+h)-u(x)}{h}\right]\right) \end{aligned}$$

Properties III and IV of limits give

$$f'(x) = \lim_{h\to 0} u(x+h)\cdot \lim_{h\to 0}\frac{v(x+h)-v(x)}{h} + \lim_{h\to 0} v(x)\cdot \lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$$

Because u is differentiable and hence continuous, it follows that $\lim_{h\to 0} u(x + h) = u(x)$, so we have $f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)$.

EXAMPLE 1

Use the Product Rule to find the derivative of $f(x) = x^2 \cdot x$.

Solution

Using the formula with $u(x) = x^2$, $v(x) = x$, we have

$$\begin{aligned} f'(x) &= u(x) \cdot v'(x) + v(x) \cdot u'(x) \\ &= x^2 \cdot 1 + x(2x) \\ &= x^2 + 2x^2 \\ &= 3x^2 \end{aligned}$$

Note that we could have found the same result by multiplying the factors and finding the derivative of $f(x) = x^3$. But we will soon see how valuable the Product Rule is.

EXAMPLE 2

Find dy/dx if $y = (2x^3 + 3x + 1)(x^2 + 4)$.

Solution

Using the Product Rule with $u(x) = 2x^3 + 3x + 1$ and $v(x) = x^2 + 4$, we have

$$\begin{aligned} \frac{dy}{dx} &= (2x^3 + 3x + 1)(2x) + (x^2 + 4)(6x^2 + 3) \\ &= 4x^4 + 6x^2 + 2x + 6x^4 + 3x^2 + 24x^2 + 12 \\ &= 10x^4 + 33x^2 + 2x + 12 \end{aligned}$$

We could, of course, avoid using the Product Rule by multiplying the two factors before taking the derivative. But multiplying the factors first may involve more work than using the Product Rule.

EXAMPLE 3

Given $f(x) = (4x^3 + 5x^2 - 6x + 5)(x^3 - 4x^2 + 1)$, find the slope of the tangent to the graph of $y = f(x)$ at $x = 1$.

Solution

$$f'(x) = (4x^3 + 5x^2 - 6x + 5)(3x^2 - 8x) + (x^3 - 4x^2 + 1)(12x^2 + 10x - 6)$$

If we substitute $x = 1$ into $f'(x)$, we find that the slope of the curve at $x = 1$ is $f'(1) = 8(-5) + (-2)(16) = -72$.

The rule for finding the derivative of a function that is the quotient of two functions requires a new formula.

Quotient Rule If $f(x) = u(x)/v(x)$, where u and v are differentiable functions of x, with $v(x) \neq 0$, then

$$f'(x) = \frac{v(x) \cdot u''(x) - u(x) \cdot v''(x)}{[v(x)]^2}$$

The preceding formula says that the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

To see that this rule is reasonable, consider the function $f(x) = x^3/x$, $x \neq 0$. Using the Quotient Rule, with $u(x) = x^3$ and $v(x) = x$, we get

$$f'(x) = \frac{x(3x^2) - x^3(1)}{x^2} = \frac{3x^3 - x^3}{x^2} = \frac{2x^3}{x^2} = 2x$$

Because $f(x) = x^3/x = x^2$ if $x \neq 0$, we see that $f'(x) = 2x$ is the correct derivative. The proof of the Quotient Rule is left for the student in Problem 41 of the exercises in this section.

EXAMPLE 4

If $f(x) = \dfrac{x^2 - 4x}{x + 5}$, find $f'(x)$.

Solution

Using the Quotient Rule with $u(x) = x^2 - 4x$ and $v(x) = x + 5$, we get

$$\begin{aligned} f'(x) &= \frac{(x+5)(2x-4) - (x^2 - 4x)(1)}{(x+5)^2} \\ &= \frac{2x^2 + 6x - 20 - x^2 + 4x}{(x+5)^2} \\ &= \frac{x^2 + 10x - 20}{(x+5)^2} \end{aligned}$$

EXAMPLE 5

If $f(x) = \dfrac{x^3 - 3x^2 + 2}{x^2 - 4}$, find $f'(x)$.

Solution

Using the Quotient Rule with $u(x) = x^3 - 3x^2 + 2$ and $v(x) = x^2 - 4$, we get

$$\begin{aligned} f'(x) &= \frac{(x^2 - 4)(3x^2 - 6x) - (x^3 - 3x^2 + 2)(2x)}{(x^2 - 4)^2} \\ &= \frac{(3x^4 - 6x^3 - 12x^2 + 24x) - (2x^4 - 6x^3 + 4x)}{(x^2 - 4)^2} \\ &= \frac{x^4 - 12x^2 + 20x}{(x^2 - 4)^2} \end{aligned}$$

EXAMPLE 6

Use the Quotient Rule to find the derivative of $y = 1/x^3$.

Solution

Letting $u(x) = 1$ and $v(x) = x^3$, we get

$$\begin{aligned} y' &= \frac{x^3(0) - 1(3x^2)}{(x^3)^2} \\ &= -\frac{3x^2}{x^6} \\ &= -\frac{3}{x^4} \end{aligned}$$

Note that we could have found the derivative more easily by writing

$$y = 1/x^3 = x^{-3}$$

so

$$y' = -3x^{-4} = -\frac{3}{x^4}$$

Recall that we proved the Powers of x Rule for positive integer powers and assumed that it was true for all real number powers. In Problem 42 of the exercises in this section, you will be asked to use the Quotient Rule to show that the Powers of x Rule applies to negative integers.

It is not necessary to use the Quotient Rule when the denominator of the function in question contains only a constant. For example, the function $y = (x^3 - 3x)/3$ can be written $y = \frac{1}{3}(x^3 - 3x)$, so the derivative is $y' = \frac{1}{3}(3x^2 - 3) = x^2 - 1$.

CHECKPOINT

1. True or false: The derivative of the product of two functions is equal to the product of the derivatives of the two functions.
2. True or false: The derivative of the quotient of two functions is equal to the quotient of the derivatives of the two functions.

3. Find $f'(x)$ for each of the following.
   (a) $f(x) = (x^{12} + 8x^5 - 7)(10x^7 - 4x + 19)$ Do not simplify.
   (b) $f(x) = \dfrac{2x^4 + 3}{3x^4 + 2}$ Simplify.
4. If $y = \frac{4}{3}(x^2 + 3x - 4)$, does finding y' require the Product Rule? Explain.
5. If $y = f(x)/c$, where c is a constant, does finding y' require the Quotient Rule? Explain.

EXAMPLE 7

Suppose that the revenue function for a product is given by

$$R(x) = 10x + \frac{100x}{3x + 5}$$

where x is the number of units sold and R is in dollars.

(a) Find the marginal revenue function.
(b) Find the marginal revenue when $x = 15$.

Solution

(a) We must use the Quotient Rule to find the marginal revenue (the derivative).

$$\overline{MR} = R'(x) = 10 + \frac{(3x+5)(100) - 100x(3)}{(3x+5)^2}$$

$$= 10 + \frac{300x + 500 - 300x}{(3x+5)^2} = 10 + \frac{500}{(3x+5)^2}$$

(b) The marginal revenue when $x = 15$ is $R'(15)$.

$$R'(15) = 10 + \frac{500}{[(3)(15) + 5]^2} = 10 + \frac{500}{(50)^2}$$

$$= 10 + \frac{500}{2500} = 10.20$$

Recall that $R'(15)$ estimates the revenue from the sale of the 16th item.

We now consider the problem posed in the Application Preview.

EXAMPLE 8

When medicine is administered, reaction (measured in change of blood pressure or temperature) can be modeled by

$$R = m^2\left(\frac{c}{2} - \frac{m}{3}\right)$$

where c is a positive constant and m is the amount of medicine absorbed into the blood. The rate of change of R with respect to m is the sensitivity of the body to medicine. Find an expression for sensitivity s as a function of m.

Solution

The sensitivity is the rate of change of R with respect to m, or the derivative

$$\frac{dR}{dm} = m^2\left(0 - \frac{1}{3}\right) + \left(\frac{c}{2} - \frac{1}{3}m\right)(2m)$$
$$= -\frac{1}{3}m^2 + mc - \frac{2}{3}m^2 = mc - m^2$$

so sensitivity is given by

$$s = \frac{dR}{dm} = mc - m^2$$

EXAMPLE 9

(a) Graph $f(x) = \dfrac{8x}{x^2 + 4}$ and its derivative on the same set of axes over an interval that contains the x-values where $f'(x) = 0$.

(b) Determine the values of x where $f'(x) = 0$ and the intervals where $f'(x) > 0$ and where $f'(x) < 0$.

(c) What is the relationship between the derivative $f'(x)$ being positive or negative and the graph of $y = f(x)$ increasing (rising) or decreasing (falling)?

Solution

We find the derivative of $f(x)$ by using the Quotient Rule.

$$f'(x) = \frac{(x^2+4)(8) - (8x)(2x)}{(x^2+4)^2}$$
$$= \frac{32 - 8x^2}{(x^2+4)^2}$$

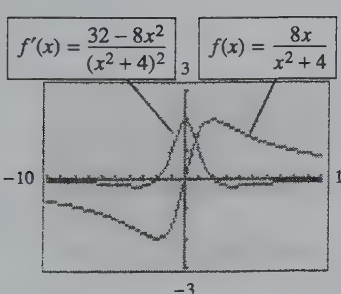


Figure 9.30

(a) The graphs of $f(x) = \dfrac{8x}{x^2+4}$ and $f'(x) = \dfrac{32 - 8x^2}{(x^2+4)^2}$ are shown in Figure 9.30.

(b) We can use TRACE or one of the solution features to find that $f'(x) = 0$ at $x = -2$ and $x = 2$. In the interval $-2 < x < 2$, the graph of $y = f'(x)$ is above the x-axis, so $f'(x)$ is positive for $-2 < x < 2$. In the intervals $x < -2$ and $x > 2$, the graph of $y = f'(x)$ is below the x-axis, so $f'(x)$ is negative there.

(c) The graph of $y = f(x)$ appears to turn at $x = -2$, and at $x = 2$, where $f'(x) = 0$. The graph of $y = f(x)$ is increasing where $f'(x) > 0$ and decreasing where $f'(x) < 0$.

CHECKPOINT SOLUTIONS

1. False. The derivative of a product is equal to the first function times the derivative of the second plus the second function times the derivative of the first. That is,

$$\frac{d}{dx}(fg) = f \cdot \frac{dg}{dx} + g \cdot \frac{df}{dx}$$

2. False. The derivative of a quotient is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator. That is,

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \cdot f' - f \cdot g'}{g^2}$$

3. (a) $f'(x) = (x^{12} + 8x^5 - 7)(70x^6 - 4) + (10x^7 - 4x + 19)(12x^{11} + 40x^4)$

(b) $$f'(x) = \frac{(3x^4+2)(8x^3) - (2x^4+3)(12x^3)}{(3x^4+2)^2} = \frac{24x^7 + 16x^3 - 24x^7 - 36x^3}{(3x^4+2)^2} = \frac{-20x^3}{(3x^4+2)^2}$$

4. No; y' can be found with the Coefficient Rule:

$$y' = \frac{4}{3}(2x+3)$$

5. No; y' can be found with the Coefficient Rule:

$$y' = \left(\frac{1}{c}\right)f'(x)$$

EXERCISE 9.5

1. Find y' if $y = (x + 3)(x^2 - 2x)$.
2. Find $f'(x)$ if $f(x) = (3x - 1)(x^3 + 1)$.
3. Find $\frac{dp}{dq}$ if $p = (3q - 1)(q^2 + 2)$.
4. Find $\frac{ds}{dt}$ if $s = (t^4 + 1)(t^3 - 1)$.
5. If $f(x) = (x^{12} + 3x^4 + 4)(4x^3 - 1)$, find $f'(x)$.
6. If $y = (3x^7 + 4)(8x^6 - 6x^4 - 9)$, find $\frac{dy}{dx}$.

In Problems 7–10, find the derivative, but do not simplify your answer.

7. $y = (7x^6 - 5x^4 + 2x^2 - 1)(4x^9 + 3x^7 - 5x^2 + 3x)$
8. $y = (9x^9 - 7x^7 - 6x)(3x^5 - 4x^4 + 3x^3 - 8)$
9. $y = (x^2 + x + 1)(\sqrt[3]{x} - 2\sqrt{x} + 5)$
10. $y = (\sqrt[5]{x} - 2\sqrt[4]{x} + 1)(x^3 - 5x - 7)$

In Problems 11 and 12, at each indicated point find
(a) the slope of the tangent line, and
(b) the instantaneous rate of change of the function.

11. $y = (x^2 + 1)(x^3 - 4x)$ at $(-2, 0)$
12. $y = (x^3 - 3)(x^2 - 4x + 1)$ at $(2, -15)$

In Problems 13–24, find the indicated derivatives.

13. y' for $y = \dfrac{x}{x^2 - 1}$
14. $f'(x)$ for $f(x) = \dfrac{x^2}{x - 3}$
15. $\dfrac{dp}{dq}$ for $p = \dfrac{q^2 + 1}{q - 2}$
16. $C'(x)$ for $C(x) = \dfrac{x^2 + 1}{x^2 - 1}$
17. $\dfrac{dy}{dx}$ for $y = \dfrac{1 - 2x^2}{x^4 - 2x^2 + 5}$
18. $\dfrac{ds}{dt}$ for $s = \dfrac{t^3 - 4}{t^3 - 2t^2 - t - 5}$
19. $\dfrac{dz}{dx}$ for $z = x^2 + \dfrac{x^2}{1 - x - 2x^2}$
20. $\dfrac{dy}{dx}$ for $y = 200x - \dfrac{100x}{3x + 1}$
21. $\dfrac{dp}{dq}$ for $p = \dfrac{3\sqrt[3]{q}}{1 - q}$
22. $\dfrac{dy}{dx}$ for $y = \dfrac{2\sqrt{x} - 1}{1 - 4\sqrt{x^3}}$
23. y' for $y = \dfrac{x(x^2 + 4)}{x - 2}$
24. $f'(x)$ for $f(x) = \dfrac{(x + 1)(x - 2)}{x^2 + 1}$

In Problems 25 and 26, at the indicated point for each function, find
(a) the slope of the tangent line, and
(b) the instantaneous rate of change of the function.

25. $y = \dfrac{x^2 + 1}{x + 3}$ at $(2, 1)$
26. $y = \dfrac{x^2 - 4x}{x^2 + 2x}$ at $\left(2, -\dfrac{1}{2}\right)$

In Problems 27–30, write the equation of the tangent line to the graph of the function at the indicated point.

27. $y = (9x^2 - 6x + 1)(1 + 2x)$ at $x = 1$
28. $y = (4x^2 + 4x + 1)(7 - 2x)$ at $x = 0$
29. $y = \dfrac{3x^4 - 2x - 1}{4 - x^2}$ at $x = 1$
30. $y = \dfrac{x^2 - 4x}{2x - x^3}$ at $x = 2$

In Problems 31–34, use the numerical derivative feature of a graphing utility to find the derivative of each function at the given x-value.

31. $y = \left(4\sqrt{x} + \dfrac{3}{x}\right)\left(3\sqrt[3]{x} - \dfrac{5}{x^2} - 25\right)$ at $x = 1$

32. $y = (3\sqrt[4]{x^5} + \sqrt[5]{x^4} - 1)\left(\dfrac{2}{x^3} - \dfrac{1}{\sqrt{x}}\right)$ at $x = 1$

33. $f(x) = \dfrac{4x - 4}{3x^{2/3}}$ at $x = 1$

34. $f(x) = \dfrac{3\sqrt[3]{x} + 1}{x + 2}$ at $x = -1$

In Problems 35–38, complete the following.
(a) Find the derivative of each function, and check your work by graphing both your calculated derivative and the numerical derivative of the function.
(b) Use your graph of the derivative to find points where the original function has horizontal tangent lines.
(c) Use a graphing utility to graph the function and indicate the points found in (b) on the graph.

35. $f(x) = (x^2 + 4x + 4)(x - 7)$

36. $f(x) = (x^2 - 14x + 49)(2x + 1)$

37. $y = \dfrac{x^2}{x - 2}$

38. $y = \dfrac{x^2 - 7}{4 - x}$

In Problems 39 and 40,
(a) find $f'(x)$.
(b) graph both $f(x)$ and $f'(x)$ with a graphing utility.
(c) identify the x-values where $f'(x) = 0$, $f'(x) > 0$, and $f'(x) < 0$.
(d) identify x-values where $f(x)$ has a maximum point or a minimum point, where $f(x)$ is increasing, and where $f(x)$ is decreasing.

39. $f(x) = \dfrac{10x^2}{x^2 + 1}$

40. $f(x) = \dfrac{8 - x^2}{x^2 + 4}$

41. Prove the Quotient Rule for differentiation. (*Hint:* Add $[-u(x) \cdot v(x) + u(x) \cdot v(x)]$ to the expanded numerator and use steps similar to those used to prove the Product Rule.)

42. Use the Quotient Rule to show that the Powers of x Rule applies to negative integer powers. That is, show that $(d/dx)x^n = nx^{n-1}$ when $n = -k$, $k > 0$, by finding the derivative of $f(x) = 1/(x^k)$.

Applications

43. ***Cost-benefit*** If the cost C of removing p percent of the particulate pollution from the exhaust gases at an industrial site is given by

$$C(p) = \frac{8100p}{100 - p}$$

find the rate of change of C with respect to p.

44. ***Cost-benefit*** If the cost C of removing p percent of the impurities from the waste water in a manufacturing process is given by

$$C(p) = \frac{9800p}{101 - p}$$

find the rate of change of C with respect to p.

45. ***Revenue*** Suppose the revenue function for a product is given by

$$R(x) = \frac{60x^2 + 74x}{2x + 2}$$

Find the marginal revenue when 49 units are sold. Interpret your result.

46. ***Revenue*** The revenue from the sale of x units of a product is given by

$$R(x) = \frac{3000}{2x + 2} + 80x - 1500$$

Find the marginal revenue when 149 units are sold. Interpret your result.

47. ***Revenue*** A travel agency will plan a group tour for groups of size 25 or larger. If the group contains exactly 25 people, the cost is \$300 per person. If each person's cost is reduced by \$10 for each additional person above the 25, then the revenue is given by

$$R(x) = (25 + x)(300 - 10x)$$

where x is the number of additional people above 25. Find the marginal revenue if the group contains 30 people. Interpret your result.

48. ***Revenue*** McRobert's TV Shop sells 200 sets per month at a price of \$400 per unit. Market research indicates that the shop can sell one additional set for each \$1 it reduces the price, and in this case the total revenue is

$$R(x) = (200 + x)(400 - x)$$

where x is the number of additional sets beyond the 200. If the shop sells a total of 250 sets, find the marginal revenue. Interpret your result.

49. ***Response to a drug*** The reaction R to an injection of a drug is related to the dosage x according to

$$R(x) = x^2\left(500 - \frac{x}{3}\right)$$

where 1000 mg is the maximum dosage. If the rate of reaction with respect to the dosage defines the sensitivity to the drug, find the sensitivity.

50. ***Nerve response*** The number of action potentials produced by a nerve, t seconds after a stimulus, is given by

$$N(t) = 25t + \frac{4}{t^2+2} - 2$$

Find the rate at which the action potentials are produced by the nerve.

51. ***Test reliability*** If a test having reliability r is lengthened by a factor n, the reliability of the new test is given by

$$R = \frac{nr}{1+(n-1)r}, \quad 0 < r \leq 1$$

Find the rate at which R changes with respect to n.

52. ***Advertising and sales*** The sales of a product s (in thousands of dollars) are related to advertising expenses (in thousands of dollars) by

$$s = \frac{200x}{x+10}$$

Find and interpret the meaning of the rate of change of sales with respect to advertising expenses when
(a) $x = 10$ (b) $x = 20$

53. ***Candidate recognition*** Suppose that the proportion P of voters who recognize a candidate's name t months after the start of the campaign is given by

$$P(t) = \frac{13t}{t^2+100} + 0.18$$

(a) Find the rate of change of P when $t = 6$ and explain its meaning.
(b) Find the rate of change of P when $t = 12$ and explain its meaning.
(c) One month prior to the election, is it better for $P'(t)$ to be positive or negative?

54. ***Endangered species population*** It is determined that a wildlife refuge can support a group of up to 120 of a certain endangered species. If 75 are introduced onto the refuge and their population after t years is given by

$$p(t) = 75\left(1 + \frac{4t}{t^2+16}\right)$$

find the rate of population growth after t years. Find the rate after each of the first 7 years.

55. ***Wind chill*** According to the National Climatic Data Center, during 1991, the lowest temperature recorded in Indianapolis, Indiana, was 0°F. If x is the wind speed in miles per hour and $x \geq 5$, then the wind chill (in degrees Fahrenheit) for an air temperature of 0°F can be approximated by the function

$$f(x) = \frac{289.173 - 58.5731x}{x+1}$$

(a) At what rate is the wind chill changing when the wind speed is 20 mph?
(b) Explain the meaning of your answer to (a).

56. ***Response to injected adrenalin*** Experimental evidence has shown that the concentration of injected adrenaline x is related to the response y of a muscle according to the equation

$$y = \frac{x}{a+bx}$$

where a and b are constants. Find the rate of change of response with respect to the concentration.

57. ***Social Security*** America's 45 million Social Security recipients got a 2.6% cost-of-living increase in 1994. That was the second-smallest increase in nearly 20 years, a reflection of the low inflation rate. The following data show the percent increases for selected years. (We will assume that $t = 0$ in 1985.)

Year	*Percent Increase in Cost of Living*
1989	4.0
1990	4.9
1991	5.1
1992	4.1
1993	3.5
1994	2.6

Source: Social Security Administration

Assume these data can be modeled by the equation

$$C(t) = (t+5)(-0.218t + 3.57) - 20.13$$

(a) Find the rate of change of the cost-of-living increase as predicted by the model for 1990.
(b) Interpret your answer to (a).
(c) Graph both the data and the model using a graphing utility.
(d) From the graph in (c), identify t-values for which the model must be invalid.

58. ***Drug use*** The percentage of high school seniors who have tried hallucinogens is shown in the table below for selected years.

Year of Graduation	*Hallucinogens*
1975	16.3
1978	14.3
1980	13.3
1983	11.9
1984	10.7
1985	10.3
1986	9.7
1987	10.3
1988	8.9
1989	9.4
1990	9.4
1991	9.6

Source: National Institute on Drug Abuse

The data from the table indicate that the percentage of high school seniors who have tried hallucinogens can be modeled with the function

$$f_h(t) = 216.074 - (t + 70)(2.782 - 0.0242t)$$

where t is the number of years past 1970.

(a) Find the function that models the instantaneous rate of change, with respect to time, of the percentage of seniors who have tried hallucinogens.
(b) Use the model in (a) to find the instantaneous rate of change in 1978 and in 1988.
(c) Interpret each rate of change found in (b).

59. ***Farm workers*** The percentage of U.S. workers in farm occupations during certain years is shown in the table.

Year	*Percent of All Workers in Farm Occupations*
1820	71.8
1850	63.7
1870	53
1900	37.5
1920	27
1930	21.2
1940	17.4
1950	11.6
1960	6.1
1970	3.6
1980	2.7
1985	2.8
1990	2.4

Source: *The World Almanac and Book of Facts*, 1993

Assume that the percentage of U.S. workers in farm occupations can be modeled with the function

$$f(t) = 1000 \cdot \frac{-8.0912t + 1558.9}{1.09816t^2 - 122.183t + 21472.6}$$

where t is the number of years past 1800.

(a) Find the function that models the instantaneous rate of change of the percentage of U.S. workers in farm occupations.
(b) Use the model in (a) to find the instantaneous rate of change in 1870 and in 1970.
(c) Interpret each of the rates of change in (b).

9.6 The Chain Rule and Power Rule

OBJECTIVES

- *To use the Chain Rule to differentiate functions*
- *To use the Power Rule to differentiate functions*

APPLICATION PREVIEW

The demand x for a product is given by

$$x = \frac{98}{\sqrt{2p + 1}} - 1$$

where p is the price per unit. To find how fast demand is changing when price is \$24, we take the derivative of x with respect to p. If we write this function with a power rather than a radical, it has the form

$$x = 98(2p + 1)^{-1/2} - 1$$

The formulas learned so far cannot be used to find this derivative. Rather than use the limit definition of derivative, we use a new formula, the **Power Rule,** to find this derivative. In this section we will discuss the **Chain Rule** and the Power Rule, which is one of the results of the Chain Rule, and we will use these formulas to solve applied problems.

Recall from Section 1.2, "Functions," that if f and g are functions, then the composite functions g of f (denoted $g \circ f$) and f of g (denoted $f \circ g$) are defined as follows:

$$(g \circ f)(x) = g(f(x)) \quad \text{and} \quad (f \circ g)(x) = f(g(x))$$

EXAMPLE 1

If $f(x) = 3x^2$ and $g(x) = 2x - 1$, find $F(x) = f(g(x))$.

Solution

Substituting $g(x) = 2x - 1$ for x in $f(x)$ gives

$$f(g(x)) = f(2x - 1) = 3(2x - 1)^2$$

Thus $F(x) = 3(2x - 1)^2$.

We could find the derivative of the function $F(x) = 3(2x - 1)^2$ by multiplying out the expression $3(2x - 1)^2$. Then

$$F(x) = 3(4x^2 - 4x + 1) = 12x^2 - 12x + 3$$

so $F'(x) = 24x - 12$. But we can also use a very powerful rule, called the **Chain Rule,** to find derivatives of functions of this type. If we write the composite function $y = f(g(x))$ in the form $y = f(u)$, where $u = g(x)$, we state the Chain Rule as follows:

Chain Rule If f and g are differentiable functions with $y = f(u)$ and $u = g(x)$, then y is a differentiable function of x, and

$$\frac{dy}{dx} = \frac{d}{du} f(u) \cdot \frac{d}{dx} g(x)$$

or, written another way,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Note that dy/du represents the derivative of $y = f(u)$ *with respect to* u and du/dx represents the derivative of $u = g(x)$ *with respect to* x. For example, if $y = 3(2x - 1)^2$, we may write $y = f(u) = 3u^2$, where $u = 2x - 1$. Then the derivative is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 6u \cdot 2 = 12u$$

To write this derivative in terms of x, we substitute $2x - 1$ for u. Thus

$$\frac{dy}{dx} = 12(2x - 1) = 24x - 12$$

Note that we get the same result by using the Chain Rule as we did by multiplying out $f(x) = 3(2x - 1)^2$. The Chain Rule is important because it is not always possible to rewrite the function as a polynomial. Consider the following example.

EXAMPLE 2

If $y = \sqrt{x^2 - 1}$, find $\frac{dy}{dx}$.

Solution

If we write this function as $y = f(u) = \sqrt{u}$, when $u = x^2 - 1$, we can find the derivative.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2} \cdot u^{-1/2} \cdot 2x = u^{-1/2} \cdot x = \frac{1}{\sqrt{u}} \cdot x = \frac{x}{\sqrt{u}}$$

To write this derivative in terms of x alone, we substitute $x^2 - 1$ for u. Then

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 - 1}}$$

Note that we could not find the derivative of a function like that of Example 2 by the methods learned previously.

EXAMPLE 3

If $y = \frac{1}{(x^2 + 3x + 1)^2}$, find $\frac{dy}{dx}$.

Solution

If we let $u = x^2 + 3x + 1$, we can write $y = f(u) = \frac{1}{u^2}$, or $y = u^{-2}$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -2u^{-3}(2x + 3) = \frac{-4x - 6}{u^3}$$

Substituting for u gives

$$\frac{dy}{dx} = \frac{-4x - 6}{(x^2 + 3x + 1)^3}$$

EXAMPLE 4

The relationship between the length L (in meters) and weight W (in kilograms) of a species of fish in the Pacific Ocean is given by $W = 10.375L^3$. The rate of growth in length is given by $\frac{dL}{dt} = 0.36 - 0.18L$, where t is measured in years.

(a) Determine a formula for the rate of growth in weight $\frac{dW}{dt}$ in terms of L.

(b) If a fish weighs 30 kilograms, approximate its rate of growth in weight using the formula found in (a).

Solution

(a) The rate of change uses the Chain Rule, as follows:

$$\frac{dW}{dt} = \frac{dW}{dL} \cdot \frac{dL}{dt} = 31.125L^2(0.36 - 0.18L) = 11.205L^2 - 5.6025L^3$$

(b) From $W = 10.375L^3$, we have

$$L = \sqrt[3]{\frac{W}{10.375}}$$

If $W = 30$ kilograms, $L = \sqrt[3]{\dfrac{30}{10.375}} = 1.4247$ meters, so the rate of growth in weight is

$$\frac{dW}{dt} = 11.205(1.4247)^2 - 5.6025(1.4247)^3 = 6.542 \text{ kilograms/year}$$

The Chain Rule is very useful and will be extremely important with functions that we will study later, but a special case of the Chain Rule, called the **Power Rule,** is useful for the algebraic functions we have studied so far.

Power Rule If $y = u^n$, where u is a differentiable function of x, then

$$\frac{dy}{dx} = nu^{n-1} \cdot \frac{du}{dx}$$

EXAMPLE 5

If $y = (x^2 - 4x)^6$, find $\dfrac{dy}{dx}$.

Solution

The right side of the equation is in the form u^n, with $u = x^2 - 4x$. Thus, by the Power Rule,

$$\frac{dy}{dx} = nu^{n-1} \cdot \frac{du}{dx} = 6u^5(2x - 4)$$

Substituting for u gives

$$\begin{aligned}\frac{dy}{dx} &= 6(x^2 - 4x)^5(2x - 4) \\ &= (12x - 24)(x^2 - 4x)^5\end{aligned}$$

EXAMPLE 6

If $y = 3\sqrt[3]{x^2 - 3x + 1}$, find y'.

Solution

Because $y = 3(x^2 - 3x + 1)^{1/3}$, we can make use of the Power Rule with $u = x^2 - 3x + 1$.

$$\begin{aligned} y' = 3\left(nu^{n-1}\frac{du}{dx}\right) &= 3\left[\frac{1}{3}u^{-2/3}(2x-3)\right] \\ &= (x^2 - 3x + 1)^{-2/3}(2x - 3) \\ &= \frac{2x-3}{(x^2 - 3x + 1)^{2/3}} \end{aligned}$$

EXAMPLE 7

If $p = \dfrac{4}{3q^2 + 1}$, find $\dfrac{dp}{dq}$.

Solution

We can use the Power Rule to find dp/dq if we write the equation in the form

$$p = 4(3q^2 + 1)^{-1}$$

Then

$$\frac{dp}{dq} = 4[-1(3q^2 + 1)^{-2}(6q)] = \frac{-24q}{(3q^2 + 1)^2}$$

The derivative of the function in Example 7 can also be found by using the Quotient Rule, but the Power Rule provides a more efficient method.

EXAMPLE 8

Find the derivative of $g(x) = \dfrac{1}{\sqrt{(x^2 + 1)^3}}$.

Solution

Writing $g(x)$ as a power gives

$$g(x) = (x^2 + 1)^{-3/2}$$

Then

$$\begin{aligned} g'(x) = -\frac{3}{2}(x^2 + 1)^{-5/2}(2x) &= -3x \cdot \frac{1}{(x^2 + 1)^{5/2}} \\ &= \frac{-3x}{\sqrt{(x^2 + 1)^5}} \end{aligned}$$

CHECKPOINT

1. (a) If $f(x) = (3x^4 + 1)^{10}$, does $f'(x) = 10(3x^4 + 1)^9$?
   (b) If $f(x) = (2x + 1)^5$, does $f'(x) = 10(2x + 1)^4$?
   (c) If $f(x) = \dfrac{[u(x)]^n}{c}$, where c is a constant, does $f'(x) = \dfrac{n[u(x)]^{n-1} \cdot u'(x)}{c}$?
2. (a) If $f(x) = \dfrac{12}{2x^2 - 1}$, find $f'(x)$ by using the Power Rule (not the Quotient Rule).
   (b) If $f(x) = \dfrac{\sqrt{x^3 - 1}}{3}$, find $f'(x)$ by using the Power Rule (not the Quotient Rule).

EXAMPLE 9

The demand x for a product is given by

$$x = 98(2p + 1)^{-1/2} - 1$$

where p is the price per unit. Find the rate of change of the demand with respect to price when $p = 24$.

Solution

The rate of change of demand with respect to price is

$$\frac{dx}{dp} = 98\left[-\frac{1}{2}(2p+1)^{-3/2}(2)\right] = -98(2p+1)^{-3/2}$$

When $p = 24$, the rate of change is

$$\left.\frac{dx}{dp}\right|_{p=24} = -98(48+1)^{-3/2} = -98 \cdot \frac{1}{49^{3/2}}$$

$$= -98 \cdot \frac{1}{343}$$

$$= -\frac{2}{7}$$

CHECKPOINT SOLUTIONS

1. (a) No, $f'(x) = 10(3x^4 + 1)^9(12x^3)$. (b) Yes (c) Yes
2. (a) $f(x) = 12(2x^2 - 1)^{-1}$

   $f'(x) = -12(2x^2 - 1)^{-2}(4x) = \dfrac{-48x}{(2x^2 - 1)^2}$

   (b) $f(x) = \dfrac{1}{3}(x^3 - 1)^{1/2}$

   $f'(x) = \dfrac{1}{6}(x^3 - 1)^{-1/2}(3x^2) = \dfrac{x^2}{2\sqrt{x^3 - 1}}$

EXERCISE 9.6

In Problems 1–4, find the indicated derivatives.

1. $\dfrac{dy}{dx}$ for $y = u^3$ and $u = x^2 + 1$
2. $\dfrac{dp}{dq}$ for $p = u^4$ and $u = q^2 + 4q$
3. $\dfrac{dy}{dx}$ for $y = u^4$ and $u = 4x^2 - x + 8$
4. $\dfrac{dr}{ds}$ for $r = u^{10}$ and $u = s^2 + 5s$

Differentiate the functions in Problems 5–20.

5. $f(x) = \dfrac{1}{(x^2 + 2)^3}$
6. $g(x) = \dfrac{1}{4x^3 + 1}$
7. $g(x) = (x^2 + 4x)^{-2}$
8. $p = (q^3 + 1)^{-5}$
9. $c(x) = (x^2 + 3x + 4)^{-3}$
10. $y = (x^2 - 8x)^{2/3}$
11. $g(x) = \dfrac{1}{(2x^3 + 3x + 5)^{3/4}}$
12. $y = \dfrac{1}{(3x^3 + 4x + 1)^{3/2}}$
13. $y = \sqrt{x^2 + 4x + 5}$
14. $y = \sqrt{x^2 + 3x}$
15. $s = 4\sqrt{3x - x^2}$
16. $y = 3\sqrt[3]{(x - 1)^2}$

17. $y = \dfrac{8(x^2-3)^5}{5}$ 18. $y = \dfrac{5\sqrt{1-x^3}}{6}$

19. $y = \dfrac{(3x+1)^5 - 3x}{7}$

20. $y = \dfrac{\sqrt{2x-1} - \sqrt{x}}{2}$

At the indicated point, for each function in Problems 21–24, find
(a) the slope of the tangent line, and
(b) the instantaneous rate of change of the function.

A graphing utility's numerical derivative feature can be used to check your work.

21. $y = (x^3 + 2x)^4$ at $x = 2$
22. $y = \sqrt{5x^2 + 2x}$ at $x = 1$
23. $y = \sqrt{x^3 + 1}$ at $(2, 3)$
24. $y = (4x^3 - 5x + 1)^3$ at $(1, 0)$

In Problems 25–28, write the equation of the line tangent to the graph of each function at the indicated point.

25. $y = (x^2 - 3x + 3)^3$ at $(1, 1)$
26. $y = (x^2 + 1)^3$ at $(2, 125)$
27. $y = \sqrt{3x^2 - 2}$ at $x = 3$
28. $y = \left(\dfrac{1}{x^3 - x}\right)^3$ at $x = 2$

In Problems 29 and 30, complete the following for each function.
(a) Find $f'(x)$.
(b) Check your result in (a) by graphing both it and the numerical derivative of the function.
(c) Find x-values for which the slope of the tangent is 0.
(d) Find points (x, y) where the slope of the tangent is 0.
(e) Use a graphing utility to graph the function and locate the points found in (d).

29. $f(x) = (x^2 - 4)^3 + 12$
30. $f(x) = 10 - (x^2 - 2x - 8)^2$

In Problems 31 and 32, do the following for each function $f(x)$.
(a) Find $f'(x)$,
(b) Graph both $f(x)$ and $f'(x)$ with a graphing utility,
(c) Determine x-values where $f'(x) = 0$, $f'(x) > 0$, $f'(x) < 0$.
(d) Determine x-values for which $f(x)$ has a maximum or minimum point, where the graph is increasing, and where it is decreasing.

31. $f(x) = 5 - 3(1 - x^2)^{4/3}$
32. $f(x) = 3 + \dfrac{1}{16}(x^2 - 4x)^4$

In Problems 33 and 34, find the derivative of each function.

33. (a) $y = \dfrac{2x^3}{3}$ (b) $y = \dfrac{2}{3x^3}$
(c) $y = \dfrac{(2x)^3}{3}$ (d) $y = \dfrac{2}{(3x)^3}$

34. (a) $y = \dfrac{3}{(5x)^5}$ (b) $y = \dfrac{3x^5}{5}$
(c) $y = \dfrac{3}{5x^5}$ (d) $y = \dfrac{(3x)^5}{5}$

Applications

35. ***Ballistics*** Ballistics experts are able to identify the weapon that fired a certain bullet by studying the markings on the bullet. Tests are conducted by firing into a bale of paper. If the distance s, in inches, that the bullet travels into the paper is given by

$$s = 27 - (3 - 10t)^3$$

for $0 \le t \le 0.3$ second, find the velocity of the bullet one-tenth of a second after it hits the paper.

36. ***Population of microorganisms*** Suppose that the population of a certain microorganism at time t (in minutes) is given by

$$P = 1000 - 1000(t + 10)^{-1}$$

Find the rate of change of population.

37. ***Revenue*** The revenue from the sale of x units of a product is

$$R = 1500x + 3000(2x + 3)^{-1} - 1000$$

where x is the number of units sold. Find the marginal revenue when 100 units are sold. Interpret your result.

38. ***Revenue*** The revenue from the sale of x units of a product is

$$R = 15(3x + 1)^{-1} + 50x - 15$$

Find the marginal revenue when 40 units are sold. Interpret your result.

39. ***Pricing and sales*** Suppose that the weekly sales volume y (in thousands of units sold) depends on the price per unit of the product according to

$$y = 32(3p + 1)^{-2/5}, \quad p > 0$$

where p is in dollars.
(a) What is the rate of change in sales volume when the price is \$21?
(b) Interpret your answer to (a).

40. ***Pricing and sales*** A chain of auto service stations has found that its monthly sales volume y (in thousands of dollars) is related to the price p (in dollars) of an oil change according to

$$y = \frac{90}{\sqrt{p+5}}, \quad p > 10$$

(a) What is the rate of change of sales volume when the price is \$20?
(b) Interpret your answer to (a).

41. ***Demand*** Suppose that the demand for a product is described by

$$p = \frac{200{,}000}{(q+1)^2}$$

(a) What is the rate of change of price with respect to the quantity demanded when $q = 49$?
(b) Interpret your answer to (a).

Stimulus-response The relation between the magnitude of a sensation y and the magnitude of the stimulus x is given by

$$y = k(x - x_0)^n$$

where k is a constant, x_0 is the threshold of effective stimulus, and n depends on the type of stimulus. Find the rate of change of sensation with respect to the amount of stimulus for each of Problems 42–44.

42. For the stimulus of visual brightness $y = k(x - x_0)^{1/3}$
43. For the stimulus of warmth $y = k(x - x_0)^{8/5}$
44. For the stimulus of electrical stimulation $y = k(x - x_0)^{7/2}$

45. ***Demand*** If the demand for a product is described by the equation

$$p = \frac{100}{\sqrt{2q+1}}$$

find the rate of change of p with respect to q.

46. ***Advertising and sales*** The daily sales S (in thousands of dollars) attributed to an advertising campaign are given by

$$S = 1 + \frac{3}{t+3} - \frac{18}{(t+3)^2}$$

where t is the number of weeks the campaign runs. What is the rate of change of sales at
(a) $t = 8$? (b) $t = 10$?
(c) Should the campaign be continued after the 10th week? Explain.

47. ***Body-heat loss*** The description of body-heat loss due to convection involves a coefficient of convection, K_c, which depends on wind velocity according to the following equation.

$$K_c = 4\sqrt{4v+1}$$

Find the rate of change of the coefficient with respect to the wind velocity.

48. ***Typing speed*** The typing speed (in words per minute) of a secretarial student is

$$S = 10\sqrt{0.8x+4}, \quad 0 \le x \le 100$$

where x is the number of hours of training he has had. What is the rate at which his speed is changing and what does this rate mean when he has had
(a) 15 hours of training?
(b) 40 hours of training?

49. ***Investments*** If an IRA is a variable-rate investment for 20 years at rate r percent per year, compounded monthly, then the future value S that accumulates from an initial investment of \$1000 is

$$S = 1000\left[1 + \frac{0.01r}{12}\right]^{240}$$

What is the rate of change of S with respect to r and what does it tell us if the interest rate is (a) 6%? (b) 12%?

50. ***Concentration of body substances*** The concentration C of a substance in the body depends on the quantity of the substance Q and the volume V through which it is distributed. For a static substance this is given by

$$C = \frac{Q}{V}$$

For a situation like that in the kidneys, where the fluids are moving, the concentration is the ratio of the rate of change of quantity with respect to time and the rate of change of volume with respect to time.
(a) Formulate the equation for concentration of a moving substance.
(b) Show that this is equal to the rate of change of quantity with respect to volume.

51. ***Public debt of the United States*** The interest paid on the public debt of the United States of America, as a percentage of federal expenditures for selected years, is shown in the table.

Assume that the percentage of federal expenditures devoted to payment of interest can be modeled with the function

$$d(t) = -0.1543(0.1t + 3)^4 + 4.2743(0.1t + 3)^3 \\ -42.1504(0.1t + 3)^2 + 175.805(0.1t + 3) \\ -251.334$$

where t is the number of years past 1930. Use this model to determine and interpret the instantaneous rate of change of the percentage of federal expenditures devoted to payment of interest on the public debt in 1960 and in 1990.

Year	*Interest Paid as a Percentage of Federal Expenditures*
1930	0
1940	10.5
1950	13.4
1955	9.4
1960	10.0
1965	9.6
1970	9.9
1975	9.8
1980	12.7
1985	18.9
1990	21.1
1995	22.0

Source: Bureau of Public Debt, Department of the Treasury

52. ***Union membership*** The table shows the percentage of U.S. workers who belonged to unions for selected years from 1930 to 1996.

Year	*Union Membership as a Percentage of the Labor Force*
1930	11.6
1935	13.2
1940	26.9
1945	35.5
1950	31.5
1955	33.2
1960	31.4
1965	28.4
1970	27.3
1975	25.5
1980	21.9
1985	18.0
1990	16.1
1993	15.8
1994	15.5
1995	14.9
1996	14.5

Source: Bureau of Labor Statistics, Department of Labor

Assume that the percentage of the labor force that belongs to unions can be modeled with the function

$$f(t) = 0.4711(0.1t + 2)^3 - 10.653(0.1t + 2)^2 \\ + 73.874(0.1t + 2) - 129.237$$

where t is the number of years past 1920. Use the model to find and interpret the instantaneous rate of change of union membership in 1940 and in 1990.

53. ***Persons living below the poverty level*** The table below shows the number of millions of people in the United States who lived below the poverty level for selected years between 1960 and 1996.

Year	*Persons Living Below the Poverty Level (in Millions)*
1960	39.9
1965	33.2
1970	25.4
1975	25.9
1980	29.3
1986	32.4
1990	33.6
1993	39.3
1994	38.1
1995	36.4
1996	36.5

Source: *The World Almanac and Book of Facts,* 1998

Assume that the number of persons below the poverty level can be modeled by

$$p(t) = -0.001773(t + 60)^3 + 0.4506(t + 60)^2 \\ -37.44(t + 60) + 1047.8$$

where t is the number of years past 1960. Use the model to find and interpret the instantaneous rate of change of the number of persons below the poverty level in 1970 and in 1990.

54. ***Inflation rate*** The annual change in the consumer price index (CPI) during certain years is shown in the table. Assume that the annual change in the CPI can be modeled with the function

$$f(t) = 0.0308469(t + 87)^3 - 8.4978(t + 87)^2 + 779.66(t + 87) - 23,820.3$$

where t is the number of years past 1987.

Year	*Consumer Price Index Annual Percent Change*
1987	3.6
1988	4.1
1989	4.8
1990	5.4
1991	4.2
1992	3.0
1993	3.0
1994	2.6
1995	2.8
1996	3.0

Source: *The World Almanac and Book of Facts*, 1998

Use the model to find and interpret the instantaneous rate of change of the CPI in 1990 and in 1995.

9.7 Using Derivative Formulas

OBJECTIVE

- *To use derivative formulas separately and in combination with each other*

APPLICATION PREVIEW

Suppose the weekly revenue function for a product is given by

$$R(x) = \frac{36,000,000x}{(2x + 500)^2}$$

where x is the number of units sold. We can find marginal revenue by finding the derivative of the revenue function. This revenue function contains both a quotient and a power, so its derivative is found by using both the Quotient Rule and the Power Rule. But before we do this, we must first decide the order in which to apply these formulas.

We have used the Power Rule to find the derivative of functions like

$$y = (x^3 - 3x^2 + x + 1)^5$$

but we have not found the derivative of functions like

$$y = [(x^2 + 1)(x^3 + x + 1)]^5$$

This function is different because the function u (which is raised to the fifth power) is the product of two functions, $(x^2 + 1)$ and $(x^3 + x + 1)$. The equation is of the form $y = u^5$, where $u = (x^2 + 1)(x^3 + x + 1)$. This means that the Product Rule should be used to find du/dx. Then

$$\begin{aligned}\frac{dy}{dx} &= 5u^4 \cdot \frac{du}{dx}\\ &= 5[(x^2+1)(x^3+x+1)]^4[(x^2+1)(3x^2+1)+(x^3+x+1)(2x)]\\ &= 5[(x^2+1)(x^3+x+1)]^4(5x^4+6x^2+2x+1)\\ &= (25x^4+30x^2+10x+5)[(x^2+1)(x^3+x+1)]^4\end{aligned}$$

A different type of problem involving the Power Rule and the Product Rule is finding the derivative of $y = (x^2 + 1)^5(x^3 + x + 1)$. We may think of y as the *product* of two functions, one of which is a power. Thus the fundamental formula we should use is the Product Rule. The two functions are $u(x) = (x^2 + 1)^5$ and $v(x) = x^3 + x + 1$. The Product Rule gives

$$\begin{aligned}\frac{dy}{dx} &= u(x) \cdot v'(x) + v(x) \cdot u'(x)\\ &= (x^2+1)^5(3x^2+1) + (x^3+x+1)[5(x^2+1)^4 2x]\end{aligned}$$

Note that the Power Rule was used to find $u'(x)$, since $u(x) = (x^2 + 1)^5$.

We can simplify dy/dx by factoring $(x^2 + 1)^4$ from both terms:

$$\begin{aligned}\frac{dy}{dx} &= (x^2+1)^4[(x^2+1)(3x^2+1) + (x^3+x+1) \cdot 5 \cdot 2x]\\ &= (x^2+1)^4(13x^4+14x^2+10x+1)\end{aligned}$$

EXAMPLE 1

If $y = \left(\dfrac{x^2}{x-1}\right)^5$, find y'.

Solution

We again have an equation of the form $y = u^n$, but this time u is a quotient. Thus we will need the Quotient Rule to find du/dx.

$$y' = nu^{n-1} \cdot \frac{du}{dx} = 5u^4\,\frac{(x-1)\cdot 2x - x^2 \cdot 1}{(x-1)^2}$$

Substituting for u and simplifying gives

$$\begin{aligned}y' &= 5\left(\frac{x^2}{x-1}\right)^4 \cdot \frac{2x^2-2x-x^2}{(x-1)^2}\\ &= \frac{5x^8(x^2-2x)}{(x-1)^6} = \frac{5x^{10}-10x^9}{(x-1)^6}\end{aligned}$$

EXAMPLE 2

Find $f'(x)$ if $f(x) = \dfrac{(x-1)^2}{(x^4+3)^3}$.

Solution

This function is the quotient of two functions, $(x - 1)^2$ and $(x^4 + 3)^3$, so we must use the Quotient Rule to find the derivative of $f(x)$, but taking the derivatives of $(x - 1)^2$ and $(x^4 + 3)^3$ will require the Power Rule.

$$f'(x) = \frac{[v(x) \cdot u'(x) - u(x) \cdot v'(x)]}{[v(x)]^2}$$

$$= \frac{(x^4+3)^3[2(x-1)(1)] - (x-1)^2[3(x^4+3)^2\,4x^3]}{[(x^4+3)^3]^2}$$

$$= \frac{2(x^4+3)^3(x-1) - 12x^3(x-1)^2(x^4+3)^2}{(x^4+3)^6}$$

We see that 2, $(x^4 + 3)^2$, and $(x - 1)$ are all factors in both terms of the numerator, so we can factor them from both terms and reduce the fraction.

$$f'(x) = \frac{2(x^4+3)^2(x-1)[(x^4+3) - 6x^3(x-1)]}{(x^4+3)^6}$$

$$= \frac{2(x-1)(-5x^4+6x^3+3)}{(x^4+3)^4}$$

EXAMPLE 3

Find $f'(x)$ if $f(x) = (x^2 - 1)\sqrt{3 - x^2}$.

Solution

The function is the product of two functions, $x^2 - 1$ and $\sqrt{3 - x^2}$. Therefore, we will use the Product Rule to find the derivative of $f(x)$, but the derivative of $\sqrt{3 - x^2} = (3 - x^2)^{1/2}$ will require the Power Rule.

$$f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)$$

$$= (x^2 - 1)\left[\frac{1}{2}(3 - x^2)^{-1/2}(-2x)\right] + (3 - x^2)^{1/2}(2x)$$

$$= (x^2 - 1)[-x(3 - x^2)^{-1/2}] + (3 - x^2)^{1/2}(2x)$$

$$= \frac{-x^3 + x}{(3 - x^2)^{1/2}} + 2x(3 - x^2)^{1/2}$$

We can combine these terms over the common denominator $(3 - x^2)^{1/2}$ as follows:

$$f'(x) = \frac{-x^3 + x}{(3 - x^2)^{1/2}} + \frac{2x(3 - x^2)^1}{(3 - x^2)^{1/2}} = \frac{-x^3 + x + 6x - 2x^3}{(3 - x^2)^{1/2}}$$

$$= \frac{-3x^3 + 7x}{(3 - x^2)^{1/2}}$$

We should note that in Example 3 we could have written $f'(x)$ in the form

$$f'(x) = (-x^3 + x)(3 - x^2)^{-1/2} + 2x(3 - x^2)^{1/2}$$

Now the factor $(3 - x^2)$, to different powers, is contained in both terms of the expression. Thus we can factor $(3 - x^2)^{-1/2}$ from both terms. (We choose the $-1/2$ power because it is the smaller of the two powers.) Dividing $(3 - x^2)^{-1/2}$ into the first term gives $(-x^3 + x)$, and dividing it into the second term gives $2x(3 - x^2)^1$. Why? Thus we have

$$f'(x) = (3 - x^2)^{-1/2}[(-x^3 + x) + 2x(3 - x^2)]$$
$$= \frac{-3x^2 + 7x}{(3 - x^2)^{1/2}}$$

which agrees with our previous answer.

CHECKPOINT

1. If a function has the form $y = [u(x)]^n \cdot v(x)$, where n is a constant, we begin to find the derivative by using the ________ Rule and then use the ________ Rule to find the derivative of $[u(x)]^n$.
2. If a function has the form $y = [u(x)/v(x)]^n$, where n is a constant, we begin to find the derivative by using the ________ Rule and then use the ________ Rule.
3. Find the derivative of each of the following and simplify.
   (a) $f(x) = 3x^4(2x^4 + 7)^5$ (b) $g(x) = \dfrac{(4x + 3)^7}{2x - 9}$

We now return to the Application Preview problem.

EXAMPLE 4

Suppose that the weekly revenue function for a product is given by

$$R(x) = \frac{36{,}000{,}000x}{(2x + 500)^2}$$

where x is the number of units sold.
(a) Find the marginal revenue function.
(b) Find the marginal revenue when 50 units are sold.

Solution

(a) $\overline{MR} = R'(x)$

$$= \frac{(2x + 500)^2(36{,}000{,}000) - 36{,}000{,}000x[2(2x + 500)^1(2)]}{(2x + 500)^4}$$
$$= \frac{36{,}000{,}000(2x + 500)(2x + 500 - 4x)}{(2x + 500)^4}$$
$$= \frac{36{,}000{,}000(500 - 2x)}{(2x + 500)^3}$$

(b) $$\overline{MR}(50) = R'(50) = \frac{36{,}000{,}000(500 - 100)}{(100 + 500)^3}$$
$$= \frac{36{,}000{,}000(400)}{(600)^3}$$
$$= \frac{200}{3} = 66.67$$

The marginal revenue is \$66.67 when 50 units are sold. That is, the predicted revenue from the sale of the 51st unit is approximately \$66.67.

It may be helpful to review the formulas needed to find the derivatives of various types of functions. Table 9.5 presents examples of different types of functions and the formulas needed to find their derivatives.

TABLE 9.5 Derivative Formulas Summary

Example	*Formula*
$f(x) = 14$	If $f(x) = c$, then $f'(x) = 0$.
$y = x^4$	If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
$g(x) = 5x^3$	If $g(x) = cf(x)$, then $g'(x) = cf'(x)$.
$y = 3x^2 + 4x$	If $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$.
$y = (x^2 - 2)(x + 4)$	If $f(x) = u(x) \cdot v(x)$, then $f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)$.
$f(x) = \dfrac{x^3}{x^2+1}$	If $f(x) = \dfrac{u(x)}{v(x)}$, then $f'(x) = \dfrac{v(x)\cdot u'(x) - u(x)\cdot v'(x)}{[v(x)]^2}$.
$y = (x^3 - 4x)^{10}$	If $y = u^n$ and $u = g(x)$, then $\dfrac{dy}{dx} = nu^{n-1} \cdot \dfrac{du}{dx}$.
$y = \left(\dfrac{x-1}{x^2+3}\right)^3$	Power Rule, then Quotient Rule to find $\dfrac{du}{dx}$, where $u = \dfrac{x-1}{x^2+3}$.
$y = (x + 1)\sqrt{x^3 + 1}$	Product Rule, then Power Rule to find $v'(x)$, where $v(x) = \sqrt{x^3 + 1}$.
$y = \dfrac{(x^2-3)^4}{x+1}$	Quotient Rule, then Power Rule to find the derivative of the numerator.

CHECKPOINT SOLUTIONS

1. Product, Power
2. Power, Quotient
3. (a) $f'(x) = 3x^4[5(2x^4 + 7)^4(8x^3)] + (2x^4 + 7)^5(12x^3)$

$$= 120x^7(2x^4 + 7)^4 + 12x^3(2x^4 + 7)^5$$

$$= 12x^3(2x^4 + 7)^4[10x^4 + (2x^4 + 7)] = 12x^3(12x^4 + 7)(2x^4 + 7)^4$$

(b) $g'(x) = \dfrac{(2x-9)[7(4x+3)^6(4)] - (4x+3)^7(2)}{(2x-9)^2}$

$$= \frac{2(4x+3)^6[14(2x-9) - (4x+3)]}{(2x-9)^2} = \frac{2(24x - 129)(4x+3)^6}{(2x-9)^2}$$

EXERCISE 9.7

Find the derivatives of the functions in Problems 1–32. Simplify and express the answer using positive exponents only.

1. $f(x) = \pi^4$
2. $f(x) = \dfrac{1}{4}$
3. $g(x) = \dfrac{4}{x^4}$
4. $y = \dfrac{x^4}{4}$
5. $g(x) = 5x^3 + \dfrac{4}{x}$
6. $y = 3x^2 + 4\sqrt{x}$
7. $y = (x^2 - 2)(x + 4)$

8. $y = (x^3 - 5x^2 + 1)(x^3 - 3)$
9. $f(x) = \dfrac{x^3 + 1}{x^2}$
10. $y = \dfrac{1 + x^2 - x^4}{1 + x^4}$
11. $y = \dfrac{(x^3 - 4x)^{10}}{10}$
12. $y = \dfrac{5}{2}(3x^4 - 6x^2 + 2)^5$
13. $y = \dfrac{5}{3}x^3(4x^5 - 5)^3$
14. $y = 3x^4(2x^5 + 1)^7$
15. $y = (x - 1)^2(x^2 + 1)$
16. $f(x) = (5x^3 + 1)(x^4 + 5x)^2$
17. $y = \dfrac{(x^2 - 4)^3}{x^2 + 1}$
18. $y = \dfrac{(x^2 - 3)^4}{x}$
19. $p = [(q + 1)(q^3 - 3)]^3$
20. $y = [(4 - x^2)(x^2 + 5x)]^4$
21. $R(x) = [x^2(x^2 + 3x)]^4$
22. $c(x) = [x^3(x^2 + 1)]^{-3}$
23. $y = \left(\dfrac{2x - 1}{x^2 + x}\right)^4$
24. $y = \left(\dfrac{5 - x^2}{x^4}\right)^3$
25. $g(x) = (8x^4 + 3)^2(x^3 - 4x)^3$
26. $y = (3x^3 - 4x)^3(4x^2 - 8)^2$
27. $f(x) = \dfrac{\sqrt[3]{x^2 + 5}}{4 - x^2}$
28. $g(x) = \dfrac{\sqrt[3]{2x - 1}}{2x + 1}$
29. $y = x^2\sqrt[4]{4x - 3}$
30. $y = 3x\sqrt[3]{4x^4 + 3}$
31. $c(x) = 2x\sqrt{x^3 + 1}$
32. $R(x) = x\sqrt[3]{3x^3 + 2}$

In Problems 33 and 34, find the derivative of each function.

33. (a) $F_1(x) = \dfrac{3(x^4 + 1)^5}{5}$ (b) $F_2(x) = \dfrac{3}{5(x^4 + 1)^5}$
(c) $F_3(x) = \dfrac{(3x^4 + 1)^5}{5}$ (d) $F_4(x) = \dfrac{3}{(5x^4 + 1)^5}$

34. (a) $G_1(x) = \dfrac{2(x^3 - 5)^3}{3}$ (b) $G_2(x) = \dfrac{(2x^3 - 5)^3}{3}$
(c) $G_3(x) = \dfrac{2}{3(x^3 - 5)^3}$ (d) $G_4(x) = \dfrac{2}{(3x^3 - 5)^3}$

Applications

35. ***Physical output*** The total physical output P of workers is a function of the number of workers, x. The function $P = f(x)$ is called the physical productivity function. Suppose that the physical productivity of x construction workers is given by

$$P = 10(3x + 1)^3 - 10$$

Find the marginal physical productivity, dP/dx.

36. ***Revenue*** Suppose that the revenue function for a certain product is given by

$$R(x) = 15(2x + 1)^{-1} + 30x - 15$$

where x is in thousands of units and R is in thousands of dollars.

(a) Find the marginal revenue when 2000 units are sold.
(b) How is revenue changing when 2000 units are sold?

37. ***Revenue*** Suppose that the revenue function for a computer is given by

$$R(x) = 60{,}000x + 40{,}000(10 + x)^{-1} - 4000$$

(a) Find the marginal revenue when 10 units are sold.
(b) How is revenue changing when 10 units are sold?

38. ***Production*** Suppose that the production of x items of a new line of products is given by

$$x = 200[(t + 10) - 400(t + 40)^{-1}]$$

where t is the number of weeks the line has been in production. Find the rate of production, dx/dt.

39. ***National consumption*** If the national consumption function is given by

$$C(y) = 2(y + 1)^{1/2} + 0.4y + 4$$

find the marginal propensity to consume, dC/dy.

40. ***Demand*** Suppose that the demand function for an appliance is given by

$$p = \frac{400(q + 1)}{(q + 2)^2}$$

Find the rate of change of price with respect to the number of appliances.

41. ***Volume*** When squares of side x are cut from the corners of a 12-inch-square piece of cardboard, an open-top box can be formed by folding up the sides. The volume of this box is given by

$$V = x(12 - 2x)^2$$

Find the rate of change of volume with respect to the size of the squares.

42. ***Advertising and sales*** Suppose that sales (in thousands of dollars) are directly related to an advertising campaign according to

$$S = 1 + \frac{3t - 9}{(t + 3)^2}$$

where t is the number of weeks of the campaign.
(a) Find the rate of change of sales after 3 weeks.
(b) Interpret the result in (a).

43. ***Advertising and sales*** An inferior product with an extensive advertising campaign does well when it is released, but sales decline as people discontinue use of the product. If the sales S after t weeks are given by

$$S(t) = \frac{200t}{(t+1)^2}, \quad t \geq 0$$

what is the rate of change of sales when $t = 9$? Interpret your result.

44. ***Advertising and sales*** An excellent film with a very small advertising budget must depend largely on word-of-mouth advertising. If attendance at the film after t weeks is given by

$$A = \frac{100t}{(t+10)^2}$$

what is the rate of change in attendance and what does it mean when (a) $t = 10$? (b) $t = 20$?

45. ***Farm workers*** The percentage of U.S. workers in farm occupations during certain years is shown in the table.

Year	*Percent*	*Year*	*Percent*
1820	71.8	1950	11.6
1850	63.7	1960	6.1
1870	53	1970	3.6
1900	37.5	1980	2.7
1920	27	1985	2.8
1930	21.2	1990	2.4
1940	17.4		

Source: *The World Almanac and Book of Facts*, 1993

Assume that the percentage of U.S. workers in farm occupations can be modeled with the function

$$f(t) = \frac{1000[-8.0912(t+20) + 1558.9]}{1.09816(t+20)^2 - 122.183(t+20) + 21472.6}$$

where t is the number of years past 1820.

(a) Find the function that models the instantaneous rate of change of the percentage of U.S. workers in farm occupations.

(b) Use the result of (a) to find the instantaneous rate of change in 1850 and in 1950.

(c) Interpret the two rates of change in (b).

9.8 Higher-Order Derivatives

OBJECTIVE

- *To find second derivatives and higher derivatives of certain functions*

APPLICATION PREVIEW

Suppose a particle travels according to the equation $s = 100t - 16t^2 + 200$, where s is the distance and t is the time. Then $\frac{ds}{dt}$ is the velocity, and the acceleration of the particle is the rate of change of the velocity of the particle, so acceleration is $\frac{dv}{dt}$. That is, acceleration is the derivative of the derivative of the distance function. This is called the second derivative. In this section we will discuss second-order and higher-order derivatives.

Because the derivative of a function is itself a function, we can take a derivative of the derivative. The derivative of a first derivative is called a **second derivative.** We can find the second derivative of a function f by differentiating it twice. If f' represents the first derivative of a function, then f'' represents the second derivative of that function.

EXAMPLE 1

If $f(x) = 3x^3 - 4x^2 + 5$, find $f''(x)$.

Solution

The first derivative is $f'(x) = 9x^2 - 8x$.
The second derivative is $f''(x) = 18x - 8$.

EXAMPLE 2

Find the second derivative of $y = x^4 - 3x^2 + x^{-2}$.

Solution

The first derivative is $y' = 4x^3 - 6x - 2x^{-3}$.
The second derivative, which we may denote by y'', is

$$y'' = 12x^2 - 6 + 6x^{-4}$$

It is also common to use $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^2}{dx^2}f(x)$ to denote the second derivative of a function.

EXAMPLE 3

If $y = \sqrt{2x-1}$, find d^2y/dx^2.

Solution

The first derivative is

$$\frac{dy}{dx} = \frac{1}{2}(2x-1)^{-1/2}(2) = (2x-1)^{-1/2}$$

The second derivative is

$$\frac{d^2y}{dx^2} = -\frac{1}{2}(2x-1)^{-3/2}(2) = -(2x-1)^{-3/2}$$

$$= \frac{-1}{(2x-1)^{3/2}} = \frac{-1}{\sqrt{(2x-1)^3}}$$

We can also find third, fourth, fifth, and higher derivatives, continuing indefinitely. The third, fourth, and fifth derivatives of a function f are denoted by f''', $f^{(4)}$, and $f^{(5)}$, respectively. Other notations for the third and fourth derivatives include

$$y''' = \frac{d^3y}{dx^3} = \frac{d^3f(x)}{dx^3} \qquad y^{(4)} = \frac{d^4y}{dx^4} = \frac{d^4f(x)}{dx^4}$$

EXAMPLE 4

Find the first four derivatives of $f(x) = 4x^3 + 5x^2 + 3$.

Solution

$f'(x) = 12x^2 + 10x, \quad f''(x) = 24x + 10, \quad f'''(x) = 24, \quad f^{(4)}(x) = 0$

Just as the first derivative, $f'(x)$, can be used to determine the rate of change of a function $f(x)$, the second derivative, $f''(x)$, can be used to determine the rate of change of $f'(x)$.

EXAMPLE 5

Let $f(x) = 3x^4 + 6x^3 - 3x^2 + 4$.

(a) How fast is $f(x)$ changing at $(1, 10)$?
(b) How fast is $f'(x)$ changing at $(1, 10)$?
(c) Is $f'(x)$ increasing or decreasing at $(1, 10)$?

Solution

(a) Because $f'(x) = 12x^3 + 18x^2 - 6x$, we have

$$f'(1) = 12 + 18 - 6 = 24$$

Thus the rate of change of $f(x)$ at $(1, 10)$ is 24.

(b) Because $f''(x) = 36x^2 + 36x - 6$, we have

$$f''(1) = 66$$

Thus the rate of change of $f'(x)$ at $(1, 10)$ is 66.

(c) Because $f''(1) = 66 > 0$, $f'(x)$ is increasing at $(1, 10)$.

EXAMPLE 6

Suppose that a particle travels according to the equation

$$s = 100t - 16t^2 + 200$$

where s is the distance and t is the time. Then ds/dt is the velocity, and $d^2s/dt^2 = dv/dt$ is the acceleration of the particle. Find the acceleration.

Solution

The velocity is $v = ds/dt = 100 - 32t$, and the acceleration is

$$\frac{dv}{dt} = \frac{d^2s}{dt^2} = -32$$

CHECKPOINT

Suppose that the vertical distance a particle travels is given by

$$s = 4x^3 - 12x^2 + 6$$

where s is in feet and x is in seconds.

1. Find the function that describes the velocity of this particle.
2. Find the function that describes the acceleration of this particle.
3. Is the acceleration always positive?
4. When does the *velocity* of this particle increase?

Graphing Utilities We can use the numerical derivative feature of a graphing utility to find the second derivative of a function at a point.

EXAMPLE 7

Find $f''(2)$ if $f(x) = \sqrt{x^3 - 1}$.

Solution

We need the derivative of the derivative function, evaluated at $x = 2$. Figure 9.31 shows how the numerical derivative feature of a graphing utility can be used to obtain this result.

```
nDeriv(nDeriv(√(
X^3-1),X,X),X,2)
                .323969225
```

Figure 9.31

Thus Figure 9.31 shows that $f''(2) = 0.323969225 \approx 0.32397$. We can check this result by calculating $f''(x)$ with formulas.

$$f'(x) = \frac{1}{2}(x^3 - 1)^{-\frac{1}{2}}(3x^2)$$

$$f''(x) = \frac{1}{2}(x^3 - 1)^{-\frac{1}{2}}(6x) + (3x^2)\left[-\frac{1}{4}(x^3 - 1)^{-\frac{3}{2}}(3x^2)\right]$$

$$f''(2) = 0.3239695483 \approx 0.32397$$

Thus we see that the numerical derivative approximation is quite accurate.

EXAMPLE 8

(a) Given $f(x) = x^4 - 12x^2 + 2$, graph $f(x)$ and its second derivative on the same set of axes over an interval that contains all x-values where $f''(x) = 0$.

(b) When the graph of $y = f(x)$ is opening downward, is $f''(x) > 0$, $f''(x) < 0$, or $f''(x) = 0$?

(c) When the graph of $y = f(x)$ is opening upward, is $f''(x) > 0$, $f''(x) < 0$, or $f''(x) = 0$?

Solution

(a) $f'(x) = 4x^3 - 24x$ and $f''(x) = 12x^2 - 24 = 12(x^2 - 2)$. Because $f''(x) = 0$ at $x = -\sqrt{2}$ and at $x = \sqrt{2}$, we use an x-range that contains $x = -\sqrt{2}$ and $\sqrt{2}$. The graph of $f(x) = x^4 - 12x^2 + 2$ and its second derivative, $f''(x) = 12x^2 - 24$, are shown in Figure 9.32.

(b) The graph of $y = f(x)$ appears to be opening downward on the same interval for which $f''(x) < 0$.

(c) The graph appears to be opening upward on the same intervals for which $f''(x) > 0$.

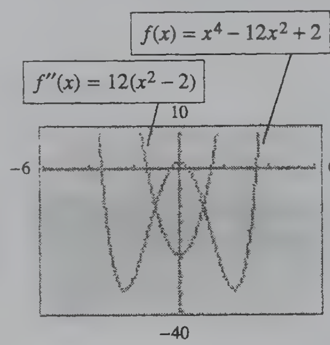


Figure 9.32

CHECKPOINT SOLUTIONS

1. The velocity is described by $s'(x) = 12x^2 - 24x$.
2. The acceleration is described by $s''(x) = 24x - 24$.
3. No; the acceleration is positive when $s''(x) > 0$—that is, when $24x - 24 > 0$. It is zero when $24x - 24 = 0$ and negative when $24x - 24 < 0$. Thus acceleration is negative when $x < 1$ second, zero when $x = 1$ second, and positive when $x > 1$ second.
4. The velocity increases when the acceleration is positive. Thus the velocity is increasing after 1 second.

EXERCISE 9.8

In Problems 1–8, find the second derivative.

1. $f(x) = 4x^3 - 15x^2 + 3x + 2$
2. $f(x) = 2x^{10} - 18x^5 - 12x^3 + 4$
3. $y = 10x^3 - x^2 + 14x + 3$
4. $y = 6x^5 - 3x^4 + 12x^2$
5. $g(x) = x^3 - \dfrac{1}{x}$
6. $h(x) = x^2 - \dfrac{1}{x^2}$
7. $y = x^3 - \sqrt{x}$
8. $y = 3x^2 - \sqrt[3]{x^2}$

In Problems 9–16, find the third derivative.

9. $y = x^5 - 16x^3 + 12$
10. $y = 6x^3 - 12x^2 + 6x$
11. $f(x) = 2x^9 - 6x^6$
12. $f(x) = 3x^5 - x^6$
13. $y = 1/x$
14. $y = 1/x^2$
15. $y = \sqrt{x}$
16. $y = \sqrt[3]{x}$

In Problems 17–28, find the indicated derivative.

17. If $y = x^5 - x^{1/2}$, find $\dfrac{d^2y}{dx^2}$.
18. If $y = x^4 + x^{1/3}$, find $\dfrac{d^2y}{dx^2}$.
19. If $f(x) = \sqrt{x + 1}$, find $f'''(x)$.
20. If $f(x) = \sqrt{x - 5}$, find $f'''(x)$.
21. Find $\dfrac{d^4y}{dx^4}$ if $y = 4x^3 - 16x$.
22. Find $y^{(4)}$ if $y = x^6 - 15x^3$.
23. Find $f^{(4)}(x)$ if $f(x) = \sqrt{x}$.
24. Find $f^{(4)}(x)$ if $f(x) = 1/x$.
25. Find $y^{(4)}$ if $y' = \sqrt{x - 1}$.
26. Find $y^{(5)}$ if $\dfrac{d^2y}{dx^2} = \sqrt[3]{x + 2}$.
27. Find $f^{(6)}(x)$ if $f^{(4)}(x) = x(x + 1)^{-1}$.
28. Find $f^{(3)}(x)$ if $f'(x) = \dfrac{x^2}{x^2 + 1}$.
29. If $f(x) = 16x^2 - x^3$, what is the rate of change of $f'(x)$ at $(1, 15)$?
30. If $y = 36x^2 - 6x^3 + x$, what is the rate of change of y' at $(1, 31)$?

In Problems 31–34, use the numerical derivative feature of a graphing utility to approximate the given second derivatives.

31. $f''(3)$ for $f(x) = x^3 - \dfrac{27}{x}$

32. $f''(-1)$ for $f(x) = \dfrac{x^2}{4} - \dfrac{4}{x^2}$

33. $f''(21)$ for $f(x) = \sqrt{x^2 + 4}$

34. $f''(3)$ for $f(x) = \dfrac{1}{\sqrt{x^2 + 7}}$

In Problems 35–38, do the following for each function $f(x)$.

(a) Find $f'(x)$ and $f''(x)$.
(b) Graph $f(x), f'(x)$, and $f''(x)$ with a graphing utility.
(c) Identify x-values where $f''(x) = 0$, $f''(x) > 0$, and $f''(x) < 0$.
(d) Identify x-values where $f'(x)$ has a maximum point or a minimum point, where $f'(x)$ is increasing, and where $f'(x)$ is decreasing.
(e) When $f(x)$ has a maximum point, is $f''(x) > 0$ or $f''(x) < 0$?
(f) When $f(x)$ has a minimum point, is $f''(x) > 0$ or $f''(x) < 0$?

35. $f(x) = x^3 - 3x^2 + 5$

36. $f(x) = 2 + 3x - x^3$

37. $f(x) = -\frac{1}{3}x^3 - x^2 + 3x + 7$

38. $f(x) = \frac{1}{3}x^3 - \dfrac{3x^2}{2} - 4x + 10$

Applications

39. ***Acceleration*** If a particle travels as a function of time according to the formula

$$s = 100 + 10t + 0.01t^3$$

find the acceleration of the particle when $t = 2$.

40. ***Acceleration*** If the formula describing the distance an object travels as a function of time is

$$s = 100 + 160t - 16t^2$$

what is the acceleration of the object when $t = 4$?

41. ***Revenue*** The revenue from sales of a certain product can be described by

$$R(x) = 100x - 0.01x^2$$

Find the instantaneous rate of change of the marginal revenue.

42. ***Revenue*** Suppose that the revenue from the sale of a product is given by

$$R = 70x + 0.5x^2 - 0.001x^3$$

where x is the number of units sold. How fast is the marginal revenue $\overline{MR}$ changing when $x = 100$?

43. ***Sensitivity*** When medicine is administered, reaction (measured in change of blood pressure or temperature) can be modeled by

$$R = m^2\left(\frac{c}{2} - \frac{m}{3}\right)$$

where c is a positive constant and m is the amount of medicine absorbed into the blood (Source: Thrall, R. M., *et al.*, *Some Mathematical Models in Biology*, U.S. Department of Commerce, 1967). The sensitivity to the medication is defined to be the rate of change of reaction R with respect to the amount of medicine m absorbed in the blood.

(a) Find the sensitivity.
(b) Find the instantaneous rate of change of sensitivity with respect to the amount of medicine absorbed in the blood.
(c) Which order derivative of reaction gives the rate of change of sensitivity?

44. ***Photosynthesis*** The amount of photosynthesis that takes place in a certain plant depends on the intensity of light x according to the equation

$$f(x) = 145x^2 - 30x^3$$

(a) Find the rate of change of photosynthesis with respect to the intensity.
(b) What is the rate of change when $x = 1$? when $x = 3$?
(c) How fast is the rate found in (a) changing when $x = 1$? when $x = 3$?

45. ***Revenue*** The revenue (in thousands of dollars) from the sale of x units of a product is

$$R = 15x + 30(4x + 1)^{-1} - 30$$

where x is the number of units sold.

(a) At what rate is the marginal revenue $\overline{MR}$ changing when the number of units being sold is 25?
(b) Interpret your result in (a).

46. ***Advertising and sales*** The sales of a product S (in thousands of dollars) are given by

$$S = \frac{600x}{x + 40}$$

where x is the advertising expenditure (in thousands of dollars).
(a) Find the rate of change of sales with respect to advertising expenditure.
(b) Use the second derivative to find how this rate is changing at $x = 20$.
(c) Interpret your result in (b).

47. ***Advertising and sales*** The daily sales S (in thousands of dollars) that are attributed to an advertising campaign is given by

$$S = 1 + \frac{3}{t + 3} - \frac{18}{(t + 3)^2}$$

where t is the number of weeks the campaign runs.
(a) Find the rate of change of sales at any time t.
(b) Use the second derivative to find how this rate is changing at $t = 15$.
(c) Interpret your result in (b).

48. ***Advertising and sales*** A product with a large advertising budget has its sales S (in millions of dollars) given by

$$S = \frac{500}{t + 2} - \frac{1000}{(t + 2)^2}$$

where t is the number of months the product has been on the market.
(a) Find the rate of change of sales at any time t.
(b) What is the rate of change of sales at $t = 2$?
(c) Use the second derivative to find how this rate is changing at $t = 2$.
(d) Interpret your result from (b) and (c).

49. ***Persons living below the poverty level*** The table below shows the number of millions of people in the United States who lived below the poverty level for selected years between 1960 and 1996.

Year	*Persons Living Below the Poverty Level (millions)*
1960	39.9
1965	33.2
1970	25.4
1975	25.9
1980	29.3
1986	32.4
1990	33.6
1993	39.3
1994	38.1
1995	36.4
1996	36.5

Source: *The World Almanac and Book of Facts*, 1998

Assume that the number of millions of people in the United States who lived below the poverty level can be modeled by the function

$$p(t) = -0.001723t^3 + 0.1291t^2 - 2.5015t + 40.656$$

where t is the number of years past 1960.
(a) Find the function that models the instantaneous rate of change of $p(t)$.
(b) Use the second derivative to find how this rate is changing in 1970 and 1990.
(c) Interpret the meaning of $p'(10)$ and $p''(10)$. Write a sentence that explains each.

50. ***Consumer debt*** The percentage of disposable income spent on consumer debt during certain years is shown in the table below.

Year	*Consumer Debt as a Percentage of Disposable Income*
1980	18.2
1982	16.9
1983	17.7
1985	20.8
1986	21.4
1988	21.0
1990	20.0
1991	18.9
1994	19.7
1995	21.3
1996	23.7

Source: Federal Reserve System

Assume that consumer debt as a percentage of disposable income can be modeled with the function

$$d(t) = 0.0028t^4 - 0.081t^3 + 0.68t^2 - 1.3t + 17.9$$

where t is the number of years past 1980.

(a) Find the function that models the instantaneous rate of change of the percentage of disposable income spent on consumer debt.

(b) Use the second derivative to determine how this rate is changing in 1985 and 1990.

(c) Write a sentence that explains the meaning of $d'(15)$ and another that explains the meaning of $d''(15)$.

51. ***Consumer price index*** The annual change in the consumer price index (CPI) during certain years is shown in the table below. Assume that the annual change in the CPI can be modeled with a cubic function, $f(t)$, where t is the number of years past 1987.

Year	*Annual Percent Change in CPI*
1987	3.6
1988	4.1
1989	4.8
1990	5.4
1991	4.2
1992	3.0
1993	3.0
1994	2.6
1995	2.8
1996	3.0

Source: *The World Almanac and Book of Facts,* 1998

(a) Find the function $f(t)$ that models the CPI.

(b) Find the function that models the instantaneous rate of change of the annual change in the CPI.

(c) Use the second derivative to determine how this rate is changing in 1991 and 1995.

(d) Write sentences that explain the meanings of $f'(8)$ and $f''(8)$.

52. ***Union membership*** The table shows the percentage of U.S. workers who belonged to unions for selected years from 1930 to 1996.

Year	*Union Membership as a Percentage of the Labor Force*
1930	11.6
1935	13.2
1940	26.9
1945	35.5
1950	31.5
1955	33.2
1960	31.4
1965	28.4
1970	27.3
1975	25.5
1980	21.9
1985	18.0
1990	16.1
1993	15.8
1994	15.5
1995	14.9
1996	14.5

Source: Bureau of Labor Statistics, Department of Labor

Assume that the percentage of the labor force that belonged to unions can be modeled with a cubic function $f(t)$, where t is the number of years past 1930.

(a) Find the function $f(t)$ that models union membership as a percentage of the labor force.

(b) Find the function that models the instantaneous rate of change of the percentage of the U.S. labor force that belonged to unions.

(c) Use the second derivative to determine how this rate was changing in 1970 and in 1990.

(d) Write sentences that explain the meanings of $f'(40)$ and $f''(40)$.

9.9 Applications of Derivatives in Business and Economics

OBJECTIVES

- *To find the marginal cost and marginal revenue at different levels of production*
- *To find the marginal profit function, given information about cost and revenue*

APPLICATION PREVIEW

In Chapter 1, "Linear Equations and Functions," we defined marginal cost as the rate of change of the total cost function. For a linear total cost function, the marginal cost was defined as the slope of the function's graph. For any total cost function defined by an equation, we can find the instantaneous rate of change of cost (the marginal cost) at any level of production by finding the derivative of the function.

In the same way, we can find the marginal revenue from the total revenue function, and we can use total revenue and total cost functions to find marginal profit.

We begin this section by defining the marginal cost function.

Marginal Cost If $C = C(x)$ is a total cost function for a commodity, then its derivative, $\overline{MC} = C'(x)$, is the **marginal cost function.**

The linear cost function with equation

$$C(x) = 300 + 6x \qquad \text{(in dollars)}$$

has marginal cost \$6 because its slope is 6. Taking the derivative of $C(x)$ gives

$$\overline{MC} = C'(x) = 6$$

which verifies that the marginal cost is \$6 at all levels of production.

The cost function

$$C(x) = 1000 + 6x + x^2$$

has derivative

$$C'(x) = 6 + 2x$$

Thus the *marginal cost* at $x = 10$ (when 10 units are produced) is

$$C'(10) = 6 + 2(10) = 26$$

and the marginal cost at 40 units is

$$C'(40) = 6 + 2(40) = 86$$

Note that when a cost function is linear, the marginal cost gives the amount by which cost would change if production were increased by 1 unit. When a cost function is *not* linear (as with $C(x) = 1000 + 6x + x^2$), the marginal cost is used to estimate the amount by which cost would change if production were increased by 1 unit. Thus at 10 units, the marginal cost is \$26, so costs would increase by

approximately \$26 if 1 more unit were produced. (Note that $C(11) - C(10) = \$27$ gives the actual increase in costs.) Also, at 40 units, the marginal cost is \$86, so costs would increase by approximately \$86 if 1 more unit were produced.

As noted previously, when the derivative of a function is positive (and thus the slope of the tangent to the curve is positive), the function is increasing, and the value of the derivative gives us a measure of how fast it is increasing. As we saw, the marginal cost for

$$C(x) = 1000 + 6x + x^2$$

is 86 at $x = 40$ and 26 at $x = 10$. This tells us that the cost is increasing faster at $x = 40$ than it is at $x = 10$.

Because producing more units can never reduce the total cost of production, the following properties are valid:

1. The total cost can never be negative. If there are fixed costs, the cost of producing 0 units is positive; otherwise, the cost of producing 0 units is 0.
2. The total cost function is always increasing; the more units produced, the higher the total cost. Thus the marginal cost is always positive.
3. There may be limitations on the units produced, such as those imposed by plant space.

The graphs of many marginal cost functions tend to be U-shaped; they eventually will rise, even though there may be an initial interval where they decrease.

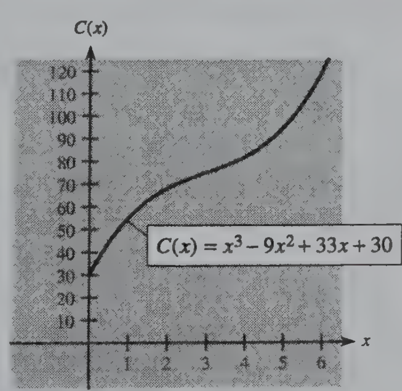
EXAMPLE 1

If the total cost function for a commodity is $C(x) = x^3 - 9x^2 + 33x + 30$, find the marginal cost.

Solution

The marginal cost is $\overline{MC} = C'(x) = 3x^2 - 18x + 33$.

The graph of the total cost function is shown in Figure 9.33(a), and the graph of the marginal cost function is shown in Figure 9.33(b).



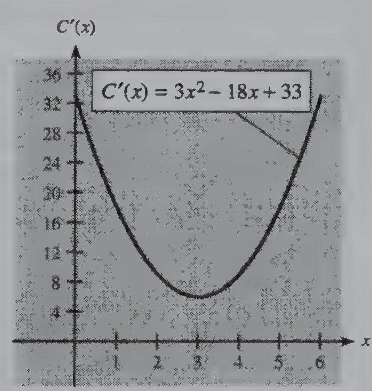


Figure 9.33 (a) Total cost function (b) Marginal cost function

As we saw in Section 9.3, "The Derivative," the instantaneous rate of change (the derivative) of the revenue function is the marginal revenue.

Marginal Revenue If $R = R(x)$ is the total revenue function for a commodity, then the **marginal revenue function** is $\overline{MR} = R'(x)$.

If the demand function for a product in a monopoly market is $p = f(x)$, then the total revenue from the sale of x units is

$$R(x) = px = f(x) \cdot x$$

EXAMPLE 2

If the demand for a product in a monopoly market is given by

$$p = 16 - 0.02x$$

where x is the number of units and p is the price per unit, (a) find the total revenue function, and (b) find the marginal revenue for this product at $x = 40$.

Solution

(a) The total revenue function is

$$\begin{aligned} R(x) &= px = (16 - 0.02x)x \\ &= 16x - 0.02x^2 \end{aligned}$$

(b) The marginal revenue function is

$$\overline{MR} = R'(x) = 16 - 0.04x$$

At $x = 40$, $R'(40) = 16 - 1.6 = 14.40$. Thus the 41st item sold will increase the total revenue by approximately \$14.40.

The marginal revenue is an approximation of the revenue gained from the sale of 1 additional unit. We have used marginal revenue in Example 2 to find that the revenue from the sale of the 41st item will be approximately \$14.40. The actual increase in revenue from the sale of the 41st item is

$$R(41) - R(40) = 622.38 - 608 = \$14.38$$

EXAMPLE 3

Use the graphs in Figure 9.34 to determine the x-value where the revenue function has its maximum. What is happening to the marginal revenue at and near this x-value?

Solution

Figure 9.34(a) shows that the total revenue function has a maximum value at $x = 400$. After that, the total revenue function decreases. This means that the total revenue will be reduced each time a unit is sold if more than 400 are produced and sold. The graph of the marginal revenue function in Figure 9.34(b) shows that the marginal revenue is positive to the left of 400. This indicates that the rate at which the total revenue is changing is positive until 400 units are sold; thus the total revenue is increasing. Then, at 400 units, the rate of change is 0. After

400 units are sold, the marginal revenue is negative, which indicates that the total revenue is now decreasing. It is clear from looking at either graph that 400 units should be produced and sold to maximize the total revenue function $R(x)$. That is, the *total revenue* function has its maximum at $x = 400$.

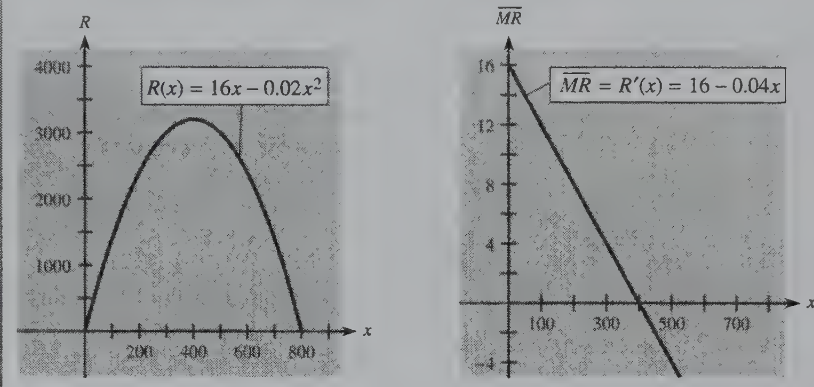


Figure 9.34 (a) Total revenue function (b) Marginal revenue function

CHECKPOINT

The total cost function for the commodity considered in Example 1 is $C(x) = x^3 - 9x^2 + 33x + 30$, and the marginal cost is $C'(x) = 3x^2 - 18x + 33$.

1. What is the marginal cost if $x = 10$ units are produced?
2. Use marginal cost to estimate the cost of producing the 11th unit.
3. Calculate $C(11) - C(10)$ to find the actual cost of producing the 11th unit.
4. True or false: For products that have linear cost functions, the actual cost of producing the $(x + 1)$st unit is equal to the marginal cost at x.

As with marginal cost and marginal revenue, the derivative of a profit function for a commodity will give us the marginal profit function for the commodity.

Marginal Profit If $P = P(x)$ is the profit function for a commodity, then the **marginal profit function** is $\overline{MP} = P'(x)$.

EXAMPLE 4

If the total profit, in thousands of dollars, for a product is given by $P(x) = 20\sqrt{x+1} - 2x$, what is the marginal profit at a production level of 15 units?

Solution

The marginal profit function is

$$\overline{MP} = P'(x) = 20 \cdot \frac{1}{2}(x+1)^{-1/2} - 2 = \frac{10}{\sqrt{x+1}} - 2$$

If 15 units are produced, the marginal profit is

$$P'(15) = \frac{10}{\sqrt{15+1}} - 2 = \frac{1}{2}$$

This means that the profit from the sale of the 16th unit is approximately $\frac{1}{2}$ (thousand dollars), or \$500.

In a **competitive market**, each firm is so small that its actions in the market cannot affect the price of the product. The price of the product is determined in the market by the intersection of the market demand curve (from all consumers) and the market supply curve (from all firms that supply this product). The firm can sell as little or as much as it desires at the given market price, which it cannot change.

Therefore, a firm in a competitive market has a total revenue function given by $R(x) = px$, where p is the market equilibrium price for the product and x is the quantity sold.

EXAMPLE 5

A firm in a competitive market must sell its product for \$200 per unit. The cost per unit (per month) is $80 + x$, where x represents the number of units sold per month. Find the marginal profit function.

Solution

If the cost per unit is $80 + x$, then the total cost of x units is given by the equation $C(x) = (80 + x)x = 80x + x^2$. The revenue per unit is \$200, so the total revenue is given by $R(x) = 200x$. Thus the profit function is

$$P(x) = R(x) - C(x) = 200x - (80x + x^2), \quad \text{or} \quad P(x) = 120x - x^2$$

The marginal profit is $P'(x) = 120 - 2x$.

The marginal profit in Example 5 is not always positive, so producing and selling a certain number of items will maximize profit. Note that the marginal profit will be negative (that is, profit will decrease) if more than 60 items per month are produced. We will discuss methods of maximizing total revenue and profit, and for minimizing average cost, in the next chapter.

CHECKPOINT

If the total profit function for a product is $P(x) = 20\sqrt{x+1} - 2x$, then the marginal profit is

$$P'(x) = \frac{10}{\sqrt{x+1}} - 2 \quad \text{and} \quad P''(x) = \frac{-5}{\sqrt{(x+1)^3}}$$

5. Is $P''(x) < 0$ for all values of $x \geq 0$?
6. Is the marginal profit decreasing for all $x \geq 0$?

EXAMPLE 6

In Example 4, we found that the profit (in thousands of dollars) for a company's products is given by $P(x) = 20\sqrt{x+1} - 2x$ and its marginal profit is given by $P'(x) = \dfrac{10}{\sqrt{x+1}} - 2$.

(a) Use the graphs of $P(x)$ and $P'(x)$ to determine the relationship between the two functions.

(b) When is the marginal profit 0? What is happening to profit at this level of production?

Solution

(a) By comparing the graphs of the two functions (shown in Figure 9.35), we see that for $x > 0$, profit $P(x)$ is increasing over the interval where the marginal profit $P'(x)$ is positive, and profit is decreasing over the interval where the marginal profit $P'(x)$ is negative.

(b) By using SOLVER, INTERSECT, or TRACE, or by using algebra, we see that $P'(x) = 0$ when $x = 24$. This level of production ($x = 24$) is where profit is maximized, at 52 (thousand dollars).

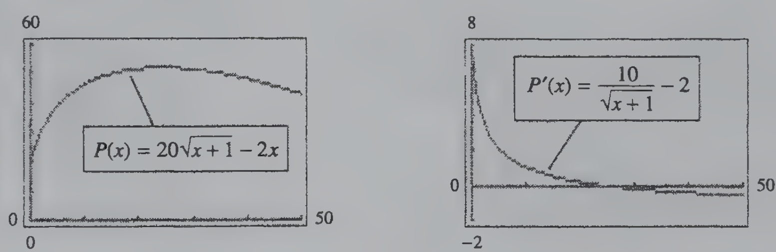


Figure 9.35 (a) (b)

CHECKPOINT SOLUTIONS

1. $C'(10) = 153$
2. $C'(10) = 153$, so it will cost approximately \$153 to produce the 11th unit.
3. $C(11) - C(10) = 635 - 460 = 175$
4. True
5. Yes
6. Yes, because $P''(x) < 0$ for $x \geq 0$.

EXERCISE 9.9

Marginal Cost, Revenue, and Profit

Find the marginal cost functions related to the cost functions in Problems 1–8.

1. $C(x) = 40 + 8x$
2. $C(x) = 200 + 16x$
3. $C(x) = 500 + 13x + x^2$
4. $C(x) = 300 + 10x + \frac{1}{100}x^2$
5. $C = x^3 - 6x^2 + 24x + 10$
6. $C = x^3 - 12x^2 + 63x + 15$
7. $C = 400 + 27x + x^3$
8. $C(x) = 50 + 48x + x^3$
9. Suppose that the cost function for a commodity is

$$C(x) = 40 + x^2$$

(a) Find the marginal cost at $x = 5$ and tell what this predicts about the cost of producing 1 additional unit.
(b) Calculate $C(6) - C(5)$ to find the actual cost of producing 1 additional unit.

10. Suppose that the cost function for a commodity is

$$C(x) = 300 + 6x + \tfrac{1}{20}x^2$$

(a) Find the marginal cost at $x = 8$ and tell what this predicts about the cost of producing 1 additional unit.
(b) Calculate $C(9) - C(8)$ to find the actual cost of producing 1 additional unit.

11. If the cost function for a commodity is

$$C(x) = x^3 - 4x^2 + 30x + 20$$

find the marginal cost at $x = 4$ and tell what this predicts about the cost of producing 1 additional unit.

12. If the cost function for a commodity is

$$C(x) = \tfrac{1}{90}x^3 + 4x^2 + 4x + 10$$

find the marginal cost at $x = 3$ and tell what this predicts about the cost of producing 1 additional unit.

13. If the cost function for a commodity is

$$C(x) = 300 + 4x + x^2$$

graph the marginal cost function.

14. If the cost function for a commodity is

$$C(x) = x^3 - 12x^2 + 63x + 15$$

graph the marginal cost function.

15. (a) If the total revenue function for a product is $R(x) = 4x$, what is the marginal revenue function for that product?
(b) What does this marginal revenue function tell us?

16. If the total revenue function for a product is $R(x) = 32x$, what is the marginal revenue for the product? What does this mean?

17. Suppose that the total revenue function for a commodity is $R = 36x - 0.01x^2$.
(a) Find $R(100)$ and tell what it represents.
(b) Find the marginal revenue function.
(c) Find the marginal revenue at $x = 100$, and tell what it predicts about the sale of the next unit.
(d) Find $R(101) - R(100)$ and explain what this value represents.

18. Suppose that the total revenue function for a commodity is $R(x) = 25x - 0.05x^2$.
(a) Find $R(50)$ and tell what it represents.
(b) Find the marginal revenue function.
(c) Find the marginal revenue at $x = 50$, and tell what it predicts about the sale of the next unit.
(d) Find $R(51) - R(50)$ and explain what this value represents.

19. (a) Graph the marginal revenue function from Problem 17.
(b) At what value of x will total revenue be maximized for Problem 17?
(c) What is the maximum revenue?

20. (a) Graph the marginal revenue function from Problem 18.
(b) Determine the number of units that must be sold to maximize total revenue.
(c) What is the maximum revenue?

21. If the total profit function is $P(x) = 5x - 25$, find the marginal profit.

22. If the total profit function is $P(x) = 16x - 32$, find the marginal profit.

23. Suppose that the total revenue function for a product is $R(x) = 32x$ and that the total cost function is $C(x) = 200 + 2x + x^2$.
(a) Find the profit from the production and sale of 20 units.
(b) Find the marginal profit function.
(c) Find $\overline{MP}$ at $x = 20$ and explain what it predicts.
(d) Find $P(21) - P(20)$ and explain what this value represents.

24. Suppose that the total revenue function is given by

$$R(x) = 46x$$

and that the total cost function is given by

$$C(x) = 100 + 30x + \tfrac{1}{10}x^2$$

(a) Find $P(100)$.
(b) Find the marginal profit function.
(c) Find $\overline{MP}$ at $x = 100$ and explain what it predicts.
(d) Find $P(101) - P(100)$ and explain what this value represents.

25. (a) Graph the marginal profit function for the profit function $P(x) = 30x - x^2 - 200$.
(b) What level of production and sales will give a 0 marginal profit?
(c) At what level of production and sales will profit be at a maximum?
(d) What is the maximum profit?

26. (a) Graph the marginal profit function for the profit function $P(x) = 16x - 0.1x^2 - 100$.
(b) What level of production and sales will give a 0 marginal profit?
(c) At what level of production and sales will profit be at a maximum?
(d) What is the maximum profit?

27. The price of a product in a competitive market is \$300. If the cost per unit of producing the product is $160 + x$, where x is the number of units produced per month, how many units should the firm produce and sell to maximize its profit?
28. The cost per unit of producing a product is $60 + 2x$, where x represents the number of units produced per week. If the equilibrium price determined by a competitive market is \$220, how many units should the firm produce and sell each week to maximize its profit?
29. If the daily cost per unit of producing a product by the Ace Company is $10 + 2x$, and if the price on the competitive market is \$50, what is the maximum daily profit the Ace Company can expect on this product?
30. The Mary Ellen Candy Company produces chocolate Easter bunnies at a cost per unit of $0.10 + 0.01x$, where x is the number produced. If the price on the competitive market for a bunny this size is \$2.50, how many should the company produce to maximize its profit?
31. The following table gives the total revenues of AT&T for selected years.*

Year	*Total Revenues (billions)*
1985	\$63.13
1986	\$69.906
1987	\$60.53
1989	\$61.1
1990	\$62.191
1991	\$63.089
1992	\$64.904
1993	\$67.156

Source: *AT&T Annual Report*, 1993

Suppose the data can be modeled by the equation

$$R(t) = 0.253t^2 - 4.03t + 76.84$$

where t is the number of years past 1980.

(a) Find $R(7)$ from the data and compare it to $R(7)$ as found from the model. What does $R(7)$ represent?

(b) Find the function that gives the instantaneous rate of change of revenue.

(c) Find the instantaneous rate of change of revenue in 1992.

(d) Interpret your result to (c).

The data in the table give sales revenues and costs and expenses for Scott Paper Company for various years.† Use these data for Problems 32–34.

Year	*Sales Revenue (billions)*	*Costs and Expenses (billions)*
1983	\$2.6155	\$2.4105
1984	2.7474	2.4412
1985	2.934	2.6378
1986	3.3131	2.9447
1987	3.9769	3.5344
1988	4.5494	3.8171
1989	4.8949	4.2587
1990	5.1686	4.8769
1991	4.9593	4.9088
1992	5.0913	4.6771
1993	4.7489	4.9025

Source: Scott Paper Company, 1993 *Annual Report*

32. Assume that sales revenues for Scott Paper can be modeled by

$$R(t) = -0.031t^2 + 0.776t + 0.179$$

where t is the number of years past 1980.

(a) Use this model to find the instantaneous rate of change of revenue in 1990.

(b) Interpret your answer to (a).

33. Assume that costs and expenses for Scott Paper Company can be modeled by

$$C(t) = -0.012t^2 + 0.492t + 0.725$$

where t is the number of years past 1980.

(a) Use this model to find the instantaneous rate of change of costs and expenses in 1990.

(b) Interpret your answer to (a) and check this interpretation against the data in the table.

34. Let income from operations, $I(t)$, be revenues minus costs.

(a) Find $I(t)$.

(b) Find the instantaneous rate of change of income in 1990.

(c) Interpret your result in (b).

(d) Would the board of directors be interested in altering this model? Explain.

*Before AT&T split off NCR and Lucent

†Before Scott merged with Kimberly-Clark

KEY TERMS AND FORMULAS

Section	Key Terms	Formula
9.1	Limit, infinite limit 0/0 indeterminate form	
9.2	Continuous function Vertical asymptote Horizontal asymptote Limit at infinity	
9.3	Average velocity Velocity Instantaneous rate of change	
	Derivative	$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$
	Marginal revenue	$\overline{MR} = R'(x)$
	Tangent line Secant line Slope of a curve Differentiability and continuity	
9.4	Powers of x Rule	$\dfrac{d(x^n)}{dx} = nx^{n-1}$
	Constant Function Rule	$\dfrac{d(c)}{dx} = 0$ for constant c
	Coefficient Rule	$\dfrac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$
	Sum Rule	$\dfrac{d}{dx}[u + v] = \dfrac{du}{dx} + \dfrac{dv}{dx}$
	Difference Rule	$\dfrac{d}{dx}[u - v] = \dfrac{du}{dx} - \dfrac{dv}{dx}$
9.5	Product Rule	$\dfrac{d}{dx}[uv] = uv' + vu'$
	Quotient Rule	$\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{vu' - uv'}{v^2}$
9.6	Chain Rule	$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$
	Power Rule	$\dfrac{d}{dx}(u^n) = nu^{n-1}\dfrac{du}{dx}$
9.8	Second derivative; third derivative; higher-order derivatives	
9.9	Marginal cost function Marginal revenue function Marginal profit function	$\overline{MC} = C'(x)$ $\overline{MR} = R'(x)$ $\overline{MP} = P'(x)$

REVIEW EXERCISES

Section 9.1

In Problems 1–6, use the graph of $y = f(x)$ in Figure 9.36 to find the functional values and limits, if they exist.

1. (a) $f(-2)$ (b) $\lim_{x\to -2} f(x)$
2. (a) $f(-1)$ (b) $\lim_{x\to -1} f(x)$
3. (a) $f(4)$ (b) $\lim_{x\to 4^-} f(x)$
4. (a) $\lim_{x\to 4^+} f(x)$ (b) $\lim_{x\to 4} f(x)$
5. (a) $f(1)$ (b) $\lim_{x\to 1} f(x)$
6. (a) $f(2)$ (b) $\lim_{x\to 2} f(x)$

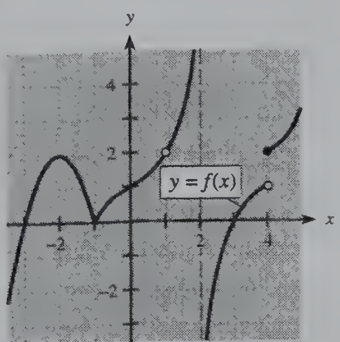


Figure 9.36

In Problems 7–20, find each limit, if it exists.

7. $\lim_{x\to 4} (3x^2 + x + 3)$
8. $\lim_{x\to 4} \frac{x^2 - 16}{x + 4}$
9. $\lim_{x\to -1} \frac{x^2 - 1}{x + 1}$
10. $\lim_{x\to 3} \frac{x^2 - 9}{x - 3}$
11. $\lim_{x\to 2} \frac{4x^3 - 8x^2}{4x^3 - 16x}$
12. $\lim_{x\to -\frac{1}{2}} \frac{x^2 - \frac{1}{4}}{6x^2 + x - 1}$
13. $\lim_{x\to 3} \frac{x^2 - 16}{x - 3}$
14. $\lim_{x\to -3} \frac{x^2 - 9}{x - 3}$
15. $\lim_{x\to 1} \frac{x^2 - 9}{x - 3}$
16. $\lim_{x\to 2} \frac{x^2 - 8}{x - 2}$
17. $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} 4 - x^2 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$
18. $\lim_{x\to -2} f(x)$ where $f(x) = \begin{cases} x^3 - x & \text{if } x < -2 \\ 2 - x^2 & \text{if } x \geq -2 \end{cases}$
19. $\lim_{h\to 0} \frac{3(x+h)^2 - 3x^2}{h}$
20. $\lim_{h\to 0} \frac{[(x+h) - 2(x+h)^2] - (x - 2x^2)}{h}$

In Problems 21 and 22, use tables to investigate each limit. Check your result analytically or graphically.

21. $\lim_{x\to 2} \frac{x^2 + 10x - 24}{x^2 - 5x + 6}$
22. $\lim_{x\to -\frac{1}{2}} \frac{x^2 + \frac{1}{6}x - \frac{1}{6}}{x^2 + \frac{5}{6}x + \frac{1}{6}}$

Section 9.2

Use the graph of $y = f(x)$ in Figure 9.36 to answer the questions in Problems 23 and 24.

23. Is $f(x)$ continuous at
    (a) $x = -1$? (b) $x = 1$?
24. Is $f(x)$ continuous at
    (a) $x = -2$? (b) $x = 2$?

In Problems 25–30, suppose that

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq 0 \\ x & \text{if } 0 < x < 1 \\ 2x^2 - 1 & \text{if } x \geq 1 \end{cases}$$

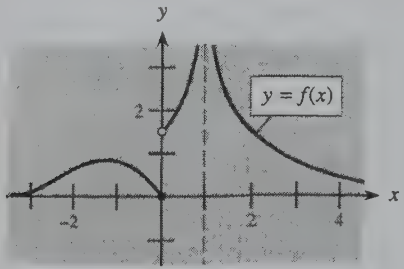
25. What is $\lim_{x\to -1} f(x)$?
26. What is $\lim_{x\to 0} f(x)$, if it exists?
27. What is $\lim_{x\to 1} f(x)$, if it exists?
28. Is $f(x)$ continuous at $x = 0$?
29. Is $f(x)$ continuous at $x = 1$?
30. Is $f(x)$ continuous at $x = -1$?

For the functions in Problems 31–34, determine which are continuous. Identify discontinuities for those that are not continuous.

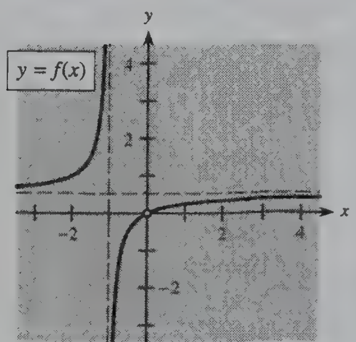
31. $y = \frac{x^2 + 25}{x - 5}$
32. $y = \frac{x^2 - 3x + 2}{x - 2}$
33. $f(x) = \begin{cases} x + 2 & \text{if } x \leq 2 \\ 5x - 6 & \text{if } x > 2 \end{cases}$
34. $y = \begin{cases} x^4 - 3 & \text{if } x \leq 1 \\ 2x - 3 & \text{if } x > 1 \end{cases}$

In Problems 35 and 36, use the graphs to find (a) the points of discontinuity, (b) $\lim_{x\to+\infty} f(x)$, and (c) $\lim_{x\to-\infty} f(x)$.

35.



36.



In Problems 37 and 38 evaluate the limits, if they exist.

37. $\lim_{x\to-\infty} \frac{2x^2}{1-x^2}$ 38. $\lim_{x\to+\infty} \frac{3x^{2/3}}{x+1}$

Section 9.3

In Problems 39 and 40, decide whether the statements are true or false.

39. $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ gives the formula for the slope of the tangent and the instantaneous rate of change of $f(x)$ at any value of x.

40. $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ gives the equation of the tangent line to $f(x)$ at $x = c$.

41. Use the definition of derivative to find $f'(x)$ for $f(x) = 3x^2 + 2x - 1$.

42. Use the definition of derivative to find $f'(x)$ if $f(x) = x - x^2$.

Use the graph of $y = f(x)$ in Figure 9.36 on the previous page to answer the questions in Problems 43 and 44.

43. Is $f(x)$ differentiable at
(a) $x = -1$? (b) $x = 1$?

44. Is $f(x)$ differentiable at
(a) $x = -2$? (b) $x = 2$?

45. Let $f(x) = \frac{\sqrt[3]{4x}}{(3x^2-10)^2}$. Approximate $f'(2)$
(a) by using the numerical derivative feature of a graphing utility, and
(b) by evaluating $\frac{f(2+h)-f(2)}{h}$ with $h = 0.0001$.

46. Use the given table of values for $g(x)$ to approximate $g'(4)$ as accurately as possible.

x	2	2.3	3.1	4	4.3	5
$g(x)$	13.2	12.1	9.7	12.2	14.3	18.1

Section 9.4

47. If $c = 4x^5 - 6x^3$, find c'.
48. If $f(x) = 4x^2 - 1$, find $f'(x)$.
49. If $p = 3q + \sqrt{7}$, find dp/dq.
50. If $y = \sqrt{x}$, find y'.
51. If $f(z) = \sqrt[3]{2^4}$, find $f'(z)$.
52. If $v(x) = 4/\sqrt[3]{x}$, find $v'(x)$.
53. If $y = \frac{1}{x} - \frac{1}{\sqrt{x}}$, find y'.
54. If $f(x) = \frac{3}{2x^2} - \sqrt[3]{x} + 4^5$, find $f'(x)$.
55. Write the equation of the line tangent to the graph of $y = 3x^5 - 6$ at $x = 1$.
56. Write the equation of the line tangent to the curve $y = 3x^3 - 2x$ at the point where $x = 2$.

In Problems 57 and 58, (a) find all x-values where the slope of the tangent equals zero, (b) find points (x, y) where the slope of the tangent equals zero, and (c) use a graphing utility to graph the function and label the points found in (b).

57. $f(x) = x^3 - 3x^2 + 1$
58. $f(x) = x^6 - 6x^4 + 8$

Section 9.5

59. If $f(x) = (3x - 1)(x^2 - 4x)$, find $f'(x)$.
60. Find y' if $y = (x^2 + 1)(3x^3 + 1)$.
61. If $p = \frac{2q-1}{q^2}$, find $\frac{dp}{dq}$.
62. Find $\frac{ds}{dt}$ if $s = \frac{\sqrt{t}}{(3t+1)}$.
63. Find $\frac{dy}{dx}$ for $y = \sqrt{x}(3x + 2)$.
64. Find $\frac{dC}{dx}$ for $C = \frac{5x^4 - 2x^2 + 1}{x^3 + 1}$.

Section 9.6

65. If $y = (x^3 - 4x^2)^3$, find y'.
66. If $y = (5x^6 + 6x^4 + 5)^6$, find y'.
67. If $y = (2x^4 - 9)^9$, find $\frac{dy}{dx}$.
68. Find $g'(x)$ if $g(x) = \frac{1}{\sqrt{x^3 - 4x}}$.

Section 9.7

69. Find $f'(x)$ if $f(x) = x^2(2x^4 + 5)^8$.

70. Find S' if $S = \dfrac{(3x + 1)^2}{x^2 - 4}$.

71. Find $\dfrac{dy}{dx}$ if $y = [(3x + 1)(2x^3 - 1)]^{12}$.

72. Find y' if $y = \left(\dfrac{x + 1}{1 - x^2}\right)^3$.

73. Find y' if $y = x\sqrt{x^2 - 4}$.

74. Find $\dfrac{dy}{dx}$ if $y = \dfrac{x}{\sqrt[3]{3x - 1}}$.

Section 9.8

In Problems 75 and 76, find the second derivatives.

75. $y = \sqrt{x} - x^2$

76. $y = x^4 - \dfrac{1}{x}$

In Problems 77 and 78, find the fifth derivatives.

77. $y = (2x + 1)^4$

78. $y = \dfrac{(1 - x)^6}{24}$

79. If $\dfrac{dy}{dx} = \sqrt{x^2 - 4}$, find $\dfrac{d^3y}{dx^3}$.

80. If $\dfrac{d^2y}{dx^2} = \dfrac{x}{x^2 + 1}$, find $\dfrac{d^4y}{dx^4}$.

Applications

Section 9.4

81. ***Demand*** Suppose that the demand x for a product is given by $x = (100/p) - 1$, where p is the price per unit of the product. Find and interpret the rate of change of demand with respect to price if the price is
(a) \$10. (b) \$20.

Section 9.6

82. ***Demand*** The demand q for a product at price p is given by

$$q = 10{,}000 - 50\sqrt{0.02p^2 + 500}$$

Find the rate of change of demand with respect to price.

83. ***Supply*** The number of units x of a product that is supplied at price p is given by

$$x = \sqrt{p - 1}, \quad p \geq 1$$

If the price p is \$10, what is the rate of change of the supply with respect to the price and what does it tell us?

Section 9.9

84. ***Cost*** If the cost function for a particular good is $C(x) = 3x^2 + 6x + 600$, what is the
(a) marginal cost function?
(b) marginal cost if 30 units are produced?
(c) interpretation of your answer in (b)?

85. ***Cost*** If the total cost function for a commodity is $C(x) = 400 + 5x + x^3$, what is the marginal cost when 4 units are produced and what does it mean?

86. ***Revenue*** The total revenue function for a commodity is $R = 40x - 0.02x^2$, with x representing the number of units.
(a) Find the marginal revenue function.
(b) At what level of production will marginal revenue be 0?

87. ***Profit*** If the total revenue function for a product is given by $R(x) = 60x$ and the total cost function is given by $C = 200 + 10x + 0.1x^2$, what is the marginal profit at $x = 10$? What does the marginal profit at $x = 10$ predict?

88. ***Revenue*** The total revenue function for a commodity is given by $R = 80x - 0.04x^2$.
(a) Find the marginal revenue function.
(b) What is the marginal revenue at $x = 100$?
(c) Interpret your answer in (b).

89. ***Revenue*** If the revenue function for a product is

$$R(x) = \frac{60x^2}{2x + 1}$$

find the marginal revenue.

90. ***Profit*** A firm has monthly costs given by

$$C = 45{,}000 + 100x + x^3$$

where x is the number of units produced per month. The firm can sell its product in a competitive market for \$4600 per unit. Find the marginal profit.

91. ***Profit*** A small business has weekly costs of

$$C = 100 + 30x + \frac{x^2}{10}$$

where x is the number of units produced each week. The competitive market price for this business's product is \$46 per unit. Find the marginal profit.

CHAPTER TEST

1. Evaluate the following limits, if they exist. Use algebraic methods.
   (a) $\lim_{x\to -2} \frac{4x - x^2}{4x - 8}$ (b) $\lim_{x\to\infty} \frac{8x^2 - 4x + 1}{2 + x - 5x^2}$
   (c) $\lim_{x\to 7} \frac{x^2 - 5x - 14}{x^2 - 6x - 7}$ (d) $\lim_{x\to -5} \frac{5x - 25}{x + 5}$
2. (a) Write the limit definition for $f'(x)$.
   (b) Use the definition from (a) to find $f'(x)$ for $f(x) = 3x^2 - x + 9$.
3. Let $f(x) = \frac{4x}{x^2 - 8x}$. Identify all x-values where $f(x)$ is *not* continuous.
4. Use derivative formulas to find the derivative of each of the following. Simplify, except for (b).
   (a) $y = \frac{3x^3}{2x^7 + 11}$
   (b) $f(x) = (3x^5 - 2x + 3)(4x^{10} + 10x^4 - 17)$
   (c) $g(x) = \frac{3}{4}(2x^5 + 7x^3 - 5)^{12}$
   (d) $y = (x^2 + 3)(2x + 5)^6$
   (e) $f(x) = 12\sqrt{x} - \frac{10}{x^2} + 17$
5. Find $\frac{d^3y}{dx^3}$ for $y = x^3 - x^{-3}$.
6. Let $f(x) = x^3 - 3x^2 - 24x - 10$.
   (a) Write the equation of the line tangent to the graph of $y = f(x)$ at $x = -1$.
   (b) Find all points (both x- and y-coordinates) where $f'(x) = 0$.
7. Use the given tables to evaluate the following limits, if they exist.
   (a) $\lim_{x\to 5} f(x)$ (b) $\lim_{x\to 5} g(x)$ (c) $\lim_{x\to 5^-} g(x)$

x	4.99	4.999	$\to 5 \leftarrow$	5.001	5.01
$f(x)$	2.01	2.001	$\to ? \leftarrow$	1.999	1.99

x	4.99	4.999	$\to 5 \leftarrow$	5.001	5.01
$g(x)$	−3.99	−3.999	$\to ? \leftarrow$	6.999	6.99

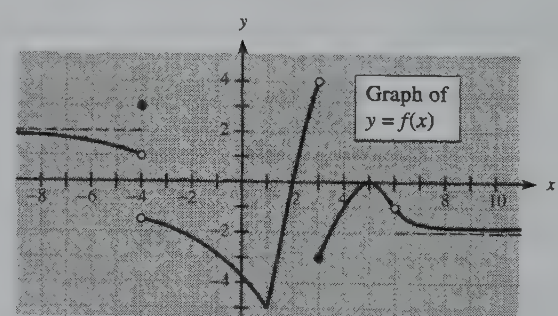
8. Use the definition of continuity to investigate whether $g(x)$ is continuous at $x = -2$. Show your work.

$$g(x) = \begin{cases} 6 - x & \text{if } x \le -2 \\ x^3 & \text{if } x > -2 \end{cases}$$

9. Suppose a company has its total cost for a product given by $C(x) = 200x + 10{,}000$ and its total revenue given by $R(x) = 250x - 0.01x^2$, where x is the number of units produced and sold.
   (a) Form the profit function for this product.
   (b) Find the marginal profit function.
   (c) Find the marginal profit when $x = 1000$, and then write a sentence that interprets this result.
10. Suppose that $f(x)$ is a differentiable function. Use the table of values to approximate $f'(3)$ as accurately as possible.

x	2	2.5	2.999	3	3.01	3.1
$f(x)$	0	18.4	44.896	45	46.05	56.18

11. Use the graph to perform the evaluations (a)–(e) and to answer (f)–(g). If no value exists, so indicate.
   (a) $f(1)$ (b) $\lim_{x\to 6} f(x)$
   (c) $\lim_{x\to 3^-} f(x)$ (d) $\lim_{x\to -4} f(x)$
   (e) $\lim_{x\to -\infty} f(x)$
   (f) Find all x-values where $f'(x)$ does not exist.
   (g) Find all x-values where $f(x)$ is not continuous.



12. Given that the line $y = \frac{2}{3}x - 8$ is tangent to the graph of $y = f(x)$ at $x = 6$, find
   (a) $f'(6)$ (b) $f(6)$
   (c) the instantaneous rate of change of $f(x)$ with respect to x at $x = 6$

Extended Applications Group Projects

I. Marginal Return to Sales

A tire manufacturer studying the effectiveness of television advertising and other promotions on sales of its GRIPPER-brand tires attempted to fit data it had gathered to the equation

$$S = a_0 + a_1x + a_2x^2 + b_1y$$

where S is sales revenue in millions of dollars, x is millions of dollars spent on television advertising, y is millions of dollars spent on other promotions, and a_0, a_1, a_2, and b_1 are constants. The data, gathered in two different regions of the country where expenditures for other promotions were kept constant (at B_1 and B_2), resulted in the following quadratic equations relating TV advertising and sales.

$$\text{Region 1:} \quad S_1 = 30 + 20x - 0.4x^2 + B_1$$
$$\text{Region 2:} \quad S_2 = 20 + 36x - 1.3x^2 + B_2$$

The company wants to know how to make the best use of its advertising dollars in the regions and whether the current allocation could be improved. Advise management about current advertising effectiveness, allocation of additional expenditures, and reallocation of current advertising expenditures by answering the following questions.

1. In the analysis of sales and advertising, **marginal return to sales** is usually used, and it is given by dS_1/dx for Region 1 and dS_2/dx for Region 2.
   (a) Find $\dfrac{dS_1}{dx}$ and $\dfrac{dS_2}{dx}$.
   (b) If \$10 million is being spent on TV advertising in each region, what is the marginal return to sales in each region?
2. Which region would benefit more from additional advertising expenditure, if \$10 million is currently being spent in each region?
3. If any additional money is made available for advertising, in which region should it be spent?
4. How could money already being spent be reallocated to produce more sales revenue?

II. Marginal Cost, Marginal Revenue, and Maximum Profit

In this chapter, we have seen how marginals for total cost, total revenue, and profit (that is, their derivatives) can be used to predict short-run future trends for each of these functions. In this project, we examine how a business might predict maximum profit by using the marginal cost and marginal revenue.

1. For each given pair of total cost and total revenue functions, complete the corresponding table. Assume that x represents the number of items produced and sold.

   (a) $C(x) = 3x + 6000$
   $R(x) = 12x - 0.001x^2$

   (b) $C(x) = 187x + 0.01x^2 + 15{,}750$
   $R(x) = 308x - 0.01x^2$

x	*Profit*	$\overline{MC} = C'(x)$	$\overline{MR} = R'(x)$
4000			
4100			
4200			
$\vdots$			
4900			
5000			

x	*Profit*	$\overline{MC} = C'(x)$	$\overline{MR} = R'(x)$
3000			
3005			
3010			
$\vdots$			
3095			
3100			

2. Examine the data you collected in each table.
   (a) Does each profit function seem to have a maximum value? If so, identify it in your table.
   (b) For each profit function, what are the values of marginal cost and marginal revenue at the x-value where profit has its maximum?
   (c) If a profit function has a maximum value at $x = a$, what seems to be the relationship between the values of marginal cost and marginal revenue at $x = a$? Answer in a summary sentence.
3. For each given total cost and total revenue pair from part 1, use a graphing utility to make the following graphs:
   (a) marginal cost and marginal revenue (graphed simultaneously)
   (b) profit

   These graphs should confirm your conclusions from part 2(c). Print or reproduce the graphs, and highlight the portion of the graphs that illustrates the relationship between the occurrence of maximum profit and the values of marginal cost and marginal revenue.
4. (a) In general, if $\overline{MR} > \overline{MC}$, is $P(x)$ increasing or decreasing? Justify your choice; use the interpretation of marginals as predictors for the next unit.
   (b) If $\overline{MR} < \overline{MC}$, is $P(x)$ increasing or decreasing? Justify your choice.
   (c) Explain how the observations in (a) and (b) provide general support for your statement relating the occurrence of maximum profit to the values of marginal cost and marginal revenue.

5. The conclusions relating marginal cost, marginal revenue, and maximum profit are valid as long as the total cost and total revenue functions are not both linear, such as with either of the following.
   (a) $C(x) = 80x + 8000$
   $R(x) = 120x$
   (b) $C(x) = 52x + 7800$
   $R(x) = 50x$
   In each of these cases, find the marginal cost and marginal revenue. In light of your answers to parts 4(a) and 4(b) above, what do these calculations tell you about the corresponding profit function? Does the profit function for either (a) or (b) have a maximum value? Explain.

Warm-up

Prerequisite Problem Type	*For Section*	*Answer*	*Section for Review*
Write $\frac{1}{3}(x^2 - 1)^{-2/3}(2x)$ with positive exponents.	**10.2**	$\dfrac{2x}{3(x^2 - 1)^{2/3}}$	0.3, 0.4 Exponents and radicals
Factor: (a) $x^3 - x^2 - 6x$ (b) $8000 - 80x - 3x^2$	**10.1** **10.2** **10.3**	(a) $x(x - 3)(x + 2)$ (b) $(40 - x)(200 + 3x)$	0.6 Factoring
(a) For what values of x is $\dfrac{2}{3\sqrt[3]{x+2}}$ undefined? (b) For what values of x is $\frac{1}{3}(x^2 - 1)^{-2/3}(2x)$ undefined?	**10.1** **10.2** **10.5**	(a) $x = -2$ (b) $x = -1, x = 1$	1.2 Domains of functions
If $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 2$, and $f'(x) = x^2 - 2x - 3$, (a) find $f(-1)$. (b) find $f'(-2)$.	**10.1**	(a) $\dfrac{11}{3}$ (b) 5	1.2 Functional notation
(a) Solve $0 = x^2 - 2x - 3$. (b) If $f'(x) = 3x^2 - 3$, what values of x make $f'(x) = 0$?	**10.1–10.5**	(a) $x = -1, x = 3$ (b) $x = -1, x = 1$	2.1 Solving quadratic equations
Does $\lim_{x \to -2} \dfrac{2x - 4}{3x + 6}$ exist?	**10.5**	No; unbounded	9.1 Limits
(a) Find $f''(x)$ if $f(x) = x^3 - 4x^2 + 3$. (b) Find $P''(x)$ if $P(x) = 48x - 1.2x^2$.	**10.2**	(a) $f''(x) = 6x - 8$ (b) $P''(x) = -2.4$	9.8 Higher-order derivatives
Find the derivatives: (a) $y = \dfrac{1}{3}x^3 - x^2 - 3x + 2$ (b) $f = x + 2\left(\dfrac{80{,}000}{x}\right)$ (c) $p(t) = 1 + \dfrac{4t}{t^2 + 16}$ (d) $y = (x + 2)^{2/3}$ (e) $y = \sqrt[3]{x^2 - 1}$	**10.1** **10.2** **10.3** **10.4**	(a) $y' = x^2 - 2x - 3$ (b) $f' = 1 - \dfrac{160{,}000}{x^2}$ (c) $p'(t) = \dfrac{64 - 4t^2}{(t^2 + 16)^2}$ (d) $y' = \dfrac{2}{3(x + 2)^{1/3}}$ (e) $y' = \dfrac{2x}{3(x^2 - 1)^{2/3}}$	9.4, 9.5, 9.6 Derivatives

Chapter 10

Applications of Derivatives

In this chapter we will discuss applications of the derivative. In particular, we will consider methods of determining when a function has a "turning point" on its graph, so that we can determine when the graph of the function reaches its highest or lowest point within a particular interval. These points are called ***relative maxima*** *and* ***relative minima,*** *respectively, and are useful in sketching the graph of a function whose equation is given and that has a maximum or minimum value within a particular interval. The endpoints of a given interval and the relative maxima and minima within the interval can be used to solve many types of applied problems. For example, we can use these points in determining the level of production that maximizes revenue or profit for a product and in minimizing the average cost of producing a product.*

In addition to using the first derivative to help graph a function, we can use the second derivative to determine where the graph will be concave up or concave down and where it will change from concave up to concave down or vice versa. Points where this change occurs are called ***points of inflection.*** *The second derivative uses concavity to determine where the graph of a function has a relative maximum or relative minimum. Knowledge of this information can be used to sketch the graph of a function or to determine the appropriate viewing window to use when graphing the function with a graphing utility. Because horizontal and vertical asymptotes are not always apparent when a graphing utility is used, we will discuss methods of finding and accounting for asymptotes when graphing functions.*

10.1 Relative Maxima and Minima; Curve Sketching

OBJECTIVES

- *To find relative maxima and minima and horizontal points of inflection of functions*
- *To sketch graphs of functions by using information about maxima, minima, and horizontal points of inflection*

APPLICATION PREVIEW

When a company initiates an advertising campaign, there is typically a surge in weekly sales. As the effect of the campaign lessens, sales attributable to it usually decrease. For example, suppose a company models its weekly sales during an advertising campaign by

$$S = \frac{100t}{t^2 + 100}$$

where t is the number of weeks since the beginning of the campaign. The company would like to determine accurately when the revenue function is increasing, when it is decreasing, and when sales revenue is maximized.

In this section we will use the derivative of a function to decide whether the function is increasing or decreasing on an interval and to find where the function has relative maximum points and relative minimum points. We will use the information about derivatives of functions to graph the functions and to solve applied problems.

Except for very simple graphs (lines and parabolas, for example), plotting points to sketch a graph may be tedious. Even when we use a graphing utility, special features of a graph of a function may be difficult to locate accurately. In addition to intercepts and asymptotes, we can use the first derivative as an aid in graphing. The first derivative identifies the "turning points" of the graph, which help us determine the general shape of the graph and choose a viewing window that includes the interesting points of the graph if a graphing utility is used.

In Figure 10.1(a) we see that the graph of $y = \frac{1}{3}x^3 - x^2 - 3x + 2$ has two "turning points," at $(-1, \frac{11}{3})$ and $(3, -7)$. The curve has a relative maximum at $(-1, \frac{11}{3})$ because this point is higher than any other point "near" it on the curve; the curve has a relative minimum at $(3, -7)$ because this point is lower than any other point "near" it on the curve. A formal definition follows.

Relative Maxima and Minima

The point $(x_1, f(x_1))$ is a **relative maximum point** for the function f if there is an interval around x_1 on which $f(x_1) \geq f(x)$ for all x in the interval. In this case, we say the relative maximum *occurs* at $x = x_1$ and the relative maximum is $f(x_1)$.

The point $(x_2, f(x_2))$ is a **relative minimum point** for the function f if there is an interval around x_2 on which $f(x_2) \leq f(x)$ for all x in the interval. In this case, we say the relative minimum *occurs* at $x = x_2$ and the relative minimum is $f(x_2)$.

In order to determine whether a turning point of a function is a maximum point or a minimum point, it is frequently helpful to know what the graph of the function does in intervals on either side of the turning point. We say a function is **increasing** on an interval if the functional values increase as the x-values

increase (that is, if the graph rises as we move from left to right on the interval). Similarly, a function is **decreasing** on an interval if the functional values decrease as the x-values increase (that is, if the graph falls as we move from left to right on the interval).

We have seen that if the slope of a line is positive, then the linear function is increasing and its graph is rising. Similarly, if $f(x)$ is differentiable over an interval and if each tangent line to the curve over that interval has positive slope, then the curve is rising over the interval and the function is increasing. Because the derivative of the function gives the slope of the tangent to the curve, we see that if $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval. A similar conclusion can be reached when the derivative is negative on the interval.

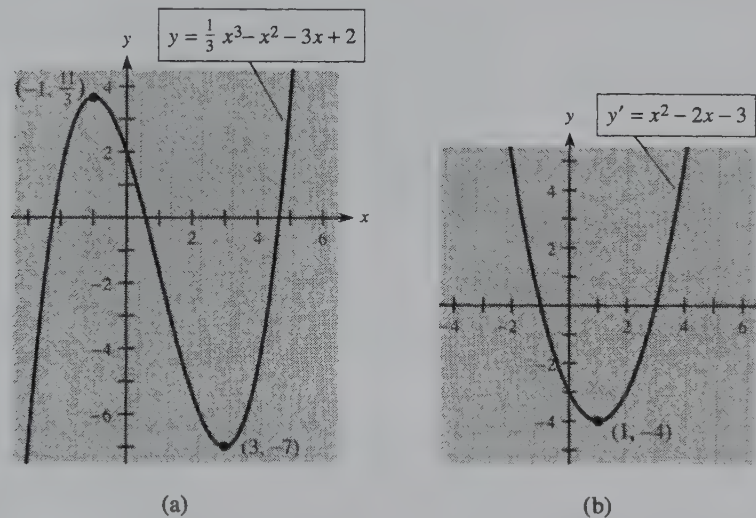


Figure 10.1

Increasing and Decreasing Functions

If f is a function that is differentiable on an interval (a, b), then

if $f'(x) > 0$ for all x in (a, b), f is increasing on (a, b).

if $f'(x) < 0$ for all x in (a, b), f is decreasing on (a, b).

Figure 10.1(a) shows the graph of a function, and Figure 10.1(b) shows the graph of its derivative. The figures show that the graph of $y = f(x)$ is increasing for the same x-values that the graph of $y' = f'(x)$ is above the x-axis (when $f'(x) > 0$). Similarly, the graph of $y = f(x)$ is decreasing for the same x-values $(-1 < x < 3)$ that the graph of $y' = f'(x)$ is below the x-axis (when $f'(x) < 0$).

The derivative $f'(x)$ can change signs only at values of x where $f'(x) = 0$ or $f'(x)$ is undefined. We call these values of x **critical values.** The point corresponding to a critical value for x is a **critical point.*** Because a curve changes from increasing to decreasing at a relative maximum (see Figure 10.1a), we have the following fact.

*There may be some critical values where $f'(x)$ and $f(x)$ are undefined. Critical points do not occur at these values, but studying the derivative on either side of such values may be of interest.

Relative Maximum If f has a relative maximum at $x = x_0$, then $f'(x_0) = 0$ or $f'(x_0)$ is undefined.

From Figure 10.2, we see that this function has two relative maxima, one at $x = x_1$ and the second at $x = x_3$. At $x = x_1$ the derivative is 0, and at $x = x_3$ the derivative does not exist. In Figure 10.2, we also see that a relative minimum occurs at $x = x_2$ and $f'(x_2) = 0$. As Figure 10.2 shows, the function changes from decreasing to increasing at a relative minimum. Thus we have the following fact.

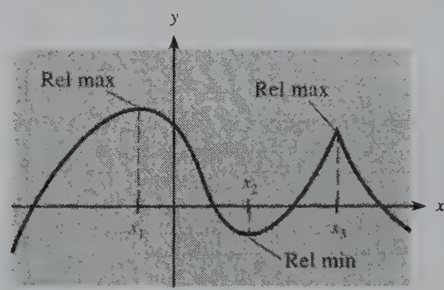


Figure 10.2

Relative Minimum If f has a relative minimum at $x = x_0$, then $f'(x_0) = 0$ or $f'(x_0)$ is undefined.

Thus we can find relative maxima and minima for a curve by finding values of x for which the function has critical points. The behavior of the derivative to the left and right of (and near) these points will tell us whether they are relative maxima, relative minima, or neither.

Because the critical values are the only values where the graph can have turning points, the derivative cannot change sign anywhere except at a critical value. Thus, in an interval between two critical values, the sign of the derivative at any value in the interval will be the sign of the derivative at all values in the interval.

Using the critical values of $f(x)$ and the sign of $f'(x)$ between those critical values, we can create a **sign diagram for $f'(x)$.** The sign diagram for the graph in Figure 10.2 is shown in Figure 10.3. This sign diagram was created from the graph of f, but it is also possible to predict the shape of a graph from a sign diagram.

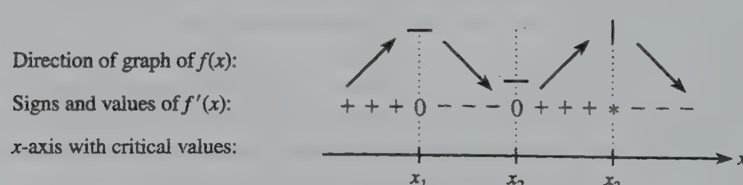


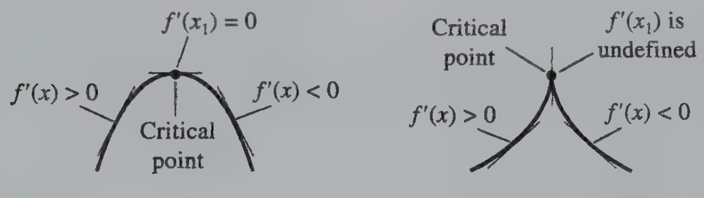
Figure 10.3

*means $f'(x_3)$ is undefined.

Suppose that the point (x_1, y_1) is a critical point. If $f'(x)$ is positive to the left of, and near, this critical point, and if $f'(x)$ is negative to the right of, and near, this critical point, then the curve is increasing to the left of the point and

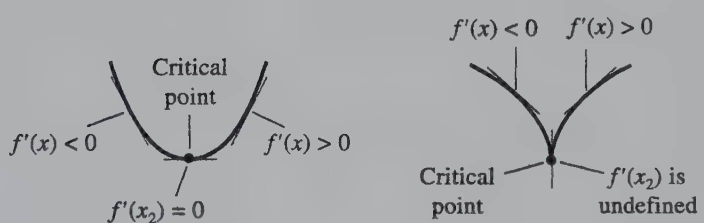
decreasing to the right. This means that a relative maximum occurs at the point. If we drew tangent lines to the curve on the left and right of this critical point, they would fit on the curve in one of the two ways shown in Figure 10.4.

Similarly, suppose that the point (x_2, y_2) is a critical point, $f'(x)$ is negative to the left of and near (x_2, y_2), and $f'(x)$ is positive to the right of and near this critical point. Then the curve is decreasing to the left of the point and increasing to the right, and a relative minimum occurs at the point. If we drew tangent lines to the curve to the left of and to the right of this critical point, they would fit on the curve in one of the two ways shown in Figure 10.5.



Relative maximum (a) Relative maximum (b)

Figure 10.4



Relative minimum (a) Relative minimum (b)

Figure 10.5

EXAMPLE 1

Show that the graph of $f(x) = 3x^2 - 2x^3$ has a relative maximum at $x = 1$.

Solution

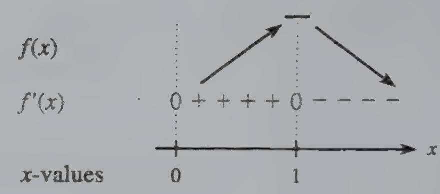
We need to show that

$$f'(1) = 0 \text{ or } f'(1) \text{ is undefined.}$$
$$f'(x) > 0 \text{ to the left of and near } x = 1.$$
$$f'(x) < 0 \text{ to the right of and near } x = 1.$$

Because $f'(x) = 6x - 6x^2 = 6x(x - 1)$ is 0 only at $x = 0$ and at $x = 1$, we can test (evaluate) the derivative at any value in the interval (0, 1) to see what the curve is doing to the left of $x = 1$, and we can test the derivative at any value to the right of $x = 1$ to see what the curve is doing for $x > 1$.

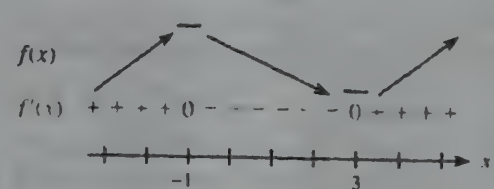
$$f'\left(\frac{1}{2}\right) = 3 - \frac{3}{2} = \frac{3}{2} > 0 \Rightarrow \text{increasing to left of } x = 1$$
$$f'(1) = 0 \Rightarrow \text{horizontal tangent at } x = 1$$
$$f'\left(\frac{3}{2}\right) = 9 - \frac{27}{2} = -\frac{9}{2} < 0 \Rightarrow \text{decreasing to right of } x = 1$$

A partial sign diagram for $f'(x)$ is given at the right.
The sign diagram shows that the graph of the function has a relative maximum at $x = 1$.



The preceding discussion suggests the following procedure for finding relative maxima and minima of a function.

First-Derivative Test

Procedure	***Example***
To find relative maxima and minima of a function:	Find the relative maxima and minima of $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 2$.
1. Find the first derivative of the function.	1. $f'(x) = x^2 - 2x - 3$
2. Set the derivative equal to 0, and solve for values of x that satisfy $f'(x) = 0$. These are called **critical values.** Values that make $f'(x)$ undefined are also critical values.	2. $0 = x^2 - 2x - 3 = (x + 1)(x - 3)$ has solutions $x = -1$, $x = 3$. No values of x make $x^2 - 2x - 3$ undefined. Critical values are -1 and 3.
3. Substitute the critical values into the *original function* to find the **critical points.**	3. $f(-1) = \frac{11}{3}$ $\quad f(3) = -7$ The critical points are $(-1, \frac{11}{3})$ and $(3, -7)$.
4. Evaluate $f'(x)$ at some value of x to the left and right of each critical point to develop a sign diagram. (a) If $f'(x) > 0$ to the left and $f'(x) < 0$ to the right of the critical value, the critical point is a relative maximum point. (b) If $f'(x) < 0$ to the left and $f'(x) > 0$ to the right of the critical value, the critical point is a relative minimum point.	4. $f'(-2) = 5 > 0$ and $f'(0) = -3 < 0$ Thus $(-1, 11/3)$ is a relative maximum point. $f'(2) = -3 < 0$ and $f'(4) = 5 > 0$ Thus $(3, -7)$ is a relative minimum point. The sign diagram for $f'(x)$ is 
5. Use the information from the sign diagram and selected points to sketch the graph.	5. The information from this sign diagram is shown in Figure 10.6(a). Plotting additional points gives the graph of the function, which is shown in Figure 10.6(b).

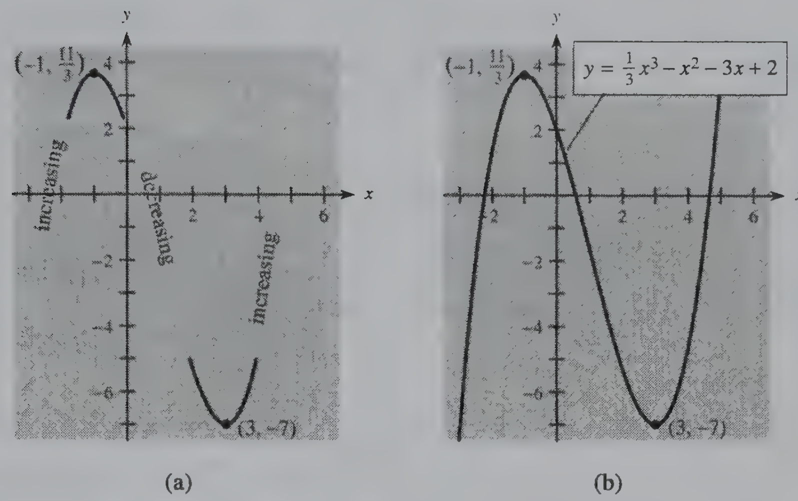


Figure 10.6

Because the critical values are the only x-values where the graph can have turning points, we can test to the left and right of each critical value by testing to the left of the smallest critical value, then testing a value *between* each two successive critical values, and then testing to the right of the largest critical value. The following example illustrates this procedure.

EXAMPLE 2

Find the relative maxima and minima of $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2 + 8$, and sketch its graph.

Solution

1. $f'(x) = x^3 - x^2 - 6x$
2. Setting $f'(x) = 0$ gives $0 = x^3 - x^2 - 6x$. Solving for x gives

$$0 = x(x - 3)(x + 2).$$

$x = 0$	$x - 3 = 0$ $x = 3$	$x + 2 = 0$ $x = -2$

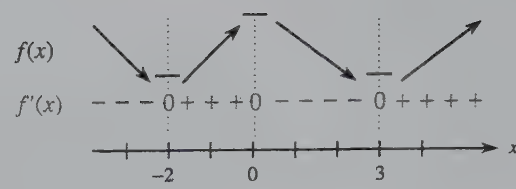
Thus the critical value are $x = 0$, $x = 3$, and $x = -2$.

3. Substituting the critical values into the original function gives the critical points:

$f(-2) = \frac{8}{3}$, so $(-2, \frac{8}{3})$ is a critical point.
$f(0) = 8$, so $(0, 8)$ is a critical point.
$f(3) = -\frac{31}{4}$, so $(3, -\frac{31}{4})$ is a critical point.

4. Testing $f'(x)$ to the left of the smallest critical value, then between the critical values, and then to the right of the largest critical value will give the sign diagram. Evaluating $f'(x)$ at the test values $x = -3$, $x = -1$, $x = 1$, and $x = 4$ gives the signs to determine relative maxima and minima.

The sign diagram for $f'(x)$ is



Thus we have

$(-2, \frac{8}{3})$ is a relative minimum.

$(0, 8)$ is a relative maximum.

$(3, -\frac{31}{4})$ is a relative minimum.

5. Figure 10.7(a) shows the graph of the function near the critical points, and Figure 10.7(b) shows the graph of the function.

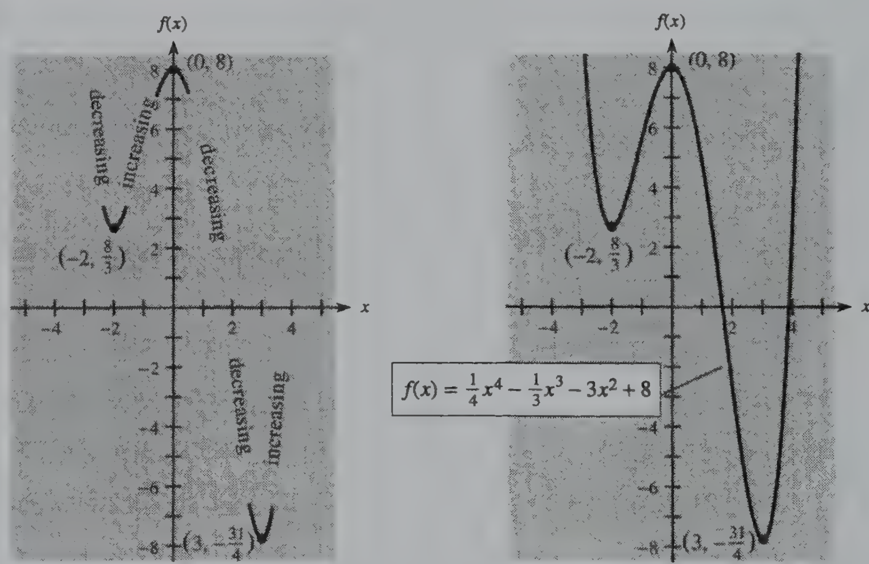


Figure 10.7 (a) (b)

Note that we substitute the critical values into the *original function* $f(x)$ to find the y-values of the critical points, but we test for relative maxima and minima by substituting values near the critical values into the *derivative of the function, $f'(x)$*.

Only four values were needed to test three critical points in Example 2. This method will work *only if* the critical values are tested in order from smallest to largest.

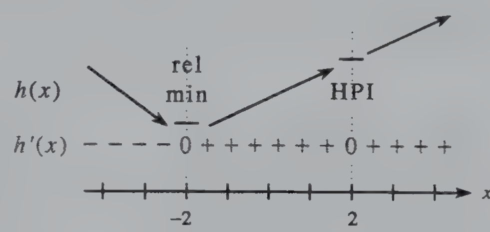
If the first derivative of f is 0 at x_0 but does not change from positive to negative or from negative to positive as x passes through x_0, then the critical point at x_0 is neither a relative maximum nor a relative minimum. In this case we say that f has a **horizontal point of inflection** (abbreviated HPI) at x_0.

EXAMPLE 3

Find the relative maxima, relative minima, and horizontal points of inflection of $h(x) = \frac{1}{4}x^4 - \frac{2}{3}x^3 - 2x^2 + 8x + 4$, and sketch its graph.

Solution

1. $h'(x) = x^3 - 2x^2 - 4x + 8$
2. $0 = x^3 - 2x^2 - 4x + 8$ or $0 = x^2(x - 2) - 4(x - 2)$. Therefore, we have $0 = (x - 2)(x^2 - 4)$. Thus $x = -2$ and $x = 2$ are solutions.
3. The critical points are $(-2, -\frac{32}{3})$ and $(2, \frac{32}{3})$.
4. Using test values, such as $x = -3$, $x = 0$, and $x = 3$ gives the sign diagram for $h'(x)$.



5. Figure 10.8(a) shows the graph of the function near the critical points, and Figure 10.8(b) shows the graph of the function.

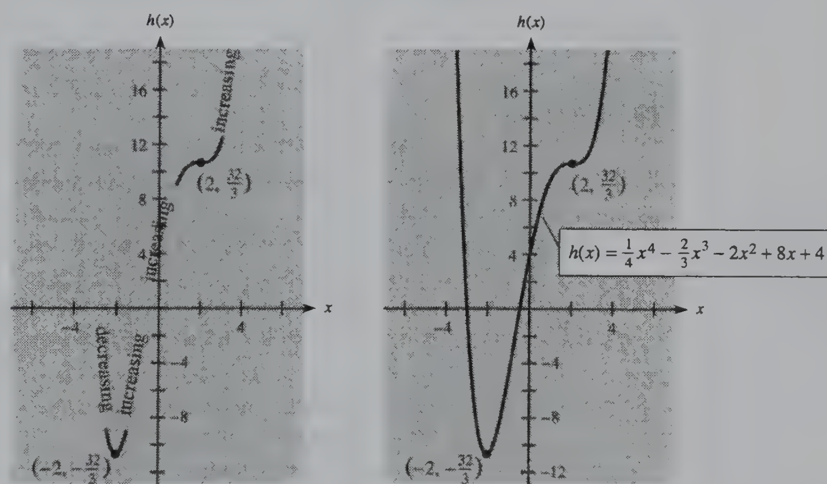


Figure 10.8 (a) (b)

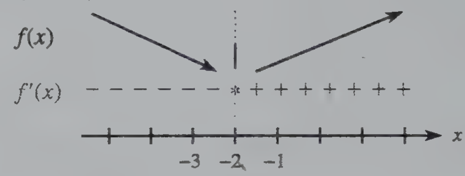
EXAMPLE 4

Find the relative maxima and minima (if any) of the graph of $y = (x + 2)^{2/3}$.

Solution

1. $y' = f'(x) = \dfrac{2}{3}(x + 2)^{-1/3} = \dfrac{2}{3\sqrt[3]{x+2}}$

2. $0 = \dfrac{2}{3\sqrt[3]{x+2}}$ has no solutions; $f'(x)$ is undefined at $x = -2$.
3. $f(-2) = 0$, so the critical point is $(-2, 0)$.
4. The sign diagram for $f'(x)$ is



* means $f'(-2)$ is undefined.

Thus a relative minimum occurs at $(-2, 0)$.

5. Figure 10.9(a) shows the graph of the function near the critical point, and Figure 10.9(b) shows the graph.

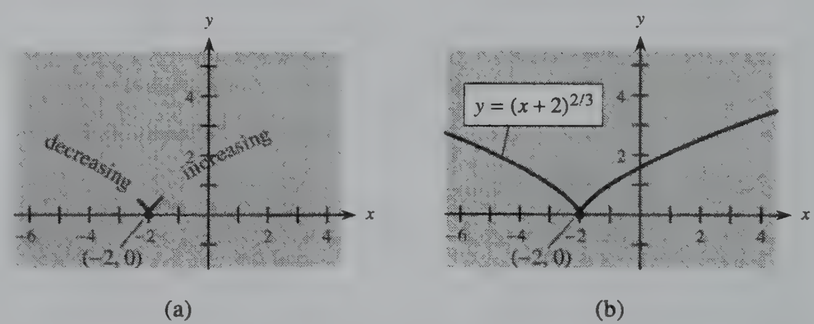
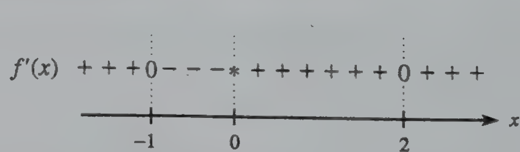


Figure 10.9

CHECKPOINT

1. The x-values of critical points are found where $f'(x)$ is ________ or ________.
2. Decide whether the following are true or false.
   (a) If $f'(1) = 7$, then $f(x)$ is increasing at $x = 1$.
   (b) If $f'(-2) = 0$, then a relative maximum or a relative minimum occurs at $x = -2$.
   (c) If $f'(-3) = 0$ and $f'(x)$ changes from positive on the left to negative on the right of $x = -3$, then a relative minimum occurs at $x = -3$.
3. If $f(x) = 7 + 3x - x^3$, then $f'(x) = 3 - 3x^2$. Use these functions to decide whether the following are true or false.
   (a) The only critical value is $x = 1$.
   (b) The critical points are $(1, 0)$ and $(-1, 0)$.
4. If $f'(x)$ has the following partial sign diagram, make a "stick-figure" sketch of $f(x)$ and label where any maxima and minima occur. Assume that $f(x)$ is defined for all real numbers.



* means $f'(0)$ is undefined.

Let us now return to the discussion of advertising and sales revenue that we began in the Application Preview.

EXAMPLE 5

The weekly sales S of a product during an advertising campaign are given by

$$S = \frac{100t}{t^2 + 100}, \qquad 0 \le t \le 20$$

where t is the number of weeks since the beginning of the campaign and S is in thousands of dollars.

(a) Over what interval are sales increasing? decreasing?
(b) What is the maximum weekly sales?
(c) Sketch the graph for $0 \le t \le 20$.

Solution

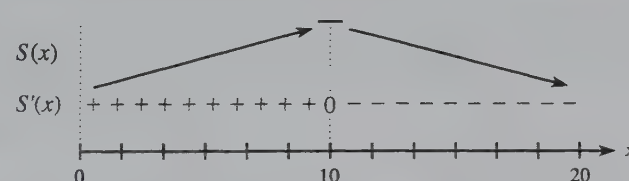
(a) To find where S is increasing, we first find $S'(t)$.

$$S''(t) = \frac{(t^2 + 100)100 - (100t)2t}{(t^2 + 100)^2}$$
$$= \frac{10{,}000 - 100t^2}{(t^2 + 100)^2}$$

We see that $S'(t) = 0$ when $10{,}000 - 100t^2 = 0$, or

$$100(100 - t^2) = 0$$
$$(10 + t)(10 - t) = 0$$
$$t = -10 \quad \text{or} \quad t = 10$$

Because $S'(t)$ is never undefined ($t^2 + 100 \neq 0$ for any real t) and because $0 \le t \le 20$, our only critical value is $t = 10$. Testing $S'(t)$ to the left and right of $t = 10$ gives the sign diagram.



Hence, S is increasing on the interval $[0, 10)$ and decreasing on the interval $(10, 20]$.

(b) Because S is increasing to the left of $t = 10$ and S is decreasing to the right of $t = 10$, the maximum value of S occurs at $t = 10$ and is

$$S = S(10) = \frac{100(10)}{10^2 + 100} = \frac{1000}{200} = 5 \text{ (thousand dollars)}$$

(c) Plotting some additional points gives the graph; see Figure 10.10.

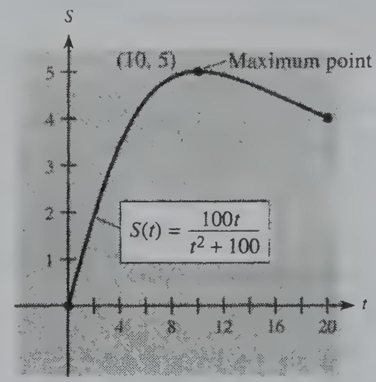


Figure 10.10

Graphing Utilities With a graphing utility, choosing an appropriate window is the key to understanding the graph of a function. We saw that the derivative can be used to determine the critical values of a function and hence the interesting points of its graph. Therefore, the derivative can be used to determine the viewing window that provides an accurate representation of the graph. The derivative can also be used to discover graphical behavior that might be overlooked in a graph with a standard window.

EXAMPLE 6

Find the critical values for $f(x) = 0.0001x^3 + 0.003x^2 - 3.6x + 5$. Use them to determine an appropriate viewing window. Then sketch the graph.

Solution

Suppose that we first graph this function using a standard viewing window. The graph of this function, for $-10 \leq x \leq 10$, is shown in Figure 10.11(a). The graph looks like a line in this window, but the function is not linear. We could explore the function by tracing or zooming, but using the critical points is more helpful in graphing the function. We begin by finding $f'(x)$.

$$f'(x) = 0.0003x^2 + 0.006x - 3.6$$

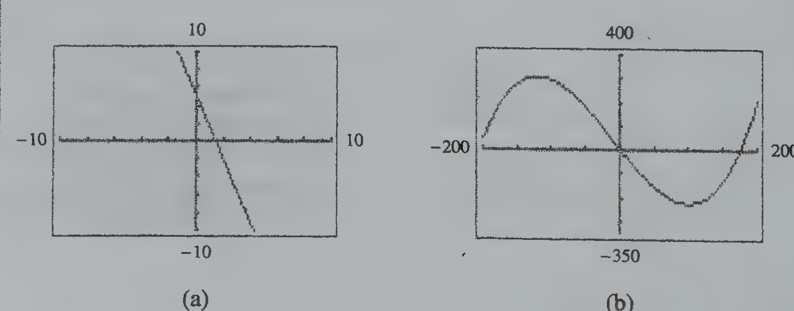


Figure 10.11

Solve $f'(x) = 0$ to find critical values.

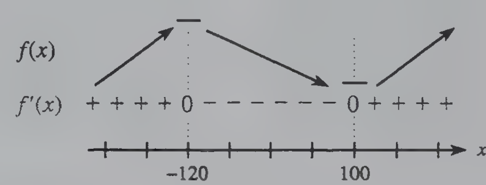
$$0 = 0.0003x^2 + 0.006x - 3.6$$
$$0 = 0.0003(x^2 + 20x - 12000)$$
$$0 = 0.0003(x + 120)(x - 100)$$
$$x = -120 \quad \text{or} \quad x = 100$$

We choose a window that includes $x = -120$ and $x = 100$, graph, and use TRACE to find $f(-120) = 307.4$ and $f(100) = -225$. We then graph $y = f(x)$ on a window that includes $y = -225$ and $y = 307.4$ (see Figure 10.11b). On this graph we can verify that $(-120, 307.4)$ is a relative maximum and that $(100, -225)$ is a relative minimum.

We can also use a table of values of $f'(x)$ to create a sign diagram that will identify the relative maxima and minima. Figure 10.12(a) shows values of $y_1 = f(x)$ and $y_2 = f'(x)$ for different values of x at and near the critical values. Using the x- and y_2-values gives the sign diagram in Figure 10.12(b), which shows that $(-120, 307.4)$ is a relative maximum and that $(100, -225)$ is a relative minimum.

X	Y1	Y2
-150	275	2.25
-120	307.4	0
0	5	-3.6
100	-225	0
150	-130	4.05
X=0		

(a)



(b)

Figure 10.12

EXAMPLE 7

The table below gives the percentage of high school seniors using marijuana for the years 1975 to 1996.

(a) Using x as the number of years from 1970, develop an equation that models the percentage.
(b) During what years does the model indicate that the maximum and minimum use occurred?

Year	*Percent Using Marijuana*	*Year*	*Percent Using Marijuana*
1975	47.3	1986	38.8
1976	52.8	1987	36.3
1977	56.4	1988	33.1
1978	59.2	1989	29.6
1979	60.4	1990	27.0
1980	60.3	1991	23.9
1981	59.5	1992	21.9
1982	58.7	1993	26.0
1983	57	1994	30.7
1984	54.9	1995	34.7
1985	54.2	1996	35.8

Source: National Institute on Drug Abuse

Solution

An equation that models the percentage is $y = 0.03743x^3 - 1.7681x^2 + 23.3761x - 32.2371$. (See Figure 10.13.)

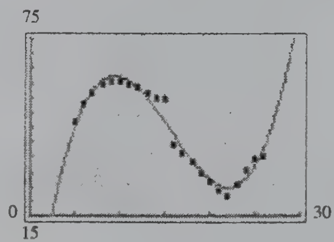


Figure 10.13

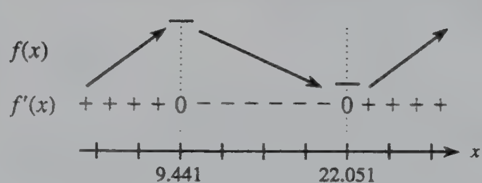
To find the critical values, we solve $y' = f'(x) = 0$, where $f'(x) = 0.11229x^2 - 3.5362x + 23.3761$. We can solve

$$0 = 0.11229x^2 - 3.5362x + 23.3761$$

with the quadratic formula or with a graphing utility. The two solutions are approximately

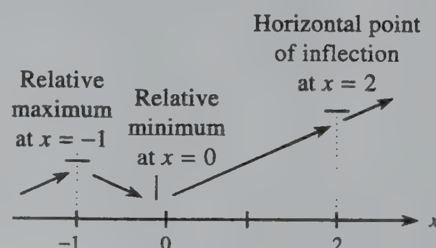
$$x = 9.441 \quad \text{or} \quad x = 22.051$$

The sign diagram shows that $x = 9.441$ gives a relative maximum (during 1979) and that $x = 22.051$ gives a relative minimum (during 1992). The data indicate that the maximum use occurred in 1979 and the minimum use in 1992.



CHECKPOINT SOLUTIONS

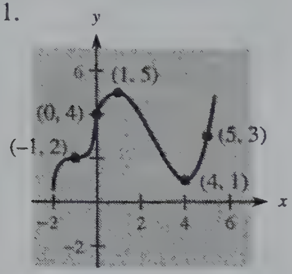
1. $f'(x) = 0$ or $f'(x)$ is undefined.
2. (a) True, $f(x)$ is increasing when $f'(x) > 0$.
   (b) False. There may be a horizontal point of inflection at $x = -2$ (see Figure 10.8 on page 741).
   (c) False. A relative maximum occurs at $x = -3$.
3. (a) False. Critical values are solutions to $3 - 3x^2 = 0$, or $x = 1$ and $x = -1$.
   (b) False, y-coordinates of critical points come from $f(x) = 7 + 3x - x^3$. Thus critical points are $(1, 9)$ and $(-1, 5)$.
4.



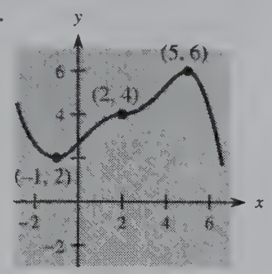
EXERCISE 10.1

In Problems 1 and 2, use the indicated points on the graph of $y = f(x)$ to identify points where $f(x)$ has (a) a relative maximum, (b) a relative minimum, and (c) a horizontal point of inflection.

1.



2.



3. Use the graph of $y = f(x)$ in Problem 1 to identify at which of the indicated points the derivative $f'(x)$ (a) changes from positive to negative, (b) changes from negative to positive, and (c) does not change sign.
4. Use the graph of $y = f(x)$ in Problem 2 to identify at which of the indicated points the derivative $f'(x)$ (a) changes from positive to negative, (b) changes from negative to positive, and (c) does not change sign.

In Problems 5 and 6, use the sign diagram of $f'(x)$ to determine (a) the critical values of $f(x)$, (b) intervals where $f(x)$ increases, (c) intervals where $f(x)$ decreases, (d) x-values where relative maxima occur, and (e) x-values where relative minima occur.

5. $f'(x)$ − − − 0 + + + + + 0 − − − x (with 0 at 3 and at 7)

6. $f'(x)$ + + + +0+ + + + + +0− − − − x (with 0 at −5 and at 8)

In Problems 7 and 8, find the critical values of the function.

7. $y = 2x^3 - 12x^2 + 6$
8. $y = x^3 - 3x^2 + 6x + 1$

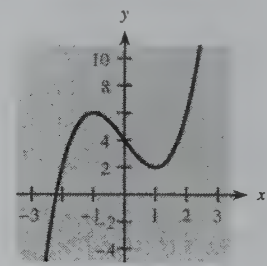
In Problems 9 and 10, make a sign diagram for the function and determine the relative maxima and minima. (See Problems 7 and 8.)

9. $y = 2x^3 - 12x^2 + 6$
10. $y = x^3 - 3x^2 + 6x + 1$

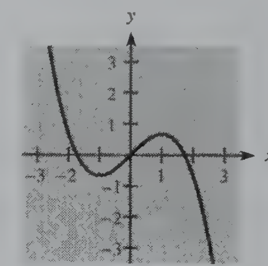
For each function and graph in Problems 11–14:

(a) Find $y' = f'(x) = \dfrac{dy}{dx}$.
(b) Use $y' = f'(x)$ to find the critical values.
(c) Find the critical points.
(d) Use the graph to classify each critical point as a relative maximum, relative minimum, or horizontal point of inflection.

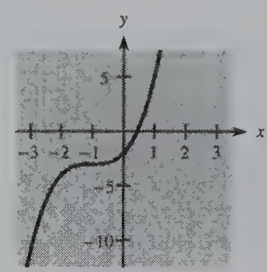
11. $y = x^3 - 3x + 4$



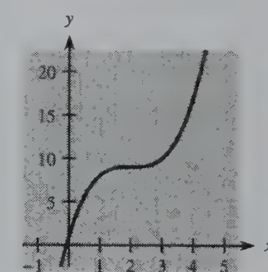
12. $y = x - \dfrac{1}{3}x^3$



13. $y = x^3 + 3x^2 + 3x - 2$



14. $y = x^3 - 6x^2 + 12x + 1$



For each function in Problems 15–20:

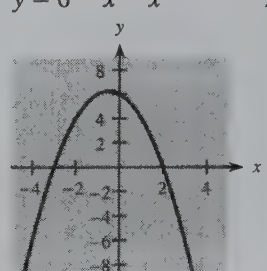
(a) Find $y' = f'(x)$.
(b) Find the critical values.
(c) Find the critical points.
(d) Find intervals of x-values where the function is increasing and where it is decreasing.
(e) Classify the critical points as relative maxima, relative minima, or horizontal points of inflection. In each case, you may check your conclusions with a graphing utility.

15. $y = \frac{1}{2}x^2 - x$
16. $y = x^2 + 4x$
17. $y = \dfrac{x^3}{3} + \dfrac{x^2}{2} - 2x + 1$
18. $y = \dfrac{x^4}{4} - \dfrac{x^3}{3} - 2$
19. $y = x^{2/3}$
20. $y = -(x - 3)^{2/3}$

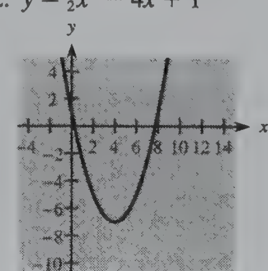
For each function and graph in Problems 21–24:

(a) Use the graph to identify x-values for which $y' > 0$, $y' < 0$, $y' = 0$, and y' does not exist.
(b) Use the derivative to check your conclusions.

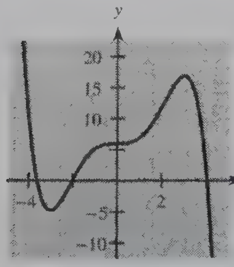
21. $y = 6 - x - x^2$



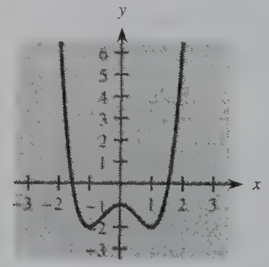
22. $y = \frac{1}{2}x^2 - 4x + 1$



23. $y = 6 + x^3 - \frac{1}{15}x^5$



24. $y = x^4 - 2x^2 - 1$



For each function in Problems 25–30, find the relative maxima, relative minima, horizontal points of inflection, and sketch the graph. You may check your graph with a graphing utility.

25. $y = \frac{1}{3}x^3 - x^2 + x + 1$
26. $y = \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 - 2$
27. $y = \frac{1}{3}x^3 + x^2 - 24x + 20$
28. $C(x) = x^3 - \frac{3}{2}x^2 - 18x + 5$
29. $y = 3x^5 - 5x^3 + 1$
30. $y = \frac{1}{6}x^6 - x^4 + 7$

In Problems 31–36, both a function and its derivative are given. Use them to find critical values, critical points, intervals where the function is increasing and decreasing, relative maxima, relative minima, and horizontal points of inflection; sketch the graph of each function.

31. $y = (x^2 - 2x)^2$ $\qquad \dfrac{dy}{dx} = 4x(x-1)(x-2)$

32. $f(x) = (x^2 - 4)^2$ $\qquad f'(x) = 4x(x+2)(x-2)$

33. $y = \dfrac{x^3(x-5)^2}{27}$ $\qquad \dfrac{dy}{dx} = \dfrac{5x^2(x-3)(x-5)}{27}$

34. $y = \dfrac{x^2(x-5)^3}{27}$ $\qquad \dfrac{dy}{dx} = \dfrac{5x(x-2)(x-5)^2}{27}$

35. $f(x) = x^{2/3}(x-5)$ $\qquad f'(x) = \dfrac{5(x-2)}{3x^{1/3}}$

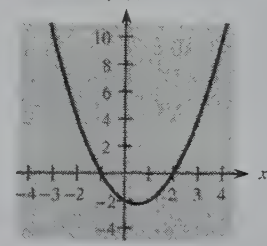
36. $f(x) = x - 3x^{2/3}$ $\qquad f'(x) = \dfrac{x^{1/3} - 2}{x^{1/3}}$

In Problems 37–42, use the derivative to locate critical points and determine a viewing window that shows all features of the graph. Use a graphing utility to sketch a complete graph.

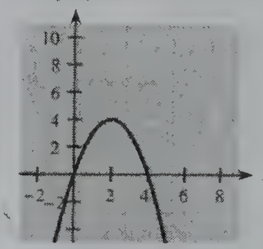
37. $f(x) = x^3 - 225x^2 + 15000x - 12000$
38. $f(x) = x^3 - 15x^2 - 16800x + 80000$
39. $f(x) = x^4 - 160x^3 + 7200x^2 - 40000$
40. $f(x) = x^4 - 240x^3 + 16200x^2 - 60000$
41. $y = 7.5x^4 - x^3 + 2$
42. $y = 2 - x^3 - 7.5x^4$

In each of Problems 43–46, a graph of $f'(x)$ is given. Use the graph to determine the critical values of $f(x)$, where $f(x)$ is increasing, where it is decreasing, and where it has relative maxima, relative minima, and horizontal points of inflection. In each case sketch a possible graph for $f(x)$ that passes through (0, 0).

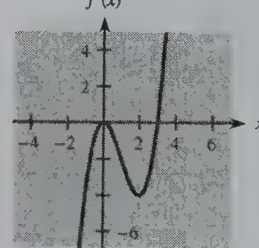
43. $f'(x) = x^2 - x - 2$



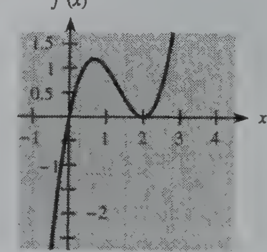
44. $f'(x) = 4x - x^2$



45. $f'(x) = x^3 - 3x^2$

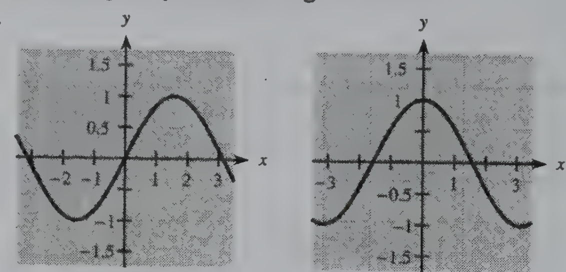


46. $f'(x) = x(x-2)^2$

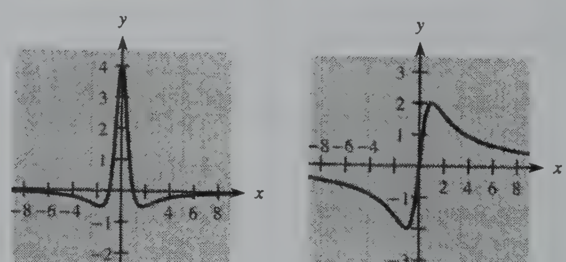


In Problems 47 and 48, two graphs are given. One is the graph of f and the other is the graph of f'. Decide which is which and explain your reasoning.

47.



48.



Applications

49. ***Advertising and sales*** Suppose that the daily sales (in dollars) t days after the end of an advertising campaign are given by

$$S = 1000 + \frac{400}{t+1}, \qquad t \geq 0$$

Does S increase for all $t \geq 0$, decrease for all $t \geq 0$, or change direction at some point?

50. ***Pricing and sales*** Suppose that a chain of auto service stations, Quick-Oil, Inc., has found that its monthly sales volume y (in thousands of dollars) is related to the price p (in dollars) of an oil change by

$$y = \frac{90}{\sqrt{p+5}}, \qquad p > 10$$

Is y increasing or decreasing for all values of $p > 10$?

51. ***Productivity*** A time study showed that, on average, the productivity of a worker after t hours on the job can be modeled by

$$P(t) = 27t + 6t^2 - t^3, \quad 0 \leq t \leq 8$$

where P is the number of units produced per hour.
(a) Find the critical values for this function.
(b) Which critical value makes sense in this model?
(c) For what values of t is P increasing?
(d) Graph the function for $0 \leq t \leq 8$.

52. ***Production*** Analysis of daily output of a factory shows that, on average, the number of units per hour y produced after t hours of production is

$$y = 70t + \frac{1}{2}t^2 - t^3, \qquad 0 \leq t \leq 8$$

(a) Find the critical values for this function.
(b) Which critical values make sense in this particular problem?
(c) For which values of t, for $0 \leq t \leq 8$, is y increasing?
(d) Graph this function.

53. ***Production costs*** Suppose that the average cost of producing a shipment of a certain product is

$$\overline{C} = 5000x + \frac{125{,}000}{x}, \qquad x > 0$$

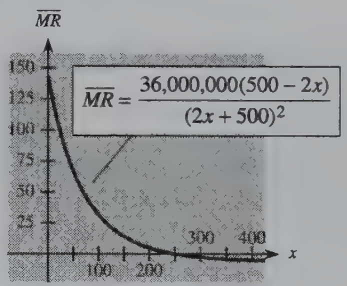
where x is the number of machines used in the production process.
(a) Find the critical values for this function.
(b) Over what interval does the average cost decrease?
(c) Over what interval does the average cost increase?

54. ***Average costs*** Suppose the average costs of a mining operation depend on the number of machines used, and average costs are given by

$$\overline{C}(x) = 2900x + \frac{1{,}278{,}900}{x}, \qquad x > 0$$

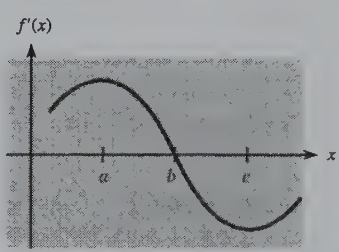
where x is the number of machines used.
(a) Find the critical values for $\overline{C}(x)$ that lie in the domain of the problem.
(b) Over what interval in the domain do average costs decrease?
(c) Over what interval in the domain do average costs increase?
(d) How many machines give minimum average costs?
(e) What are the minimum average costs?

55. ***Marginal revenue*** Suppose the weekly marginal revenue function for selling x units of a product is given by the graph in the figure.



(a) At each of $x = 150$, $x = 250$, and $x = 350$, what is happening to revenue?
(b) Over what interval is revenue increasing?
(c) How many units must be sold to maximize revenue?

56. ***Earnings*** Suppose that the rate of change $f'(x)$ of the average annual earnings of new car salespersons is shown in the figure.



(a) If a, b, and c represent certain years, what is happening to $f(x)$, the average annual earnings of the salespersons, at a, b, and c?
(b) Over what interval (involving a, b, or c) is there an increase in $f(x)$, the average annual earnings of the salespersons?

57. ***Revenue*** The weekly revenue of a certain recently released film is given by

$$R(t) = \frac{50t}{t^2 + 36}, \qquad t \geq 0$$

where R is in millions of dollars and t is in weeks.
(a) Find the critical values.
(b) For how many weeks will weekly revenue increase?

58. ***Medication*** Suppose that the concentration C of a medication in the bloodstream t hours after an injection is given by

$$C(t) = \frac{0.2t}{t^2 + 1}$$

(a) Determine the number of hours before C attains its maximum.
(b) Find the maximum concentration.

59. ***Candidate recognition*** Suppose that the proportion P of voters who recognize a candidate's name t months after the start of the campaign is given by

$$P(t) = \frac{13t}{t^2 + 100} + 0.18$$

(a) How many months after the start of the campaign is recognition at its maximum?
(b) To have greatest recognition on November 1, when should a campaign be launched?

60. ***Medication*** The number of milligrams x of a medication in the bloodstream t hours after a dosage is taken can be modeled by

$$x(t) = \frac{2000t}{t^2 + 16}$$

(a) For what t-values is x increasing?
(b) Find the t-value at which x is maximum.
(c) Find the maximum value for x.

61. ***Residential remodeling expenditures*** According to the data provided by *Kitchen & Bath Design News*, the number of billions of dollars spent on residential remodeling (in 1992 dollars) can be modeled by

$$y = 0.003652x^4 - 0.1408x^3 + 1.6083x^2 - 4.4349x + 56.7387$$

where x is the number of years from 1980.
(a) To the nearest year from 1980 to 1999, find when this model predicts the highest expenditure for remodeling.
(b) Find the year in this interval when the model gives the minimum expenditure.

62. ***Poverty*** The table shows the number of millions of the people in the United States who lived below the poverty level for selected years from 1960 to 1996.
(a) Find a model that approximately fits the data, using x as years past 1900.
(b) Find the year when poverty is a minimum according to the model.

Year	*Persons Living Below the Poverty Level (millions)*
1960	39.9
1965	33.2
1970	25.4
1975	25.9
1980	29.3
1986	32.4
1989	31.5
1990	33.6
1991	35.7
1992	38
1993	39.3
1994	38.1
1995	36.4
1996	36.5

Source: Bureau of the Census, U.S. Dept. of Commerce

63. ***Budget deficit*** The table gives the yearly budget deficit, in billions of dollars, for the years 1990–1997, with White House estimates for 1998 and 1999.
(a) Use the data for 1990 to 1997 to find the equation that models the yearly deficit, with x as the number of years past 1990.
(b) Find the year when the model indicates that the deficit is a maximum, and compare it to the data.

Year	*Yearly Deficit (billions)*
1990	\$221.2
1991	269.4
1992	290.4
1993	255.0
1994	203.1
1995	163.9
1996	107.3
1997	22.6
1998	22.0
1999	0

Source: *USA Today*, Jan. 7, 1998

64. ***Total capital*** The table gives the percent earned total capital for Eli Lilly & Co. for the years 1987–1998.
(a) Find a cubic equation that models the data, using $x = 0$ in 1987.
(b) Use the model to find the year during which the maximum percent total return occurred.

Year	Percent Earned Total Capital	Year	Percent Earned Total Capital
1987	18.8	1993	25.4
1988	21.5	1994	17.4
1989	23.6	1995	17.3
1990	30.4	1996	17.9
1991	25.0	1997	27.0
1992	25.8	1998	28.0

Source: Value Line Publishing Company, Oct. 31, 1997

10.2 Concavity; Points of Inflection

OBJECTIVES

- *To find points of inflection of graphs of functions*
- *To use the second-derivative test to graph functions*

APPLICATION PREVIEW

Suppose that in 1996 a retailer wishes to sell his store and uses the graph in Figure 10.14 to show how profits have increased since he opened the store and the potential for profit in the future. Can we conclude that profits will continue to grow, or should we be concerned about future earnings?

Note that although profits are still increasing in 1995, they are increasing more slowly than in previous years. Indeed, they appear to have been growing at a decreasing rate since about 1990. This means that 1990 was the year that the rate of change of profits was maximum, and since then the rate of change of profits has been diminishing. For this reason, 1990 is called the *point of diminishing returns.* We say that this profit curve is **concave down** over the interval from 1990 to 1995. Judging by Figure 10.14, it would be unwise to expect a large increase in profit after 1995.

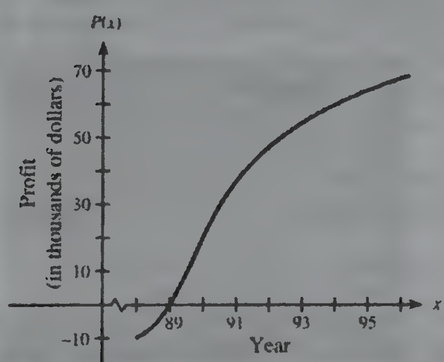


Figure 10.14

Just as we used the first derivative to determine whether a curve was increasing or decreasing on a given interval, we can use the second derivative to determine whether the curve is concave up or concave down on an interval.

A curve is said to be **concave up** on an interval $[a, b]$ if at each point on the interval the curve is above its tangent at the point (Figure 10.15a). If the curve is below all its tangents on a given interval, it is **concave down** on the interval (Figure 10.15b).

Looking at Figure 10.15(a), we see that the *slopes* of the tangent lines increase over the interval where the graph is concave up. Because $f'(x)$ gives the slopes of those tangents, it follows that $f'(x)$ is increasing over the interval where $f(x)$ is concave up. However, if we know that $f'(x)$ is increasing, then its derivative, $f''(x)$, must be positive. That is, the second derivative is positive if the curve is concave up. Conversely, it can be shown that the graph of a function is concave up if the second derivative is positive.

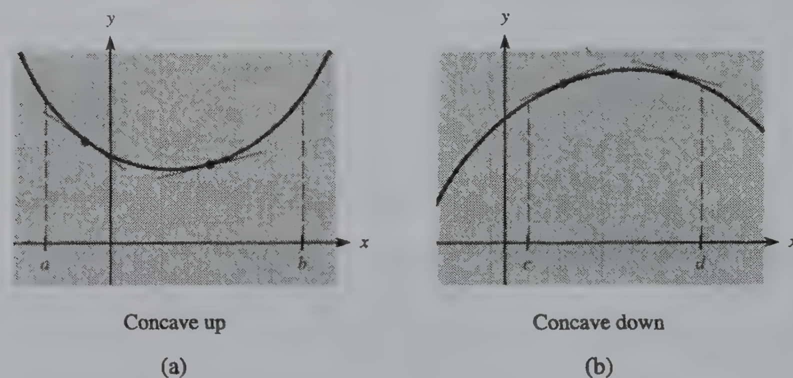


Figure 10.15

Similarly, if the second derivative of a function is negative over an interval, the slopes of the tangents to the graph decrease over that interval. This happens when the tangent lines are above the graph, as in Figure 10.15(b), so the graph must be concave down on this interval.

Thus we see that the second derivative can be used to determine the concavity of a curve.

Concave Up and Concave Down

Assume that the first and second derivatives of function f exist. If $f''(x) > 0$ on an interval I, the graph of f is **concave up** on the interval. If $f''(x) < 0$ on an interval I, then the graph of f is **concave down** on I. We also say that the graph of $y = f(x)$ is concave up at $(a, f(a))$ if $f''(a) > 0$ and that the graph is concave down at $(b, f(b))$ if $f''(b) < 0$.

EXAMPLE 1

Is the graph of $f(x) = x^3 - 4x^2 + 3$ concave up or down at the point

(a) $(1, 0)$? (b) $(2, -5)$?

Solution

(a) We must find $f''(x)$ before we can answer this question.

$$f'(x) = 3x^2 - 8x \qquad f''(x) = 6x - 8$$

Then $f''(1) = 6(1) - 8 = -2$, so the graph is concave down at $(1, 0)$.

(b) Because $f''(2) = 6(2) - 8 = 4$, the graph is concave up at $(2, -5)$. The graph of $f(x) = x^3 - 4x^2 + 3$ is shown in Figure 10.16(a).

Looking at the graph of $y = x^3 - 4x^2 + 3$ (Figure 10.16a), we see that the curve is concave down on the left and concave up on the right. Thus it has changed from concave down to concave up. Figure 10.16(b) shows the graph of $y'' = f''(x) = 6x - 8$, and we can see that $y'' < 0$ for $x < \frac{4}{3}$ and $y'' > 0$ for $x > \frac{4}{3}$.

Thus the second derivative changes sign at $x = \frac{4}{3}$, so the concavity of the graph of $y = f(x)$ changes at $x = \frac{4}{3}$, $y = -\frac{47}{27}$. The point where concavity changes is called a **point of inflection.**

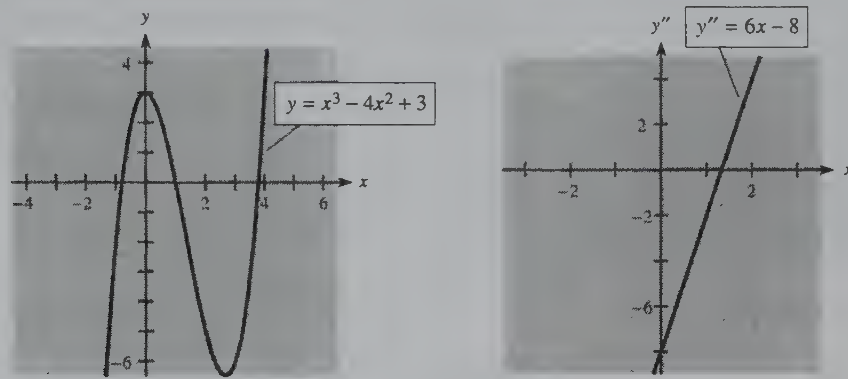


Figure 10.16 (a) (b)

Point of Inflection A point (x_0, y_0) on the graph of a function f is called a **point of inflection** if the curve is concave up on one side of the point and concave down on the other side. The second derivative at this point, $f''(x_0)$, will be 0 or undefined.

In general, we can find points of inflection and information about concavity as follows.

Finding Points of Inflection and Concavity

Procedure	***Example***
To find the point(s) of inflection of a curve and intervals where it is concave upward and where it is concave downward:	Find the points of inflection and concavity of the graph of $y = \dfrac{x^4}{2} - x^3 + 5$.
1. Find the second derivative of the function.	1. $y' = f'(x) = 2x^3 - 3x^2$ $y'' = f''(x) = 6x^2 - 6x$
2. Set the second derivative equal to 0, and solve for x. Potential points of inflection occur at these values of x or at values of x where $f(x)$ is defined and $f''(x)$ is undefined.	2. $0 = 6x^2 - 6x = 6x(x - 1)$ has solutions $x = 0, x = 1$. $f''(x)$ is defined everywhere.
3. Find the potential points of inflection.	3. $(0, 5)$ and $(1, \frac{9}{2})$ are potential points of inflection.
4. If the second derivative has opposite signs on the two sides of one of these values of x, a point of inflection occurs. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$.	4. $\left.\begin{array}{l} f''(-1) = 12 > 0 \\ f''(0) = 0 \\ f''(\frac{1}{2}) = -\frac{3}{2} < 0 \end{array}\right\} \Rightarrow$ $(0, 5)$ is a point of inflection. $\left.\begin{array}{l} f''(\frac{1}{2}) = -\frac{3}{2} < 0 \\ f''(1) = 0 \\ f''(2) = 12 > 0 \end{array}\right\} \Rightarrow$ $(1, \frac{9}{2})$ is a point of inflection. See the graph in Figure 10.17 and the sign diagram on the next page.

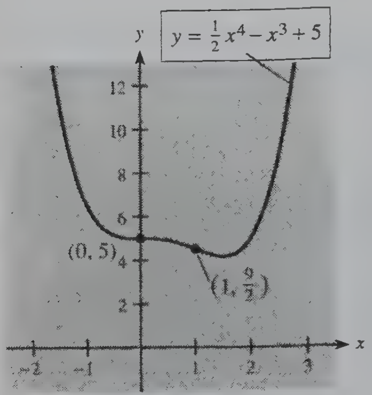
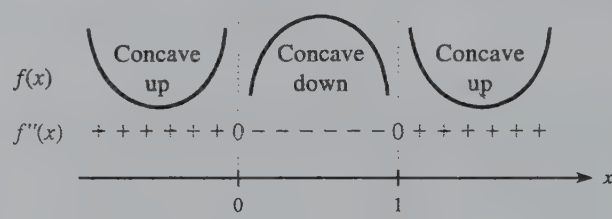


Figure 10.17

The graph of $y = \frac{1}{2}x^4 - x^3 + 5$ is shown in Figure 10.17. Note the points of inflection at (0, 5) and $(1, \frac{9}{2})$. The point of inflection at (0, 5) is a horizontal point of inflection because $f'(x)$ is also 0 at $x = 0$. A **sign diagram for $f''(x)$,** the second derivative of this function, is shown below. The changes in the sign of $f''(x)$ correspond to changes in concavity and occur at points of inflection.



EXAMPLE 2

Suppose that a real estate developer wishes to remove pollution from a small lake so that she can sell lakefront homes on a "crystal clear" lake. The graph in Figure 10.18 shows the relation between dollars spent on cleaning the lake and the purity of the water. The point of inflection on the graph is called the **point of diminishing returns** on her investment because it is where the *rate* of return on her investment changes from increasing to decreasing. Show that the rate of change in the purity of the lake, $f'(x)$, is maximized at this point, $x = c$. Assume that $f(c)$, $f'(c)$, and $f''(c)$ are defined.

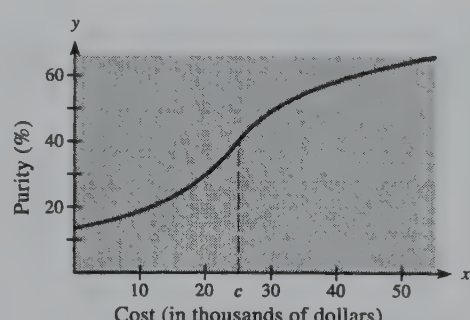


Figure 10.18

Solution

Because $x = c$ is a point of inflection for $f(x)$, we know that the concavity must change at $x = c$. From the figure we see the following.

$x < c$: $f(x)$ is concave up, so $f''(x) > 0$.
$f''(x) > 0$ means that $f'(x)$ is increasing.
$x > c$: $f(x)$ is concave down, so $f''(x) < 0$.
$f''(x) < 0$ means that $f'(x)$ is decreasing.

Thus $f'(x)$ has $f'(c)$ as its relative maximum.

EXAMPLE 3

Suppose that the daily sales (in thousands of dollars) of a product is given by

$$S = \frac{(-x^3 + 9x^2 + 6)}{6}$$

where x is thousands of dollars spent on advertising. Find the point of diminishing returns for money spent on advertising.

Solution

We seek the point where the graph of this function changes from concave up to concave down, if such a point exists.

$$\frac{dS}{dx} = S'(x) = \frac{1}{6}(-3x^2 + 18x)$$

$$S''(x) = \frac{1}{6}(-6x + 18) = -x + 3$$

$$S''(x) = 0 \quad \text{when} \quad 0 = -x + 3 \quad \text{or} \quad x = 3$$

Thus $x = 3$ is a possible point of inflection. We test $S''(x)$ to the left and right of $x = 3$.

$$S''(2) = 1 > 0 \Rightarrow \text{concave up to the left of } x = 3$$

$$S''(4) = -1 < 0 \Rightarrow \text{concave down to the right of } x = 3$$

Thus the point of diminishing returns occurs when $x = 3$ (thousand dollars) and $S = 10$ (thousand dollars). Figure 10.19 shows the graphs of S, S', and S''. At $x = 3$, we can see that the point of diminishing returns on the graph of S corresponds to the maximum point of the graph of S' and the zero (or x-intercept) of the graph of S''.

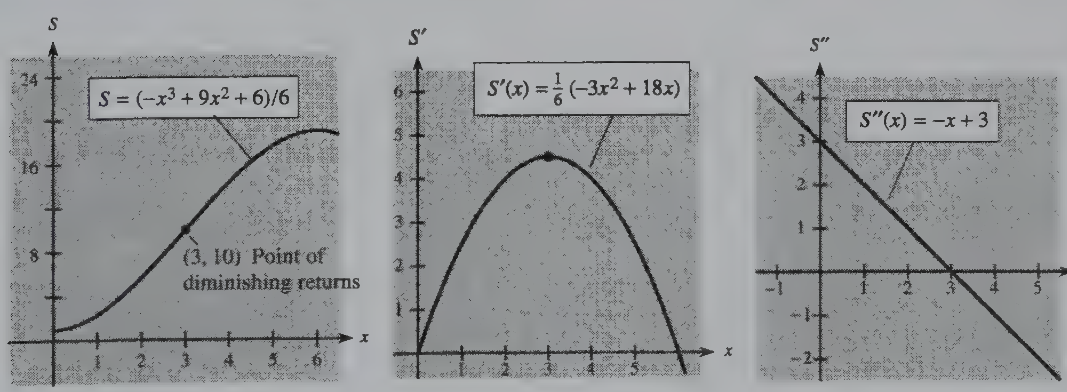


Figure 10.19 (a) (b) (c)

We can use information about points of inflection and concavity to help sketch graphs. For example, if we know that the curve is concave up at a critical point where $f'(x) = 0$, then the point must be a relative minimum because the tangent to the curve is horizontal at the critical point, and only a point at the bottom of a "concave up" curve could have a horizontal tangent (see Figure 10.20a).

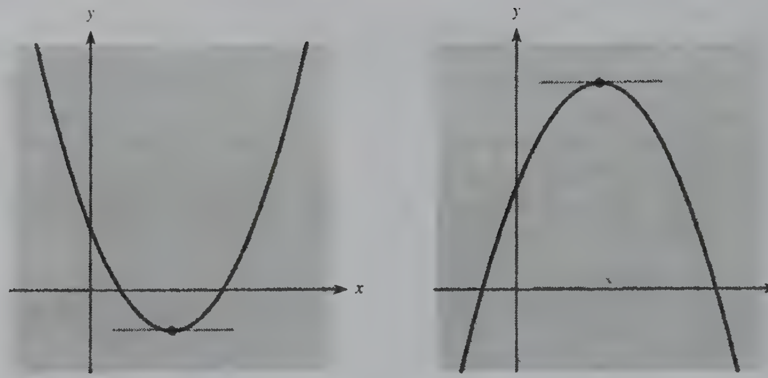


Figure 10.20 (a) (b)

On the other hand, if the curve is concave down at a critical point where $f'(x) = 0$, then the point is a relative maximum (see Figure 10.20b).

Thus we can use the **second-derivative test** to determine whether a critical point where $f'(x) = 0$ is a relative maximum or minimum.

Second-Derivative Test

Procedure	***Example***
To find relative maxima and minima of a function:	Find the relative maxima and minima of $y = f(x) = \frac{1}{3}x^3 - x^2 - 3x + 2$.
1. Find the critical values of the function.	1. $f'(x) = x^2 - 2x - 3$ $0 = (x - 3)(x + 1)$ has solutions $x = -1$ and $x = 3$. No values of x make $x^2 - 2x - 3$ undefined.
2. Substitute the critical values into $f(x)$ to find the critical points.	2. $f(-1) = \frac{11}{3}$ $f(3) = -7$ The critical points are $\left(-1, \frac{11}{3}\right)$ and $(3, -7)$.
3. Evaluate $f''(x)$ at each critical value for which $f'(x) = 0$. (a) If $f''(x_0) < 0$, a relative maximum occurs at x_0. (b) If $f''(x_0) > 0$, a relative minimum occurs at x_0. (c) If $f''(x_0) = 0$, or $f''(x_0)$ is undefined, the second-derivative test fails; use the first-derivative test.	3. $f''(x) = 2x - 2$ $f''(-1) = 2(-1) - 2 = -4 < 0$, so $\left(-1, \frac{11}{3}\right)$ is a relative maximum point. $f''(3) = 2(3) - 2 = 4 > 0$, so $(3, -7)$ is a relative minimum point. (The graph is shown in Figure 10.21 on the next page.)

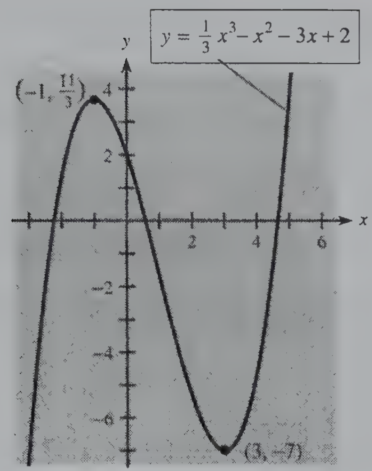
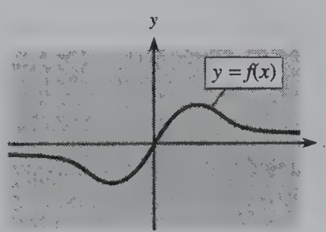


Figure 10.21

CHECKPOINT

1. If $f''(x) > 0$, then $f(x)$ is concave ______.
2. Where do possible points of inflection occur?
3. On the graph below, locate any points of inflection (approximately) and label where the curve satisfies $f''(x) > 0$ and $f''(x) < 0$.



4. Determine whether the following is true or false. If $f''(0) = 0$, then $f(x)$ has a point of inflection at $x = 0$.

EXAMPLE 4

Find the relative maxima and minima and points of inflection of $y = 3x^4 - 4x^3$.

Solution

$$y' = f'(x) = 12x^3 - 12x^2$$

Solving $0 = 12x^3 - 12x^2 = 12x^2(x - 1)$ gives $x = 1$ and $x = 0$. Thus the critical points are $(1, -1)$ and $(0, 0)$.

$$y'' = f''(x) = 36x^2 - 24x$$
$$f''(1) = 12 > 0 \Rightarrow (1, -1) \text{ is a relative minimum point.}$$
$$f''(0) = 0 \Rightarrow \text{the second-derivative test fails.}$$

Because the second-derivative test fails, we must use the first-derivative test at the critical point $(0, 0)$.

$$\left.\begin{aligned} f'(-1) &= -24 < 0 \\ f'(\tfrac{1}{2}) &= -\tfrac{3}{2} < 0 \end{aligned}\right\} \Rightarrow (0, 0) \text{ is a horizontal point of inflection.}$$

We look for points of inflection by setting $f''(x) = 0$ and solving for x. We find that $0 = 36x^2 - 24x$ has solutions $x = 0$ and $x = \frac{2}{3}$.

$$\left.\begin{array}{l} f''(-1) = 60 > 0 \\ f''(\frac{1}{2}) = -3 < 0 \end{array}\right\} \Rightarrow (0, 0) \text{ is a point of inflection.}$$

Thus we see again that (0, 0) is a horizontal point of inflection. This is a special point, where the curve changes concavity *and* has a horizontal tangent (see Figure 10.22). Testing for concavity on either side of $x = \frac{2}{3}$ gives

$$\left.\begin{array}{l} f''(\frac{1}{2}) = -3 < 0 \\ f''(1) = 12 > 0 \end{array}\right\} \Rightarrow (\tfrac{2}{3}, -\tfrac{16}{27}) \text{ is a point of inflection.}$$

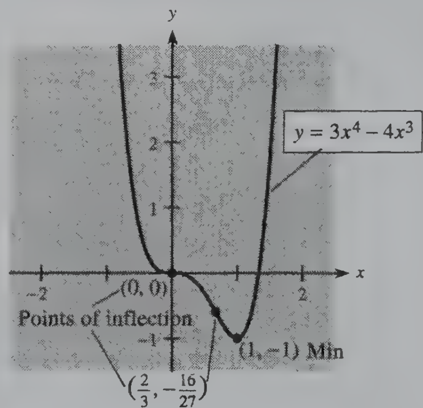


Figure 10.22

Graphing Utilities

We can use graphing utilities to explore the relationships among f, f', and f'', as we did in the previous section for f and f'.

EXAMPLE 5

Figure 10.23 shows the graph of $f(x) = \frac{1}{6}(2x^3 - 3x^2 - 12x + 12)$.

(a) From the graph, identify points where $f''(x) = 0$.
(b) From the graph, observe intervals where $f''(x) > 0$ and where $f''(x) < 0$.
(c) Check the conclusions from (a) and (b) by calculating $f''(x)$ and graphing it.

Solution

(a) From Figure 10.23, we can make an initial estimate of the x-value of the point of inflection. It appears to be near $x = \frac{1}{2}$, so we expect $f''(x) = 0$ at (or very near to) $x = \frac{1}{2}$.
(b) We see that the graph is concave downward (so $f''(x) < 0$) to the left of the point of inflection. That is, $f''(x) < 0$ when $x < \frac{1}{2}$. Similarly, $f''(x) > 0$ when $x > \frac{1}{2}$.
(c) $f(x) = \frac{1}{6}(2x^3 - 3x^2 - 12x + 12)$
$f'(x) = \frac{1}{6}(6x^2 - 6x - 12) = x^2 - x - 2$
$f''(x) = 2x - 1$
Thus $f''(x) = 0$ when $x = \frac{1}{2}$.

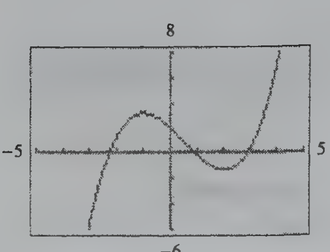


Figure 10.23

Figure 10.24 shows the graph of $f''(x) = 2x - 1$. We see that the graph crosses the x-axis ($f''(x) = 0$) when $x = \frac{1}{2}$, is below the x-axis ($f''(x) < 0$) when $x < \frac{1}{2}$, and is above the x-axis ($f''(x) > 0$) when $x > \frac{1}{2}$. This verifies our conclusions from (a) and (b).

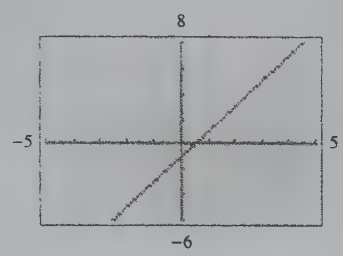


Figure 10.24

EXAMPLE 6

Figure 10.25 shows the graph of $f'(x) = -x^2 - 2x$. Use the graph of $f'(x)$ to do the following.

(a) Find intervals where $f(x)$ is concave downward and where it is concave upward.

(b) Find x-values where $f(x)$ has a point of inflection.

(c) Check the conclusions from (a) and (b) by finding $f''(x)$ and graphing it.

(d) For $f(x) = \frac{1}{3}(9 - x^3 - 3x^2)$, calculate $f'(x)$ to verify that this could be $f(x)$.

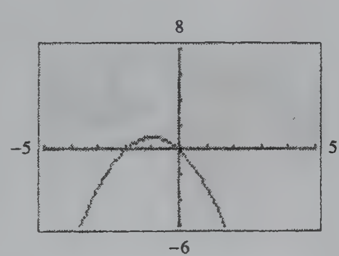


Figure 10.25

Solution

(a) Concavity for $f(x)$ can be found from the sign of $f''(x)$. Because $f''(x)$ is the first derivative of $f'(x)$, wherever the graph of $f'(x)$ is increasing, it follows that $f''(x) > 0$. Thus $f''(x) > 0$ and $f(x)$ is concave upward when $x < -1$. Similarly, $f''(x) < 0$, and $f(x)$ is concave downward when $f'(x)$ is decreasing—that is, when $x > -1$.

(b) From (a) we know that $f''(x)$ changes sign at $x = -1$, so $f(x)$ has a point of inflection at $x = -1$. Note that $f'(x)$ has its maximum at the x-value where $f(x)$ has a point of inflection. In fact, points of inflection for $f(x)$ will correspond to relative extrema for $f'(x)$.

(c) For $f'(x) = -x^2 - 2x$, we have $f''(x) = -2x - 2$. Figure 10.26 shows the graph of $y = f''(x)$ and verifies our conclusions from (a) and (b).

(d) If $f(x) = \frac{1}{3}(9 - x^3 - 3x^2)$, then $f'(x) = \frac{1}{3}(-3x^2 - 6x) = -x^2 - 2x$. Figure 10.27 shows the graph of $f(x) = \frac{1}{3}(9 - x^3 - 3x^2)$. Note that the point of inflection and the concavity correspond to what we discovered in (a) and (b).

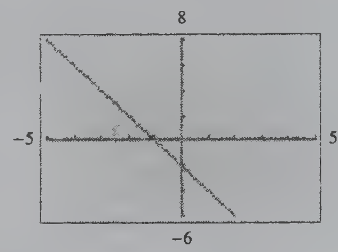


Figure 10.26

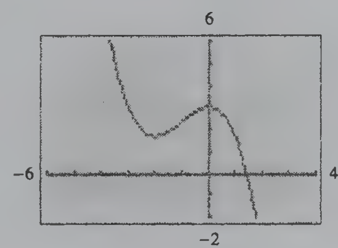


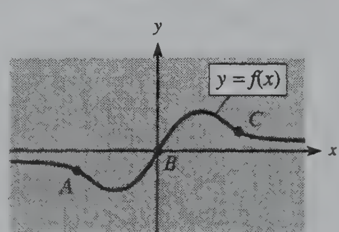
Figure 10.27

The relationship among $f(x)$, $f'(x)$, and $f''(x)$ that we explored in Example 6 can be summarized as follows.

$f(x)$	Concave Upward	Concave Downward	Point of Inflection	
$f'(x)$	increasing	decreasing	maximum	minimum
$f''(x)$	positive (+)	negative (−)	(+) to (−)	(−) to (+)

CHECKPOINT SOLUTIONS

1. up
2. Possible points of inflection occur where $f''(x) = 0$ or $f''(x)$ is undefined.
3. Points of inflection at A, B, and C
   $f''(x) < 0$ to the left of A and between B and C
   $f''(x) > 0$ between A and B and to the right of C



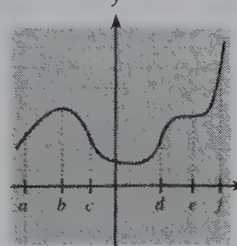
4. (a) False. For example, if $f(x) = x^4$, then $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Note that $f''(x) = 12x^2$ does not change sign from $x < 0$ to $x > 0$.

EXERCISE 10.2

In Problems 1–4, determine whether each function is concave upward or concave downward at the indicated points.

1. $f(x) = x^3 - 3x^2 + 1$ at (a) $x = -2$ (b) $x = 3$
2. $f(x) = x^3 + 6x - 4$ at (a) $x = -5$ (b) $x = 7$
3. $y = 2x^3 + 4x - 8$ at (a) $x = -1$ (b) $x = 4$
4. $y = 4x^3 - 3x^2 + 2$ at (a) $x = 0$ (b) $x = 5$

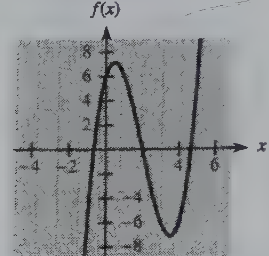
In Problems 5–10, use the indicated x-values on the graph of $y = f(x)$ to find the following.



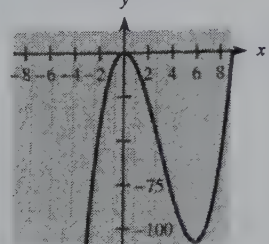
5. Find intervals over which the graph is concave down.
6. Find intervals over which the graph is concave up.
7. Find intervals where $f''(x) > 0$.
8. Find intervals where $f''(x) < 0$.
9. Find the x-coordinates of three points of inflection.
10. Find the x-coordinate of a horizontal point of inflection.

In Problems 11–14, a function and its graph are given. Use the second derivative to determine intervals where the function is concave upward, to determine intervals where it is concave downward, and to locate points of inflection. Check these results against the graph shown.

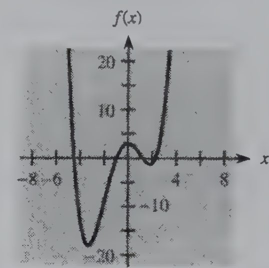
11. $f(x) = x^3 - 6x^2 + 5x + 6$



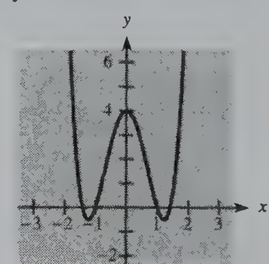
12. $y = x^3 - 9x^2$



13. $f(x) = \frac{1}{4}x^4 + \frac{1}{2}x^3 - 3x^2 + 3$



14. $y = 2x^4 - 6x^2 + 4$



Find the relative maxima, relative minima, and points of inflection, and sketch the graph of the functions in Problems 15–20.

15. $y = x^2 - 4x + 2$
16. $y = x^3 - x^2$
17. $y = \frac{1}{3}x^3 - 2x^2 + 3x + 2$
18. $y = x^3 - 3x^2 + 6$
19. $y = x^4 - 16x^2$
20. $y = x^4 - 8x^3 + 16x^2$

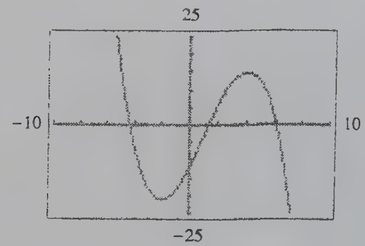
In Problems 21–24, a function and its first and second derivatives are given. Use these to find critical values, relative maxima, relative minima, and points of inflection; sketch the graph of each function.

21. $f(x) = 3x^5 - 20x^3$
$f'(x) = 15x^2(x - 2)(x + 2)$
$f''(x) = 60x(x^2 - 2)$
22. $f(x) = x^5 - 5x^4$
$f'(x) = 5x^3(x - 4)$
$f''(x) = 20x^2(x - 3)$
23. $y = x^{1/3}(x - 4)$
$y' = \dfrac{4(x-1)}{3x^{2/3}}$
$y'' = \dfrac{4(x+2)}{9x^{5/3}}$
24. $y = x^{4/3}(x - 7)$
$y' = \dfrac{7x^{1/3}(x-4)}{3}$
$y'' = \dfrac{28(x-1)}{9x^{2/3}}$

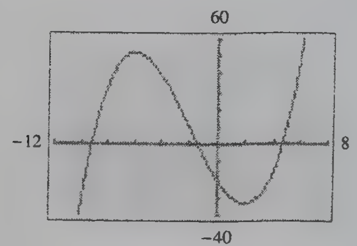
In Problems 25 and 26, a function and its graph are given.

(a) From the graph, estimate where $f''(x) > 0$, where $f''(x) < 0$, and where $f''(x) = 0$.
(b) Use (a) to decide where $f'(x)$ has its relative maxima and relative minima.
(c) Verify your results in (a) and (b) by finding $f'(x)$ and $f''(x)$ and then graphing each with a graphing utility.

25. $f(x) = -\frac{1}{3}x^3 + x^2 + 8x - 12$



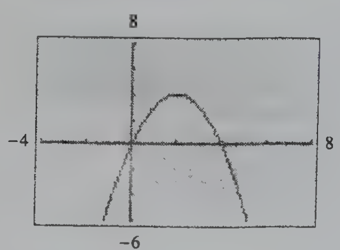
26. $f(x) = \frac{1}{3}x^3 + 2x^2 - 12x - 20$



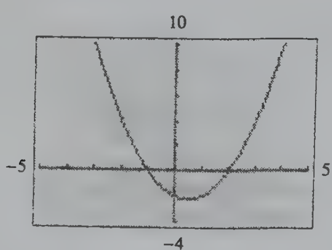
In Problems 27 and 28, $f'(x)$ and its graph are given. Use the graph of $f'(x)$ to determine the following.

(a) Where is the graph of $f(x)$ concave upward and where is it concave downward?
(b) Where does $f(x)$ have any points of inflection?
(c) Find $f''(x)$ and graph it. Then use that graph to check your conclusions from (a) and (b).
(d) Sketch a possible graph for $f(x)$.

27. $f'(x) = 4x - x^2$



28. $f'(x) = x^2 - x - 2$



In Problems 29 and 30, use the graph shown in Figure 10.28 and identify points from A through I that satisfy the given conditions.

29. (a) $f'(x) > 0$ and $f''(x) > 0$
(b) $f'(x) < 0$ and $f''(x) < 0$
(c) $f'(x) = 0$ and $f''(x) > 0$
(d) $f'(x) > 0$ and $f''(x) = 0$
(e) $f'(x) = 0$ and $f''(x) = 0$

30. (a) $f'(x) > 0$ and $f''(x) < 0$
(b) $f'(x) < 0$ and $f''(x) > 0$
(c) $f'(x) = 0$ and $f''(x) < 0$
(d) $f'(x) < 0$ and $f''(x) = 0$

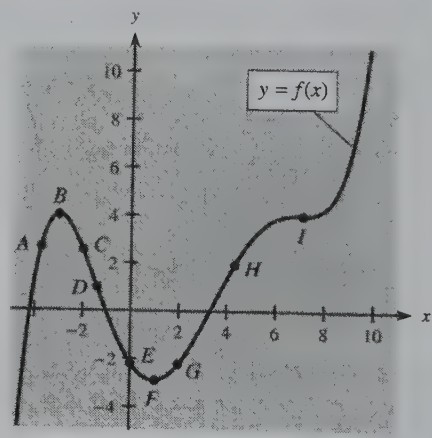
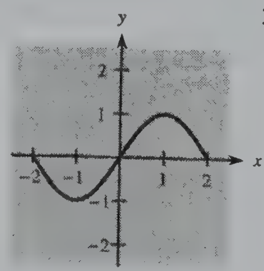


Figure 10.28

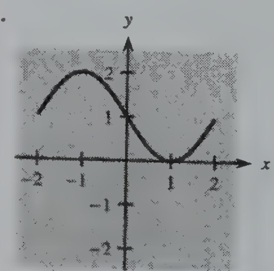
In Problems 31 and 32, a graph is given. Tell where $f(x)$ is concave upward, where it is concave downward, and where it has points of inflection on the interval $-2 < x < 2$, if the given graph is the graph of

(a) $f(x)$ (b) $f'(x)$ (c) $f''(x)$

31.

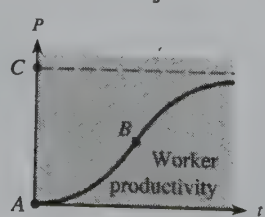


32.



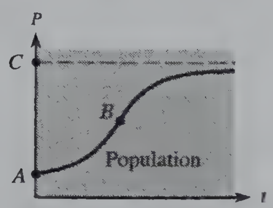
Applications

33. ***Productivity—diminishing returns*** The figure below is a typical graph of worker productivity as a function of time on the job.



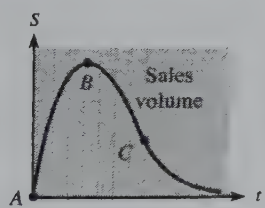
(a) If P represents the productivity and t represents the time, write a symbol that represents the rate of change of productivity with respect to time.
(b) Which of A, B, and C is the critical point for the rate of change found in (a)? This point actually corresponds to the point at which the rate of production is maximized, or the point for maximum worker efficiency. In economics, this is called the point of diminishing returns.
(c) Which of A, B, and C corresponds to the upper limit of production?

34. ***Population growth*** The figure below shows the growth of a population as a function of time.



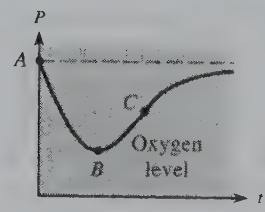
(a) If P represents the population and t represents the time, write a symbol that represents the rate of change (growth rate) of the population with respect to time.
(b) Which of A, B, and C corresponds to the point at which the growth *rate* attains its maximum?
(c) Which of A, B, and C corresponds to the upper limit of population?

35. ***Advertising and sales*** The figure below shows the daily sales volume S as a function of time t since an ad campaign began.



(a) Which of A, B, and C is the point of inflection for the graph?
(b) On which side of C is $d^2S/dt^2 > 0$?
(c) Does the *rate of change* of sales volume attain its minimum at C?

36. ***Oxygen purity*** The figure below shows the oxygen level P (for purity) in a lake t months after an oil spill.



(a) Which of A, B, and C is the point of inflection for the graph?
(b) On which side of C is $d^2P/dt^2 < 0$?
(c) Does the *rate of change* of purity attain its maximum at C?

37. ***Production*** Suppose that the total number of units produced by a worker in t hours of an 8-hour shift can be modeled by the production function $P(t)$:

$$P(t) = 27t + 12t^2 - t^3$$

(a) Find the number of hours before production is maximized.
(b) Find the number of hours before the rate of production is maximized. That is, find the point of diminishing returns.

38. ***Poiseuille's law—velocity of blood*** According to Poiseuille's law, the speed S of blood through an artery of radius r at a distance x from the artery wall is given by

$$S = k[r^2 - (r - x)^2]$$

where k is a constant. Find the distance x that maximizes the speed.

39. ***Advertising and sales—diminishing returns*** Suppose that a company's daily sales volume attributed to an advertising campaign is given by

$$S(t) = \frac{3}{t+3} - \frac{18}{(t+3)^2} + 1$$

(a) Find how long it will be before sales volume is maximized.
(b) Find how long it will be before the rate of change of sales volume is minimized. That is, find the point of diminishing returns.

40. ***Oxygen purity—diminishing returns*** Suppose that the oxygen level P (for purity) in a body of water t months after an oil spill is given by

$$P(t) = 500\left[1 - \frac{4}{t+4} + \frac{16}{(t+4)^2}\right]$$

(a) Find how long it will be before the oxygen level reaches its minimum.

(b) Find how long it will be before the rate of change of P is maximized. That is, find the point of diminishing returns.

41. ***Inflation rate*** The table below gives the annual percent change in the consumer price index (CPI) for the years 1980–1996. The equation that models the data is

$$y = 0.002138x^4 - 0.08399x^3 + 1.12630x^2 - 5.9860x + 14.1484$$

where x is number of years past 1980. Use the model to find the year between 1985 and 1996 in which it indicates that the increase is a maximum.

Year	Annual Percent Change in CPI	Year	Annual Percent Change in CPI
1980	13.5	1989	4.8
1981	10.3	1990	5.4
1982	6.2	1991	4.2
1983	3.2	1992	3.0
1984	4.3	1993	3.0
1985	3.6	1994	2.6
1986	1.9	1995	2.8
1987	3.6	1996	2.9
1988	4.1		

Source: Bureau of Labor Statistics, U.S. Dept. of Labor

42. ***Personal savings*** The following table gives the personal savings (in billions of dollars) in the United States for selected years.

Year	Personal Savings (billions)	Year	Personal Savings (billions)
1980	\$153.8	1988	155.7
1985	189.3	1989	152.1
1986	187.5	1990	175.6
1987	142.0	1991	199.6

Source: *Survey of Current Business*, March 1993

Assume that personal savings is modeled by

$$PS(t) = 0.55t^3 - 25.6t^2 + 384t - 1678$$

where t is the number of years past 1970.

(a) Find a function that models the instantaneous rate of change of personal savings.

(b) Use the model from (a) to find when the rate of personal savings is decreasing, when it is increasing, and when it is a minimum.

(c) Find the point of inflection of the function $PS(t)$, and state to what in (b) it corresponds.

43. ***Murder rates*** The table below gives the murder rates per 1000 people from 1979 to 1994.

(a) Using x as the number of years past 1970, write the cubic function that models these data.

(b) Use this model to find the years during this period when the murders were at a minimum and when they were at a maximum.

(c) Does the model yield the same years for the minimum and maximum as the data?

Year	Number of Murders per 1000 People
1979	21.5
1980	23.0
1981	22.5
1982	21.0
1983	19.3
1984	18.7
1985	19.0
1986	20.6
1987	20.1
1988	20.7
1989	21.5
1990	23.4
1991	24.7
1992	23.8
1993	24.5
1994	23.3

Source: U.S. FBI

44. ***Union membership*** The data in the table can be modeled with the function

$$u(x) = 0.0003869x^3 - 0.08917x^2 + 6.2503x - 105.9$$

where $x = 0$ in 1900 and $u(x)$ is the percentage of U.S. workers who belonged to unions. During what year does the model indicate union membership was maximized?

Year	Percent of Workers in Unions	Year	Percent of Workers in Unions
1930	11.6	1989	16.4
1940	26.9	1990	16.1
1950	31.5	1991	16.1
1960	31.4	1992	15.8
1970	27.3	1993	15.8
1975	25.5	1994	15.5
1980	21.9	1995	14.9
1985	18.0	1996	14.5

Source: Bureau of Labor Statistics, U.S. Dept. of Labor

45. ***Gross national product*** The following table gives the U.S. gross national product (GNP) for the years 1913–1922.

(a) Using $x = 0$ in 1912, find the function that models the data.

(b) During what year in this time interval does the model predict a minimum GNP?

(c) During what year in this time interval does the model predict a maximum GNP?

Year	*Gross National Product (GNP)*
1913	39.6
1914	38.6
1915	40.0
1916	48.3
1917	60.4
1918	76.4
1919	84.0
1920	91.5
1921	69.6
1922	74.1

Source: *National Debt in Perspective,* Oscar Falconi, Wholesale Nutrition (Internet)

10.3 Optimization in Business and Economics

OBJECTIVES

- *To find absolute maxima and minima*
- *To maximize revenue, given the total revenue function*
- *To minimize the average cost, given the total cost function*
- *To find the maximum profit from total cost and total revenue functions, or from a profit function*

APPLICATION PREVIEW

Most companies are interested in obtaining the greatest possible profit (instead of just making a profit that is relatively large). Similarly, manufacturers of products are concerned about producing their products for the lowest possible average cost per unit. Therefore, rather than just finding the relative maxima or relative minima of a function, we will consider where the **absolute maximum** or **absolute minimum** of a function occurs in a given interval. This requires evaluating the function at the endpoints of the given interval as well as finding the relative extrema.

As their name implies, **absolute extrema** are the functional values that are the largest or smallest values over the entire domain of the function (or over the interval of interest).

Absolute Extrema

The value $f(a)$ is the **absolute maximum** for f if $f(a) \geq f(x)$ for all x in the domain of f (or over the interval of interest).

The value $f(b)$ is the **absolute minimum** for f if $f(b) \leq f(x)$ for all x in the domain of f (or over the interval of interest).

In this section we will discuss how to find the absolute extrema of a function and then use these techniques to solve applications involving revenue, cost, and profit.

Let us begin by considering the graph of $y = (x - 1)^2$, shown in Figure 10.29(a). This graph has a relative minimum at (1, 0). Note that the relative minimum is the lowest point on the graph. In this case, the point (1, 0) is an **absolute minimum point,** and 0 is the absolute minimum for the function. Similarly, when there is a point that is the highest point on the graph over the domain of the function, we call the point an **absolute maximum point** for the function.

In Figure 10.29(a), we see that there is no relative maximum. However, if the domain of the function is restricted to the interval $\left[\frac{1}{2}, 2\right]$, then we get the graph shown in Figure 10.29(b). In this case, there is an absolute maximum of 1 at the point (2, 1), and the absolute minimum of 0 is still at (1, 0).

If the domain of $y = (x - 1)^2$ is restricted to the interval [2, 3], the resulting graph is that shown in Figure 10.29(c). In this case, the absolute minimum is 1 and occurs at the point (2, 1), and its absolute maximum is 4 and occurs at (3, 4).

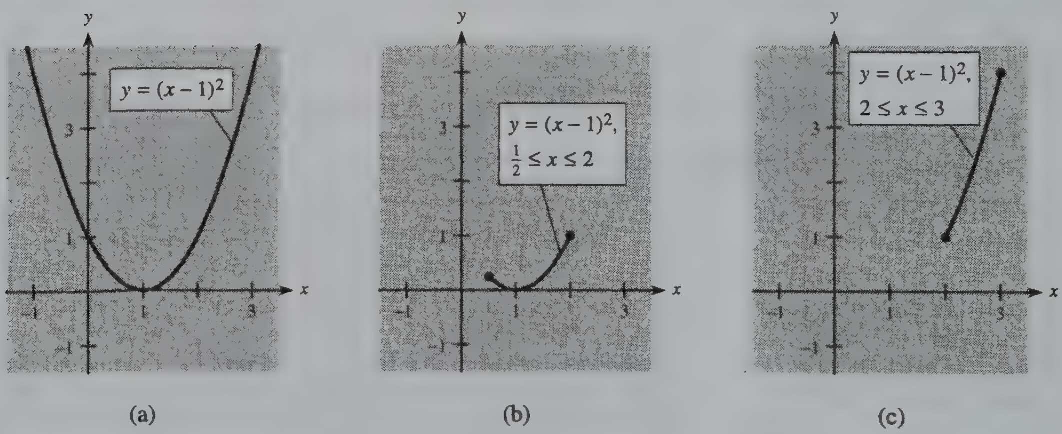


Figure 10.29

As the preceding discussion indicates, if the domain of a function is limited, an absolute maximum or minimum may occur at the endpoints of the domain. In testing functions with limited domains for absolute maxima and minima, we must compare the endpoints of the domain with the relative maxima and minima found by taking derivatives. In applications to the management, life, and social sciences, a limited domain occurs very often, because many quantities are required to be positive, or at least nonnegative.

Maximizing Revenue

Because the marginal revenue is the first derivative of the total revenue, it should be obvious that the total revenue function will have a critical point at the point where the marginal revenue equals 0. With the total revenue function $R(x) = 16x - 0.02x^2$, the point where $R'(x) = 0$ is clearly a maximum because $R(x)$ is a parabola that opens downward. But the revenue function may not always be a parabola, and the critical point may not always be a maximum, so it is wise to verify that the maximum value occurs at the critical point.

EXAMPLE 1

If total revenue for a firm is given by

$$R(x) = 8000x - 40x^2 - x^3$$

where x is the number of units sold, find the number of units that must be sold to maximize revenue. Find the maximum revenue.

Solution

$R'(x) = 8000 - 80x - 3x^2$, so we must solve $8000 - 80x - 3x^2 = 0$ for x.

$$(40 - x)(200 + 3x) = 0$$

$$40 - x = 0 \quad 200 + 3x = 0$$

so

$$x = 40 \quad \text{or} \quad x = -\frac{200}{3}$$

Now we reject the negative value for x, but we must verify that $x = 40$ will yield maximum revenue.

$$\left.\begin{aligned} R'(0) &= 8000 > 0 \\ R'(100) &= 8000 - 8000 - 30{,}000 < 0 \end{aligned}\right\} \Rightarrow \text{relative maximum}$$

This test shows that a relative maximum occurs at $x = 40$, giving revenue $R(40) = \$192{,}000$. Because $R'(x) < 0$ for all $x > 40$, the revenue function is decreasing to the right of $x = 40$. The revenue at $x = 0$ is $R(0) = 0$, so $R = \$192{,}000$ at $x = 40$ is the (absolute) maximum revenue.

EXAMPLE 2

A travel agency will plan tours for groups of 25 or larger. If the group contains exactly 25 people, the cost is \$300 per person. However, each person's cost is reduced by \$10 for each additional person above the 25. What size group will produce the largest revenue for the agency?

Solution

The total revenue is

$$R = (\text{number of people})(\text{cost per person})$$

If 25 people go, the total revenue will be

$$R = 25 \cdot \$300 = \$7500$$

But if x additional people go, the number of people will be $25 + x$, and the cost per person will be $(300 - 10x)$ dollars. Then the total revenue will be a function of x,

$$R = R(x) = (25 + x)(300 - 10x)$$

or

$$R(x) = 7500 + 50x - 10x^2$$

This function will have its maximum where $\overline{MR} = R'(x) = 0$; $R'(x) = 50 - 20x$, and the solution to $0 = 50 - 20x$ is $x = 2.5$. Thus adding 2.5 people to the group should maximize the total revenue. But we cannot add half a person, so we will test the total revenue function for 27 people and 28 people. This will determine the most profitable number because $R(x)$ is concave downward for all x.

For $x = 2$ (giving 27 people) we get $R(2) = 7500 + 50(2) - 10(2)^2 = 7560$. For $x = 3$ (giving 28 people) we get $R(3) = 7500 + 50(3) - 10(3)^2 = 7560$. Note that both 27 and 28 people give the same total revenue and that this revenue is greater than the revenue for 25 people. Thus the revenue is maximized at either 27 or 28 people in the group.

Minimizing Average Cost

Because the total cost function is always increasing, we cannot find the number of units that will make the total cost a minimum (except for producing 0 units, which is an absolute minimum). However, we usually can find the number of units that will make the average cost per unit a minimum.

Average Cost If the total cost is represented by $C = C(x)$, then the **average cost per unit** is

$$\overline{C} = \frac{C(x)}{x}$$

For example, if $C = 3x^2 + 4x + 2$ is the total cost function for a commodity, the **average cost function** is

$$\overline{C} = \frac{3x^2 + 4x + 2}{x} = 3x + 4 + \frac{2}{x}$$

Note that the average cost per unit is undefined if no units are produced.

We can use derivatives to find the minimum of the average cost function, as the following example shows.

EXAMPLE 3

If the total cost function for a commodity is given by $C = \frac{1}{4}x^2 + 4x + 100$, where x represents the number of units produced, producing how many units will result in a minimum *average cost* per unit? Find the minimum average cost.

Solution

The average cost function is given by

$$\overline{C} = \frac{\frac{1}{4}x^2 + 4x + 100}{x} = \frac{1}{4}x + 4 + \frac{100}{x}$$

Then

$$\overline{C}' = \overline{C}'(x) = \frac{1}{4} - \frac{100}{x^2}$$

Setting $\overline{C}' = 0$ gives

$$0 = \frac{1}{4} - \frac{100}{x^2}$$

$$0 = x^2 - 400, \quad \text{or} \quad x = \pm 20$$

Because the quantity produced must be positive, 20 units should minimize the average cost per unit. We show it is an absolute minimum by using the second derivative.

$$\overline{C}''(x) = \frac{200}{x^3} \quad \text{so} \quad \overline{C}''(x) > 0 \quad \text{when } x > 0$$

Thus the minimum average cost per unit occurs if 20 units are produced. The graph of the average cost per unit is shown in Figure 10.30. The minimum average cost per unit is $\overline{C}(20) = \$14$.

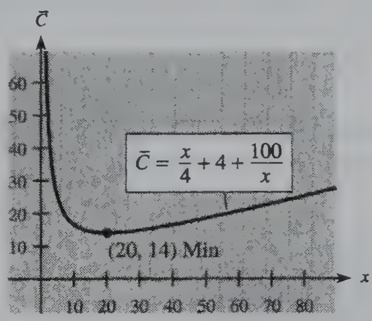


Figure 10.30

Maximizing Profit

In the previous chapter, we defined the marginal profit function as the derivative of the profit function. That is,

$$\overline{MP} = P'(x)$$

In this chapter we have seen how to use the derivative to find maxima and minima for various functions. Now we can apply those same techniques in the context of **profit maximization.** We can use marginal profit to maximize profit functions.

If there is a physical limitation on the number of units that can be produced in a given period of time, then the endpoints of the interval caused by these limitations should also be checked.

EXAMPLE 4

Suppose that the production capacity for a certain commodity cannot exceed 30. If the total profit function for this commodity is

$$P(x) = 4x^3 - 210x^2 + 3600x - 200$$

where x is the number of units sold, find the number of items that will maximize profit.

Solution

The restrictions on capacity mean that $P(x)$ is restricted by $0 \le x \le 30$. The marginal profit function is

$$P'(x) = 12x^2 - 420x + 3600$$

Setting $P'(x)$ equal to 0, we get

$$0 = 12(x - 15)(x - 20)$$

so $P'(x) = 0$ at $x = 15$ *and* $x = 20$. Testing to the right and left of these values (as in the first-derivative test), we get

$$\left.\begin{array}{l} P'(0) = 3600 > 0 \\ P'(18) = -72 < 0 \\ P'(25) = 600 > 0 \end{array}\right\} \begin{array}{l} \Rightarrow \text{relative maximum at } x = 15 \\ \Rightarrow \text{relative minimum at } x = 20 \end{array}$$

Thus, at (15, 20050) the total profit function has a *relative* maximum but we must check the endpoints (0 and 30) before deciding whether it is the absolute maximum.

$$P(0) = -200 \text{ and } \quad P(30) = 26{,}800$$

Thus the absolute maximum profit is \$26,800, and it occurs at the endpoint, $x = 30$. Figure 10.31 shows the graph of the profit function.

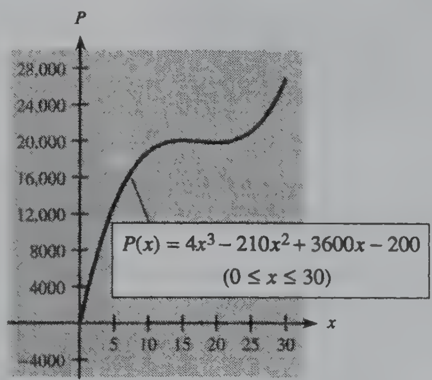


Figure 10.31

In a **monopolistic market,** the seller who has a monopoly can control the price by regulating the supply of the product. The seller controls the supply, so he or she can force the price higher by limiting supply.

If the demand function for the product is $p = f(x)$, total revenue for the sale of x units is $R(x) = px = f(x) \cdot x$. Note that the price p is fixed by the market in a competitive market but varies with output for the monopolist.

If $\overline{C} = \overline{C}(x)$ represents the average cost per unit sold, then the total cost for the x units sold is $C = \overline{C} \cdot x = \overline{C}x$. Because we have both total cost and total revenue as a function of the quantity, x, we can maximize the profit function, $P(x) = px - \overline{C}x$, where p represents the demand function $p = f(x)$ and $\overline{C}$ represents the average cost function $\overline{C} = \overline{C}(x)$.

EXAMPLE 5

The daily demand function for a product is

$$p = 168 - 0.2x$$

A monopolist finds that the average cost is

$$\overline{C} = 120 + x$$

(a) How many units must be sold to maximize profit?
(b) What is the selling price at this "optimal" level of production?
(c) What is the maximum possible profit?

Solution

(a) The total revenue function for the product is

$$R(x) = px = (168 - 0.2x)x = 168x - 0.2x^2$$

and the total cost function is

$$C(x) = \overline{C} \cdot x = (120 + x)x = 120x + x^2$$

Thus the profit function is

$$P(x) = R(x) - C(x) = 168x - 0.2x^2 - (120x + x^2)$$

or

$$P(x) = 48x - 1.2x^2$$

Then $P'(x) = 48 - 2.4x$, so $P'(x) = 0$ when $x = 20$. We see that $P''(20) = -2.4$, so by the second-derivative test, $P(x)$ has a maximum at $x = 20$. That is, selling 20 units will maximize profit.

(b) The selling price is determined by $p = 168 - 0.2x$, so the price that will result from supplying 20 units per day is $p = 168 - 0.2(20) = 164$. That is, the "optimal" selling price is \$164 per unit.

(c) The profit at $x = 20$ is $P(20) = 48(20) - 1.2(20)^2 = 960 - 480 = 480$. Thus the maximum possible profit is \$480 per day.

In a **competitive market,** each firm is so small that its actions in the market cannot affect the price of the product. The price of the product is determined in the market by the intersection of the market demand curve (from all consumers) and market supply curve (from all firms that supply this product). The firm can sell as little or as much as it desires at the market equilibrium price.

Therefore, a firm in a competitive market has a total revenue function given by $R(x) = px$, where p is the market equilibrium price for the product and x is the quantity sold.

EXAMPLE 6

A firm in a competitive market must sell its product for \$200 per unit. The average cost per unit (per month) is $\overline{C} = 80 + x$, where x is the number of units sold per month. How many units should be sold to maximize profit?

Solution

If the average cost per unit is $\overline{C} = 80 + x$, then the total cost of x units is given by $C(x) = (80 + x)x = 80x + x^2$. The revenue per unit is \$200, so the total revenue is given by $R(x) = 200x$. Thus the profit function is

$$P(x) = R(x) - C(x) = 200x - (80x + x^2), \quad \text{or} \quad P(x) = 120x - x^2$$

Then $P'(x) = 120 - 2x$. Setting $P'(x) = 0$ and solving for x gives $x = 60$. Because $P''(60) = -2$, the profit is maximized when the firm sells 60 units per month.

CHECKPOINT

1. True or false: If $R(x)$ is the revenue function, we find all possible points where $R(x)$ could be maximized by solving $\overline{MR} = 0$ for x.
2. If $C(x) = \dfrac{x^2}{20} + 10x + 2500$, form $\overline{C}(x)$, the average cost function.
3. (a) If $p = 5000 - x$ gives the demand function in a monopoly market, find $R(x)$, if it is possible with this information.
   (b) If $p = 5000 - x$ gives the demand function in a competitive market, find $R(x)$, if it is possible with this information.

Graphing Utilities

As we have seen, graphing utilities can be used to locate maximum values. In addition, if it is difficult to determine critical values algebraically, we may be able to find them graphically.

EXAMPLE 7

Suppose total revenue and total costs for a company are given by

$$R(x) = 3000 - \frac{3000}{x+1}$$
$$C(x) = 500 + 12x + x^2$$

where x is thousands of units and revenue and costs are in thousands of dollars. Graph $P(x)$ and $P'(x)$ to determine the number of units that yields maximum profit and the amount of the maximum profit.

Solution

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 3000 - \frac{3000}{x+1} - (500 + 12x + x^2) \\ &= 2500 - \frac{3000}{x+1} - 12x - x^2 \\ P'(x) &= \frac{3000}{(x+1)^2} - 12 - 2x \end{aligned}$$

Finding the critical values by solving $P'(x) = 0$ is very difficult in this case (try it). That is why we are using a graphing approach. Figure 10.32(a) shows the graph of $P(x)$ and Figure 10.32(b) shows the graph of $P'(x)$. These figures indicate that the maximum profit occurs near $x = 10$.

By adjusting the range for $P'(x)$, we obtain the graph in Figure 10.33. This shows that $P'(x) = 0$ when $x = 9$ (or when 9000 units are sold). The maximum profit is

$$\begin{aligned} P(9) &= 2500 - 300 - 108 - 81 \\ &= 2011 \text{ (thousands of dollars)} \end{aligned}$$

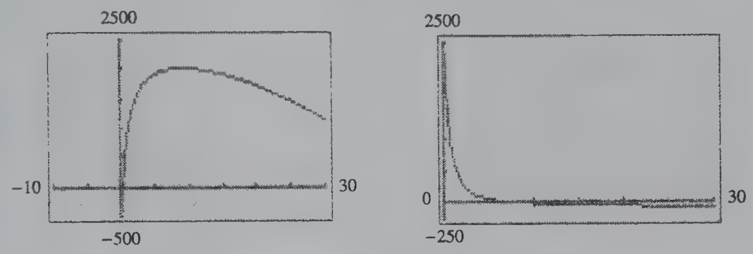


Figure 10.32

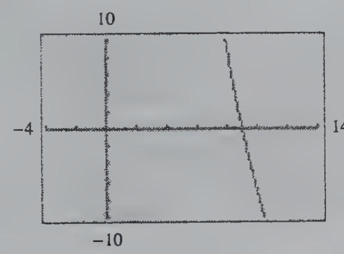


Figure 10.33

CHECKPOINT SOLUTIONS

1. False. $\overline{MR} = R'(x)$, but there may also be critical points where $R'(x)$ is undefined, or $R(x)$ may be maximized at endpoints of a restricted domain.
2. $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{x^2/20 + 10x + 2500}{x} = \dfrac{x}{20} + 10 + \dfrac{2500}{x}$
3. (a) $R(x) = p \cdot x = (5000 - x)x = 5000x - x^2$
   (b) In a competitive market, $R(x) = p \cdot x$, where p is the constant equilibrium price. Thus we need to know the supply function and find the equilibrium price before we can form $R(x)$.

EXERCISE 10.3

In Problems 1–4, find the absolute maxima and minima for $f(x)$ on the interval $[a, b]$.

1. $f(x) = x^3 - 2x^2 - 4x + 2, [-1, 3]$
2. $f(x) = x^3 - 3x + 3, [-3, 1.5]$
3. $f(x) = x^3 + x^2 - x + 1, [-2, 0]$
4. $f(x) = x^3 - x^2 - x, [-0.5, 2]$

Maximizing Revenue

5. (a) If the total revenue function for a radio is $R = 36x - 0.01x^2$, then sale of how many units will maximize the total revenue? Find the maximum revenue.
   (b) Find the maximum revenue if production is limited to at most 1500 radios.

6. (a) If the total revenue function for a blender is $R(x) = 25x - 0.05x^2$, sale of how many units will provide the maximum total revenue? Find the maximum revenue.
   (b) Find the maximum revenue if production is limited to at most 200 blenders.
7. If the total revenue function for a computer is $R(x) = 2000x - 20x^2 - x^3$, find the level of sales that maximizes revenue and find the maximum revenue.
8. A firm has total revenues given by

$$R(x) = 2800x - 8x^2 - x^3$$

for a product. Find the maximum revenue from sales of that product.
9. An agency charges \$10 per person for a trip to a concert if 30 people travel in a group. But for each person above the 30, the charge will be reduced by \$.20. How many people will maximize the total revenue for the agency if the trip is limited to at most 50 people?
10. A company handles an apartment building with 50 units. Experience has shown that if the rent for each of the units is \$360 per month, all the units will be filled, but 1 unit will become vacant for each \$10 increase in the monthly rate. What rent should be charged to maximize the total revenue from the building if the upper limit on the rent is \$450 per month?
11. A cable TV company has 1000 customers paying \$20 each month. If each \$1 reduction in price attracts 100 new customers, find the price that yields maximum revenue. Find the maximum revenue.
12. If club members charge \$5 admission to a classic car show, 1000 people will attend, and for each \$1 increase in price, 100 fewer people will attend. What price will give the maximum revenue for the show? Find the maximum revenue.
13. The function $\overline{R}(x) = R(x)/x$ defines the average revenue for selling x units. For

$$R(x) = 2000x + 20x^2 - x^3$$

   (a) find the maximum average revenue.
   (b) show that $\overline{R}(x)$ attains its maximum at an x-value where $\overline{R}(x) = \overline{MR}$.
14. For the revenue function given by

$$R(x) = 2800x + 8x^2 - x^3$$

   (a) find the maximum average revenue.
   (b) show that $\overline{R}(x)$ attains its maximum at an x-value where $\overline{R}(x) = \overline{MR}$.

Minimizing Average Cost

15. If the total cost function for a lamp is $C(x) = 25 + 13x + x^2$, producing how many units will result in a minimum average cost per unit? Find the minimum average cost.
16. If the total cost function for a product is $C(x) = 300 + 10x + 0.03x^2$, producing how many units will result in a minimum average cost per unit? Find the minimum average cost.
17. If the total cost function for a product is $C(x) = 100 + x^2$, producing how many units will result in a minimum average cost per unit? Find the minimum average cost.
18. If the total cost function for a product is $C(x) = 250 + 6x + 0.1x^2$, producing how many units will minimize the average cost? Find the minimum average cost.
19. If the total cost function for a product is $C(x) = (x + 4)^3$, where x represents the number of hundreds of units produced, producing how many units will minimize average cost? Find the minimum average cost.
20. If the total cost function for a product is $C(x) = (x + 5)^3$, where x represents the number of hundreds of units produced, producing how many units will minimize average cost? Find the minimum average cost.
21. For the cost function $C(x) = 25 + 13x + x^2$, show that average costs are minimized at the x-value where

$$\overline{C}(x) = \overline{MC}$$

22. For the cost function $C(x) = 300 + 10x + 0.03x^2$, show that average costs are minimized at the x-value where

$$\overline{C}(x) = \overline{MC}$$

Maximizing Profit

23. If the profit function for a product is $P(x) = 5600x + 85x^2 - x^3 - 200{,}000$, selling how many items will produce a maximum profit? Find the maximum profit.
24. If the profit function for a commodity is $P = 6400x - 18x^2 - \frac{1}{3}x^3 - 40{,}000$, selling how many units will result in a maximum profit? Find the maximum profit.

25. A manufacturer estimates that x units of its product can be produced at a total cost of $C(x) = 45{,}000 + 100x + x^3$. If the manufacturer's total revenue from the sale of x units is $R(x) = 4600x$, determine the level of production x that will maximize the profit. Find the maximum profit.
26. A product can be produced at a total cost $C(x) = 800 + 100x^2 + x^3$, where x is the number produced. If the total revenue is given by $R(x) = 60{,}000x - 50x^2$, determine the level of production that will maximize the profit. Find the maximum profit.
27. A firm can produce only 1000 units per month. The monthly total cost is given by $C(x) = 300 + 200x$, where x is the number produced. If the total revenue is given by $R(x) = 250x - \frac{1}{100}x^2$, how many items should the firm produce for maximum profit? Find the maximum profit.
28. A firm can produce 100 units per week. If its total cost function is $C = 500 + 1500x$ and its total revenue function is $R = 1600x - x^2$, how many units should it produce to maximize its profit? Find the maximum profit.
29. A firm has monthly average costs given by

$$\overline{C} = \frac{45{,}000}{x} + 100 + x$$

where x is the number of units produced per month. The firm can sell its product in a competitive market for \$1600 per unit. If production is limited to 600 units per month, find the number of units that gives maximum profit, and find the maximum profit.
30. A small business has weekly average costs of

$$\overline{C} = \frac{100}{x} + 30 + \frac{x}{10}$$

where x is the number of units produced each week. The competitive market price for this business's product is \$46 per unit. If production is limited to 150 units per week, find the level of production that yields maximum profit, and find the maximum profit.
31. The weekly demand function for a product sold by only one firm is $p = 600 - \frac{1}{2}x$, and the average cost of production and sale is $\overline{C} = 300 + 2x$.
    (a) Find the quantity that will maximize profit.
    (b) Find the selling price at this optimal quantity.
    (c) What is the maximum profit?
32. The monthly demand function for a product sold by a monopoly is $p = 8000 - x$, and its average cost is $\overline{C} = 4000 + 5x$.
    (a) Determine the quantity that will maximize profit.
    (b) Determine the selling price at the optimal quantity.
    (c) Determine the maximum profit.
33. The monthly demand function for a product sold by a monopoly is $p = 1960 - \frac{1}{3}x^2$, and the average cost is $\overline{C} = 1000 + 2x + x^2$. Production is limited to 1000 units and x is in hundreds of units.
    (a) Find the quantity that will give maximum profit.
    (b) Find the maximum profit.
34. The monthly demand function for a product sold by a monopoly is $p = 5900 - \frac{1}{2}x^2$, and its average cost is $\overline{C} = 3020 + 2x$. If production is limited to 100 units, find the number of units that maximizes profit. Will the maximum profit result in a profit or loss?
35. An industry with a monopoly on a product has its average weekly costs given by

$$\overline{C} = \frac{10{,}000}{x} + 60 - 0.03x + 0.00001x^2$$

The weekly demand for the product is given by $p = 120 - 0.015x$. Find the price the industry should set and the number of units it should produce to obtain maximum profit. Find the maximum profit.
36. A large corporation with monopolistic control in the marketplace has its average daily costs given by

$$\overline{C} = \frac{800}{x} + 100x + x^2$$

The daily demand for its product is given by $p = 60{,}000 - 50x$. Find the quantity that gives maximum profit, and find the maximum profit. What selling price should the corporation set for its product?
37. A company handles an apartment building with 50 units. Experience has shown that if the rent for each of the units is \$360 per month, all of the units will be filled, but 1 unit will become vacant for each \$10 increase in this monthly rate. If the monthly cost of maintaining the apartment building is \$6 per rented unit, what rent should be charged per month to maximize the profit?
38. A travel agency will plan a tour for groups of size 25 or larger. If the group contains exactly 25 people, the cost is \$500 per person. However, each person's cost is reduced by \$10 for each additional person above the 25. If the travel agency incurs a cost of \$125 per person for the tour, what size group will give the agency the maximum profit?

Miscellaneous Applications

39. ***Sales revenue*** The data in the table give sales revenues for Scott Paper Company* for various years.

Year	Sales Revenue (billions)	Year	Sales Revenue (billions)
1983	\$2.6155	1989	4.8949
1984	2.7474	1990	5.1686
1985	2.934	1991	4.9593
1986	3.3131	1992	5.0913
1987	3.9769	1993	4.7489
1988	4.5494		

Source: Scott Paper Company, 1993 *Annual Report*

Assume that sales revenues for Scott Paper can be modeled by

$$R(t) = -0.031t^2 + 0.776t + 0.179$$

where t is the number of years past 1980.

(a) Use the model to find when maximum revenue occurs and what maximum revenue it predicts.

(b) Check your result in (a) against the data in the table.

40. ***Revenue*** The following table gives the total revenues of AT&T† for selected years.

Year	Total Revenues (billions)	Year	Total Revenues (billions)
1985	\$63.13	1990	\$62.191
1986	\$69.906	1991	\$63.089
1987	\$60.53	1992	\$64.904
1989	\$61.1	1993	\$67.156

Source: *AT&T Annual Report*, 1993

Suppose the data can be modeled by the equation

$$R(t) = 0.253t^2 - 4.03t + 76.84$$

where t is the number of years past 1980.

(a) Use the model to find the year in which revenue was minimum and find the minimum predicted revenue.

(b) Check your result in (a) against the data in the table.

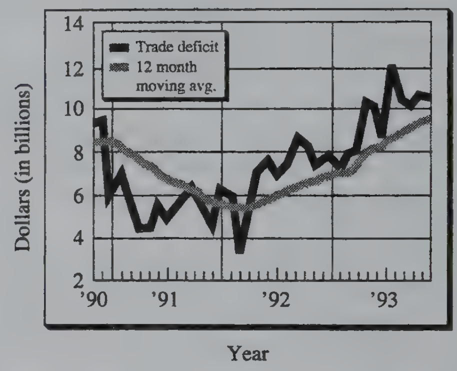
41. ***U.S. trade deficit*** The figure shows the U.S. trade deficit, in billions of dollars, from late 1990 to late 1993.

(a) Approximately when did the trade deficit reach its absolute minimum, and what was the minimum?

(b) Approximately when did the 12-month moving average reach its absolute minimum, and what was the minimum?

(c) Approximately when did the trade deficit reach its absolute maximum, and what was the maximum?

(d) Approximately when did the 12-month moving average reach its absolute maximum, and what was the maximum?



Source: *Wall Street Journal*, December 17, 1993

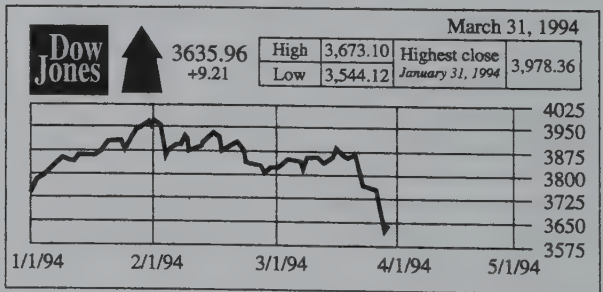
42. ***Dow Jones average*** The figures show the Dow Jones average for early 1994 (January through March) as well as the average through the day on March 31, 1994.

(a) Approximate the time of day on March 31, 1994, when the Dow Jones average reached its absolute maximum and when it reached its absolute minimum.

(b) Find the day of the year in early 1994 when the Dow Jones average reached its absolute maximum, and find its absolute maximum.

(c) Find the approximate day of the year in early 1994 when the Dow Jones average reached its absolute minimum, and find its absolute minimum.

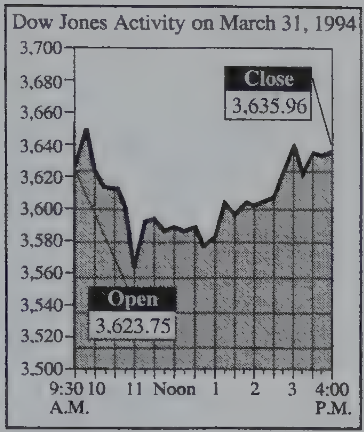
Dow Jones Closing Averages—Early 1994



Source: *Oil City Derrick*, April 1, 1994

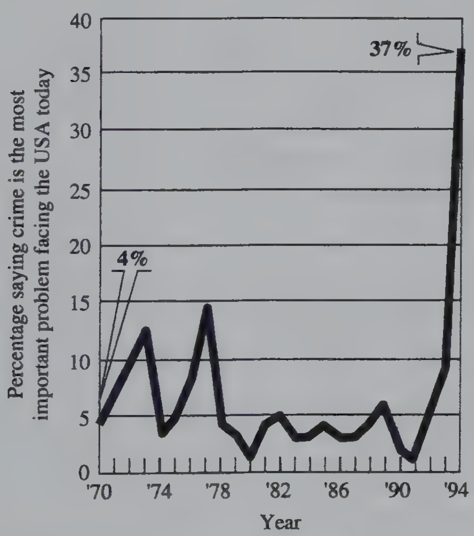
*Before Scott Paper merged with Kimberly Clark

†Before AT&T split off Lucent and NCR



From *Oil City Derrick*, April 1, 1994
Source: Telerate Systems, Inc.

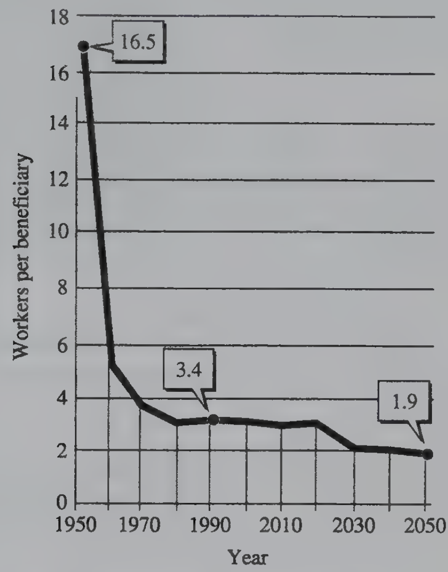
43. ***Crime concern*** The graph below shows that the percentage p of people in the United States who believe crime is the most serious domestic problem is a function of time t (in calendar years) and, of course, of the events that occur during those years. Denote this function by $p = f(t)$, and use the graph to answer the following.



Source: *USA Today*, January 25, 1994

(a) When did $f(t)$ achieve its absolute minimum?
(b) What is the absolute maximum point for $f(t)$?
(c) Is it possible to have $f(t) < 0$? Explain.
(d) What effect do you think a foreign conflict (such as a war) would have on the graph?
(e) Do you think a candidate for public office should have based a campaign extensively on the "law and order" issue in 1980? in 1994? Explain.

44. ***Social Security support*** The graph below shows the number of workers, W, still in the workforce per Social Security beneficiary (historically and projected into the future) as a function of time t, in calendar years. Denote this function by $W = f(t)$, and use the graph to answer the following.



Source: Social Security Administration

(a) What is the absolute maximum for $f(t)$?
(b) What is the absolute minimum for $f(t)$?
(c) Does this graph suggest that Social Security taxes might rise or might fall in the early 21st century? Explain.

10.4 Applications of Maxima and Minima

OBJECTIVE

- *To apply the procedures for finding maxima and minima to solve problems from the management, life, and social sciences*

APPLICATION PREVIEW

Manufacturers make production runs to restock their inventories. Because costs are associated with both the production of items and their storage (placement into inventory), a typical question in these **inventory cost models** is "How many items should be produced in each production run to minimize the total costs of production and storage?" This question is a typical example of the kinds of questions and important business applications that require the use of the derivative for finding maxima and minima. As managers, workers, or consumers, we may be interested in such things as maximum revenue, maximum profit, minimum cost, maximum medical dosage, maximum utilization of resources, and so on.

If we have functions that model cost, revenue, or population growth, we can apply the methods of this chapter to find the maxima and minima of the functions.

EXAMPLE 1

Suppose that a new company begins production in 1995 with eight employees and the growth of the company over the next 10 years is predicted by

$$N = N(t) = 8\left(1 + \frac{160t}{t^2 + 16}\right), \qquad 0 \le t \le 10$$

where N is the number of employees t years after 1995.

(a) In what year will the number of employees be maximized?
(b) What will be the maximum number of employees?

Solution

This function will have a relative maximum when $N'(t) = 0$.

$$\begin{aligned} N'(t) &= 8\left[\frac{(t^2 + 16)(160) - (160t)(2t)}{(t^2 + 16)^2}\right] \\ &= 8\left[\frac{160t^2 + 2560 - 320t^2}{(t^2 + 16)^2}\right] \\ &= 8\left[\frac{2560 - 160t^2}{(t^2 + 16)^2}\right] \end{aligned}$$

Because $N'(t) = 0$ when its numerator is 0 (note that the denominator is never 0), we must solve

$$2560 - 160t^2 = 0$$
$$160(4 + t)(4 - t) = 0$$

so

$$t = -4 \quad \text{or} \quad t = 4$$

We are interested only in positive t-values, so we test $t = 4$.

$$\left.\begin{aligned} N'(0) &= 8\left[\frac{2560}{256}\right] > 0 \\ N'(10) &= 8\left[\frac{-13{,}440}{(116)^2}\right] < 0 \end{aligned}\right\} \Rightarrow \textit{relative maximum}$$

The relative maximum is

$$N(4) = 8\left(1 + \frac{640}{32}\right) = 168$$

At $t = 0$, the number of employees is $N(0) = 8$, and it increases to $N(4) = 168$. After $t = 4$, $N(t)$ decreases to $N(10) = 118$ (approximately), so $N(4) = 168$ is the maximum number of employees. Figure 10.34 verifies these conclusions.

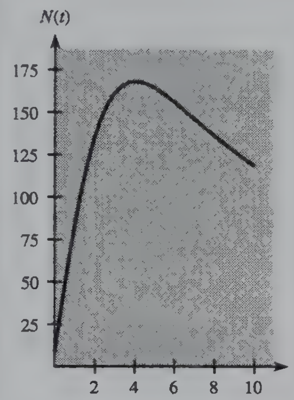


Figure 10.34

Sometimes we must develop the function we need from the statement of the problem. In this case, it is important to understand what is to be maximized or minimized and to express that quantity as a function of *one* variable.

EXAMPLE 2

A farmer needs to enclose a rectangular pasture containing 1,600,000 square feet. Suppose that along the road adjoining his property he wants to use a more expensive fence and that he needs no fence on one side perpendicular to the road because a river bounds his property on that side. If the fence costs \$15 per foot along the road and \$10 per foot along the two remaining sides that must be fenced, what dimensions of his rectangular field will minimize his cost? (See Figure 10.35.)

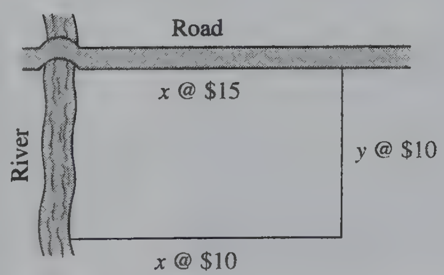


Figure 10.35

Solution

In Figure 10.35, x represents the length of the pasture along the road (and parallel to the road) and y represents the width. The cost function for the fence used is

$$C = 15x + 10y + 10x = 25x + 10y$$

We cannot use a derivative to find where C is minimized unless we write C as a function of x or y only. Because the area of the rectangular field must be 1,600,000 square feet, we have

$$A = xy = 1{,}600{,}000$$

Solving for y in terms of x and substituting give

$$y = \frac{1{,}600{,}000}{x}$$

$$C = 25x + 10\left(\frac{1{,}600{,}000}{x}\right) = 25x + \frac{16{,}000{,}000}{x}$$

The derivative of C with respect to x is

$$C'(x) = 25 - \frac{16{,}000{,}000}{x^2}$$

and we find the relative minimum of C as follows:

$$\begin{aligned} 0 &= 25 - \frac{16{,}000{,}000}{x^2} \\ 0 &= 25x^2 - 16{,}000{,}000 \\ 25x^2 &= 16{,}000{,}000 \\ x^2 &= 640{,}000 \\ x &= 800 \text{ feet} \end{aligned}$$

Testing to see if $x = 800$ gives the minimum cost, we find

$$C''(x) = \frac{32{,}000{,}000}{x^3}$$

$C''(x) > 0$ for $x > 0$, so $C(x)$ is concave up for all positive x. Thus $x = 800$ gives the absolute minimum, and $C(800)$ is the minimum cost. The other dimension of the rectangular field is $x = 1{,}600{,}000/800 = 2000$ feet. Figure 10.36 verifies that $C(x)$ reaches its minimum (of 40,000) at $x = 800$.

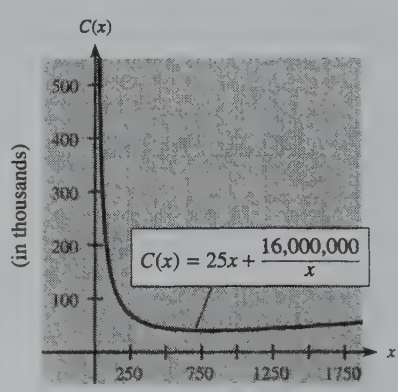


Figure 10.36

EXAMPLE 3

Postal restrictions limit the size of packages sent through the mail. If the restrictions are that the length plus the girth may not exceed 108 in., find the volume of the largest box with square cross section that can be mailed.

Solution

Let l equal the length of the box, and let s equal a side of the square end. See Figure 10.37(a). The volume we seek to maximize is given by

$$V = s^2 l$$

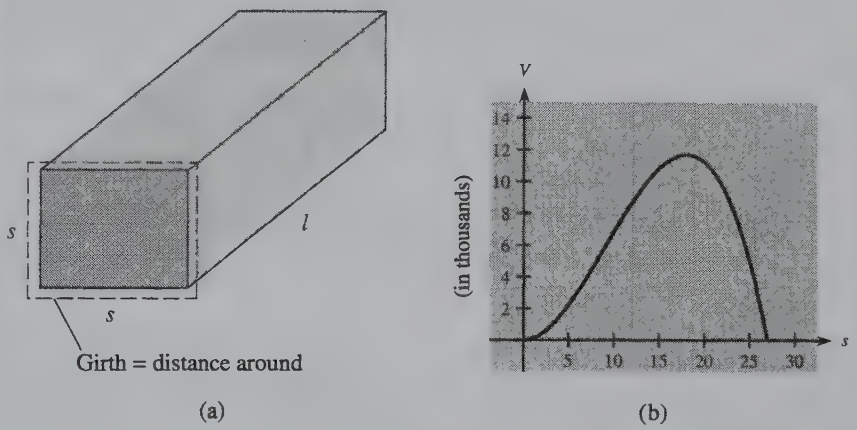


Figure 10.37

We can use the restriction that girth plus length equals 108,

$$4s + l = 108$$

to express V as a function of s or l. Because $l = 108 - 4s$, the equation for V becomes

$$V = s^2(108 - 4s)$$

or

$$V = 108s^2 - 4s^3$$

Thus we can use dV/ds to find the critical values.

$$\frac{dV}{ds} = 216s - 12s^2$$
$$0 = s(216 - 12s)$$

The critical values are $s = 0$, $s = \frac{216}{12} = 18$. The critical value $s = 0$ will not maximize the volume, for in this case, $V = 0$. Testing to the left and right of $s = 18$ gives

$$V'(17) > 0 \quad \text{and} \quad V'(19) < 0$$

Thus $s = 18$ in. and $l = 108 - 4(18) = 36$ in. yield a maximum volume of 11,664 cubic inches. Once again we can verify our results graphically. Figure 10.37(b) shows that $V = s^2(108 - 4s)$ achieves its maximum when $s = 18$.

CHECKPOINT

Suppose we want to find the minimum value of $C = 5x + 2y$ and we know that x and y must be positive and that $xy = 1000$.

1. What equation do we differentiate to solve this problem?
2. Find the critical values.
3. Find the minimum value of C.

Consider the **inventory cost model** in which x items are produced in each run and items are removed from inventory at a fixed constant rate. Then the number of units in storage changes with time and is illustrated in Figure 10.38. To see how these inventory cost models work, consider the following example.

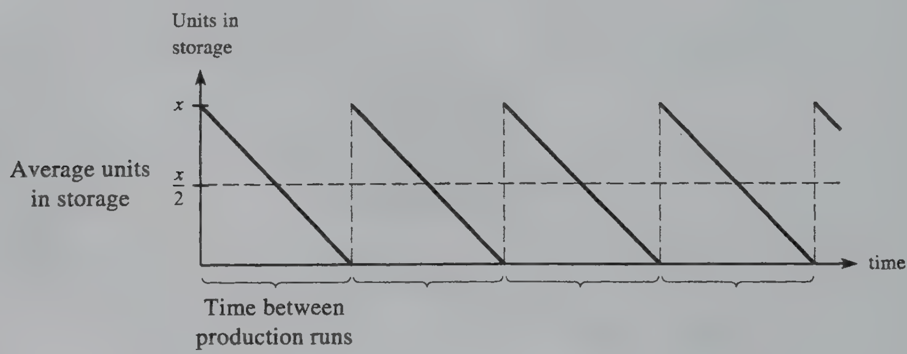


Figure 10.38

EXAMPLE 4

Suppose that a company needs 1,000,000 items during a year and that preparation costs are \$800 for each production run. Suppose further that it costs the company \$6 to produce each item and \$1 to store an item for up to a year. If each production run consists of x items, find x so that the total costs of production and storage are minimized.

Solution

The total production costs are given by

$$\begin{pmatrix}\text{No. of}\\ \text{runs}\end{pmatrix}\begin{pmatrix}\text{cost}\\ \text{per run}\end{pmatrix} + \begin{pmatrix}\text{no. of}\\ \text{items}\end{pmatrix}\begin{pmatrix}\text{cost}\\ \text{per item}\end{pmatrix}$$

$$= \left(\frac{1{,}000{,}000}{x}\right)(\$800) + (1{,}000{,}000)(\$6)$$

The total storage costs are

$$\begin{pmatrix}\text{Average}\\ \text{no. stored}\end{pmatrix}\begin{pmatrix}\text{storage cost}\\ \text{per item}\end{pmatrix} = \left(\frac{x}{2}\right)(\$1)$$

Thus the total costs of production and storage are

$$C = \left(\frac{1{,}000{,}000}{x}\right)(800) + 6{,}000{,}000 + \frac{x}{2}$$

We wish to find x so that C is minimized.

$$C' = \frac{-800{,}000{,}000}{x^2} + \frac{1}{2}$$

If $x > 0$, critical values occur when $C' = 0$.

$$0 = \frac{-800{,}000{,}000}{x^2} + \frac{1}{2}$$

$$\frac{800{,}000{,}000}{x^2} = \frac{1}{2}$$

$$1{,}600{,}000{,}000 = x^2$$

$$x = \pm\, 40{,}000$$

Because x must be positive, we test $x = 40{,}000$ with the second derivative.

$$C''(x) = \frac{1{,}600{,}000{,}000}{x^3}, \quad \text{so} \quad C''(40{,}000) > 0$$

Note that $x = 40{,}000$ yields an absolute minimum value for C, because $C'' > 0$ for all $x > 0$. That is, production runs of 40,000 items yield minimum total costs for production and storage.

Graphing Utilities

Problems of the types we've studied in this section could also be solved (at least approximately) with a graphing utility. With this approach, our first goal is still to express the quantity to be maximized or minimized as a function of one variable. Then that function can be graphed, and the (at least approximate) optimal value can be obtained from the graph.

EXAMPLE 5

A homeowner has to pay for utility line installation to her house from a transformer on the street at the corner of her property. Because of local restrictions, the lines must be underground on her property. Suppose that the costs are \$5/foot along the street and \$10/foot underground. How far from the transformer should she enter the property to minimize installation costs? See Figure 10.39(a).

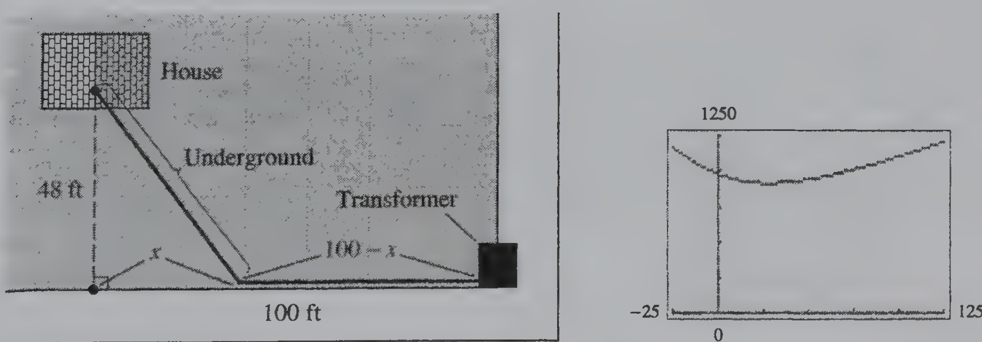


Figure 10.39 (a) (b)

Solution

If the homeowner had the cable placed underground from the house perpendicular to the street and then to the transformer, the cost would be

$$\$10(48) + \$5(100) = \$980$$

It may be possible to save some money by placing the cable on a diagonal to the street. By using the Pythagorean Theorem, we find that the length of the underground cable that meets the street x feet closer to the transformer is

$$\sqrt{48^2 + x^2} = \sqrt{2304 + x^2} \quad \text{feet}$$

Thus the cost C of installation is given by

$$C = 10\sqrt{2304 + x^2} + 5(100 - x) \quad \text{dollars}$$

Figure 10.39(b) shows the graph of this function over the possible interval for x ($0 \le x \le 100$). Because any extrema must occur in this interval, we can find the minimum by using TRACE or the MIN command on a calculator or computer. The minimum cost is \$915.70, when $x = 28$ feet (that is, when the cable meets the street 72 feet from the transformer). We can use the derivative of C to verify that $x = 28$ gives the minimum cost. By finding the derivative and graphing it, we can see that the only relative extremum occurs at $x = 28$. (Try this.) Because the cost is lower at $x = 28$ than at either endpoint of the interval, the lowest possible cost occurs there.

CHECKPOINT SOLUTIONS

1. We must differentiate C, but first C must be expressed as a function of one variable: $xy = 1000$ means that $y = 1000/x$. If we substitute $1000/x$ for y in $C = 5x + 2y$, we get

$$C(x) = 5x + 2\left(\frac{1000}{x}\right) = 5x + \frac{2000}{x}$$

   Find $C'(x)$ to solve the problem.

2. $C'(x) = 5 - 2000/x^2$, so $C'(x) = 0$ when

$$5 - \frac{2000}{x^2} = 0, \quad \text{or} \quad x = \pm 20$$

   Because x must be positive, the only critical value is $x = 20$.

3. $C''(x) = 4000/x^3$, so $C''(x) > 0$ for all $x > 0$. Thus $x = 20$ yields the minimum value. Also, when $x = 20$, we have $y = 50$, so the minimum value of C is $C = 5(20) + 2(50) = 200$.

EXERCISE 10.4

Applications

1. ***Return to sales*** The manufacturer of GRIPPER tires modeled its return on sales from television advertising expenditures in two regions, as follows:

$$\text{Region 1:}\quad S_1 = 30 + 20x_1 - 0.4x_1^2$$
$$\text{Region 2:}\quad S_2 = 20 + 36x_2 - 1.3x_2^2$$

where S_1 and S_2 are the sales revenue in millions of dollars and x_1 and x_2 are millions of dollars of expenditures for television advertising.
   (a) What advertising expenditures would maximize sales revenue in each district?
   (b) How much money will be needed to maximize sales revenue in both districts?

2. ***Projectiles*** A ball thrown into the air from a building 100 ft high travels along a path described by

$$y = \frac{-x^2}{110} + x + 100$$

where y is its height and x is the horizontal distance from the building. What is the maximum height the ball will reach?

3. ***Profit*** The profit from a grove of orange trees is given by $x(800 - x)$, where x is the number of orange trees per acre. How many trees per acre will maximize the profit?

4. ***Reaction rates*** The velocity v of an autocatalytic reaction can be represented by the equation

$$v = x(a - x)$$

where a is the amount of material originally present and x is the amount that has been decomposed at any given time. Find the maximum velocity of the reaction.

5. ***Productivity*** Analysis of daily output of a factory during an 8-hour shift shows that the hourly number of units y produced after t hours of production is

$$y = 70t + \tfrac{1}{2}t^2 - t^3, \qquad 0 \le t \le 8$$

   (a) After how many hours will the hourly number of units be maximized?
   (b) What is the maximum hourly output?

6. ***Productivity*** A time study showed that, on average, the productivity of a worker after t hours on the job can be modeled by

$$P = 27t + 6t^2 - t^3, \quad 0 \le t \le 8$$

where P is the number of units produced per hour. After how many hours will productivity be maximized? What is the maximum productivity?

7. ***Consumer expenditure*** Suppose that the demand x for a product is $x = 10{,}000 - 100p$, where p dollars is the market price per unit. Then the consumer expenditure for the product is

$$E = px = 10{,}000p - 100p^2$$

For what market price will expenditure be greatest?

8. ***Production costs*** Suppose that the monthly cost of mining a certain ore is related to the number of pieces of equipment purchased, according to

$$C = 25{,}000x + \frac{870{,}000}{x}, \qquad x > 0$$

where x is the number of pieces of equipment used. Using how many pieces of equipment will minimize the cost?

Medication For Problems 9 and 10, consider that when medicine is administered, reaction (measured in change of blood pressure or temperature) can be modeled by

$$R = m^2\left(\frac{c}{2} - \frac{m}{3}\right)$$

where c is a positive constant and m is the amount of medicine absorbed into the blood (Source: Thrall, R. M., *et al., Some Mathematical Models in Biology,* U.S. Dept. of Commerce, 1967).

9. Find the amount of medicine that is being absorbed into the blood when the reaction is maximum.
10. The rate of change of reaction R with respect to the amount of medicine m is defined to be the sensitivity.
   (a) Find the sensitivity, S.
   (b) Find the amount of medicine that is being absorbed into the blood when the sensitivity is maximum.

11. ***Advertising and sales*** An inferior product with a large advertising budget sells well when it is introduced, but sales fall as people discontinue use of the product. Suppose that the weekly sales S are given by

$$S = \frac{200t}{(t+1)^2}, \qquad t \geq 0$$

where S is in millions of dollars and t is in weeks. After how many weeks will sales be maximized?

12. ***Revenue*** A newly released film has its weekly revenue given by

$$R(t) = \frac{50t}{t^2 + 36}, \qquad t \geq 0$$

where R is in millions of dollars and t is in weeks.
(a) After how many weeks will the weekly revenue be maximized?
(b) What is the maximum weekly revenue?

13. ***News impact*** Suppose that the percentage p (as a decimal) of people who could correctly identify two of eight defendants in a drug case t days after their trial began is given by

$$p(t) = \frac{6.4t}{t^2 + 64} + 0.05$$

Find the number of days before the percentage is maximized, and find the maximum percentage.

14. ***Candidate recognition*** Suppose that in an election year the proportion p of voters who recognize a certain candidate's name t months after the campaign started is given by

$$p(t) = \frac{7.2t}{t^2 + 36} + 0.2$$

After how many months is the proportion maximized?

15. ***Minimum fence*** Two equal rectangular lots are enclosed by fencing the perimeter of a rectangular lot and then putting a fence across its middle. If each lot is to contain 1200 square feet, what is the minimum amount of fence needed to enclose the lots (include the fence across the middle)?

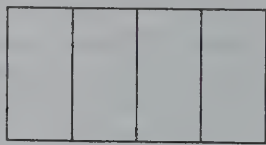
16. ***Minimum fence*** The running yard for a dog kennel must contain at least 900 square feet. If a 20-foot side of the kennel is used as part of one side of a rectangular yard with 900 square feet, what dimensions will require the least amount of fencing?

17. ***Minimum cost*** A rectangular field with one side along a river is to be fenced. Suppose that no fence is needed along the river, the fence on the side opposite the river costs \$20 per foot, and the fence on the other sides costs \$5 per foot. If the field must contain 45,000 square feet, what dimensions will minimize costs?

18. ***Minimum cost*** From a tract of land a developer plans to fence a rectangular region and then divide it into two identical rectangular lots by putting a fence down the middle. Suppose that the fence for the outside boundary costs \$5 per foot and the fence for the middle costs \$2 per foot. If each lot contains 13,500 square feet, find the dimensions of each lot that yield the minimum cost for the fence.

19. ***Optimization at a fixed cost*** A rectangular area is to be enclosed and divided into thirds. The family has \$800 to spend for the fencing material. The outside fence costs \$10 per running foot installed, and the dividers cost \$20 per running foot installed. What are the dimensions that will maximize the area enclosed? (The answer contains a fraction.)

20. ***Minimum cost*** A kennel of 640 square feet is to be constructed as shown. The cost is \$4 per running foot for the sides and \$1 per running foot for the ends and dividers. What are the dimensions of the kennel that will minimize the cost?



21. ***Minimum cost*** The base of a rectangular box is to be twice as long as it is wide. The volume of the box is 256 cubic inches. The material for the top costs \$0.10 per square inch and the material for the sides and bottom costs \$0.05 per square inch. Find the dimensions that will make the cost a minimum.

22. ***Velocity of air during a cough*** According to B. F. Visser, the velocity v of air in the trachea during a cough is related to the radius r of the trachea according to

$$v = ar^2(r_0 - r)$$

where a is a constant and r_0 is the radius of the trachea in a relaxed state. Find the radius r that produces the maximum velocity of air in the trachea during a cough.

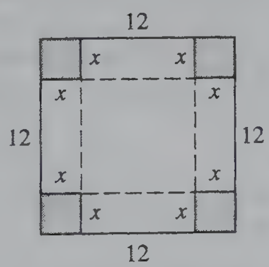
23. ***Inventory cost model*** Suppose that a company needs 1,500,000 items during a year and that preparation for each production run costs \$600. Suppose also that it costs \$15 to produce each item and \$2 per year to store an item. Use the inventory cost model to find the number of items in each production run so that the total costs of production and storage are minimized.

24. ***Inventory cost model*** Suppose that a company needs 60,000 items during a year and that preparation for each production run costs \$400. Suppose further that it costs \$4 to produce each item and \$0.75 to store an item for one year. Use the inventory cost model to find the number of items in each production run that will minimize the total costs of production and storage.

25. ***Inventory cost model*** A company needs 150,000 items per year. It costs the company \$360 to prepare a production run of these items and \$7 to produce each item. If it also costs the company \$0.75 per year for each item stored, find the number of items that should be produced in each run so that total costs of production and storage are minimized.

26. ***Inventory cost model*** A company needs 450,000 items per year. Production costs are \$500 to prepare for a production run and \$10 for each item produced. Inventory costs are \$2 per item per year. Find the number of items that should be produced in each run so that the total costs of production and storage are minimized.

27. ***Volume*** A rectangular box with a square base is to be formed from a square piece of metal with 12-inch sides. If a square piece with side x is cut from the corners of the metal and the sides are folded up to form an open box, the volume of the box is $V = (12 - 2x)^2x$. What value of x will maximize the volume of the box?



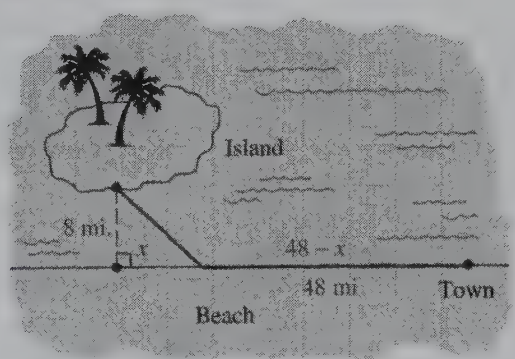
28. ***Volume*** A square piece of cardboard 36 centimeters on a side is to be formed into a rectangular box by cutting squares with length x from each corner and folding up the sides. What is the maximum volume possible for the box?

29. ***Revenue*** The owner of an orange grove must decide when to pick one variety of oranges. She can sell them for \$8 a bushel if she sells them now, with each tree yielding an average of 5 bushels. The yield increases by half a bushel per week for the next 5 weeks, but the price per bushel decreases by \$0.50 per bushel each week. When should the oranges be picked for maximum return?

30. ***Minimum material*** A box with an open top and a square base is to be constructed to contain 4000 cubic inches. Find the dimensions that will require the minimum amount of material to construct the box.

31. ***Minimum cost*** A printer has a contract to print 100,000 posters for a political candidate. He can run the posters by using any number of plates from 1 to 30 on his press. If he uses x metal plates, they will produce x copies of the poster with each impression of the press. The metal plates cost \$2.00 to prepare, and it costs \$12.50 per hour to run the press. If the press can make 1000 impressions per hour, how many metal plates should the printer make to minimize costs?

32. ***Shortest time*** A vacationer on an island 8 miles offshore from a point that is 48 miles from town must travel to town occasionally. (See the figure.) The vacationer has a boat capable of traveling 30 mph and can go by auto along the coast at 55 mph. At what point should the car be left to minimize the time it takes to get to town?



10.5 Asymptotes; More Curve Sketching

OBJECTIVES

- *To locate horizontal asymptotes*
- *To locate vertical asymptotes*
- *To sketch graphs of functions that have vertical and/or horizontal asymptotes*

APPLICATION PREVIEW

If the total daily cost of producing plastic rafts for swimming pools is given by

$$C = 500 + 8x + 0.05x^2$$

where x is the number of units produced per day, then the average cost per unit produced is given by

$$\overline{C} = \frac{500 + 8x + 0.05x^2}{x} = \frac{500}{x} + 8 + 0.05x, \qquad \text{for } x > 0$$

The graph of this function is very useful in understanding how many rafts should be produced to keep the average daily cost under control. The graph of this function contains a vertical asymptote at $x = 0$. We will discuss graphs and applications involving asymptotes in this section.

The procedures for using the first-derivative test and the second-derivative test are given in previous sections, but none of the graphs discussed in those sections contains vertical asymptotes or horizontal asymptotes. In this section, we consider how to use information about asymptotes along with the first and second derivatives, and we present a unified approach to curve sketching.

Asymptotes

In Section 2.4, "Special Functions and Their Graphs," we first discussed asymptotes and saw that they are important features of the graphs that have them. Then, in our discussion of limits, we discovered the relationship between certain limits and asymptotes. Limits are used to define and locate asymptotes precisely.

Because a horizontal asymptote tells us the behavior of the functional values (y-coordinates) when x increases or decreases without bound, we use limits at infinity to determine the existence of horizontal asymptotes.

Horizontal Asymptote The graph of a rational function $y = f(x)$ will have a **horizontal asymptote** at $y = b$, for a constant b, if

$$\lim_{x\to+\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

Otherwise, the graph has no horizontal asymptote.

For a rational function f, $\lim_{x\to+\infty} f(x) = b$ if and only if $\lim_{x\to-\infty} f(x) = b$, so we only need to find one of these limits to locate a horizontal asymptote. Just as with horizontal asymptotes, the formal definition of vertical asymptotes uses limits.

Vertical Asymptote The line $x = x_0$ is a **vertical asymptote** of the graph of $y = f(x)$ if the values of $f(x)$ approach $+\infty$ or $-\infty$ as x approaches x_0 (from the left or the right).

From our work with limits, recall that a vertical asymptote will occur on the graph of a function at an x-value where the function has its denominator (but not its numerator) equal to zero. These observations allow us to determine where vertical asymptotes occur.

Vertical Asymptote of a Rational Function The graph of the rational function

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$ if $g(c) = 0$ and $f(c) \neq 0$.

EXAMPLE 1

Find any vertical and horizontal asymptotes for

(a) $f(x) = \dfrac{2x - 1}{x + 2}$ (b) $f(x) = \dfrac{x^2 + 3}{1 - x}$

Solution

(a) The denominator of this function is 0 at $x = -2$, and because this value does not make the numerator 0, there is a vertical asymptote at $x = -2$.

Because the function is rational, we can find horizontal asymptotes by evaluating

$$\lim_{x\to+\infty} \frac{2x-1}{x+2} \quad \text{or} \quad \lim_{x\to-\infty} \frac{2x-1}{x+2}$$

We will evaluate both.

$$\lim_{x\to+\infty} \frac{2x-1}{x+2} = \lim_{x\to+\infty} \frac{2 - 1/x}{1 + 2/x} = \frac{2-0}{1+0} = 2$$

$$\lim_{x\to-\infty} \frac{2x-1}{x+2} = \lim_{x\to-\infty} \frac{2 - 1/x}{1 + 2/x} = \frac{2-0}{1+0} = 2$$

Thus there is a horizontal asymptote at $y = 2$. The graph is shown in Figure 10.40(a).

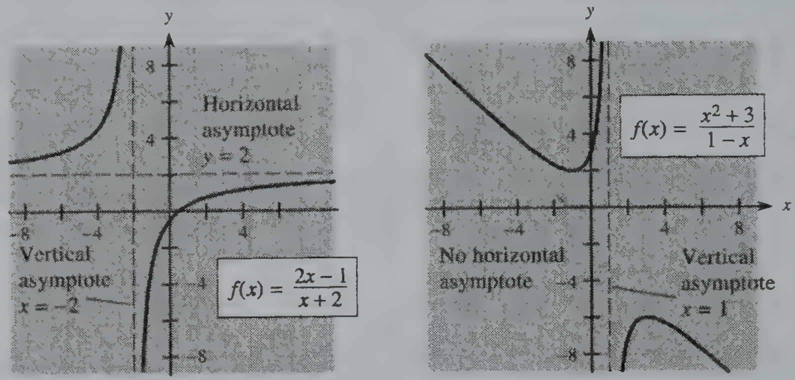


Figure 10.40 (a) (b)

(b) The denominator of this function is 0 at $x = 1$, and because this value does not make the numerator 0, there is a vertical asymptote at $x = 1$. To find horizontal asymptotes, we evaluate the following.

$$\lim_{x \to +\infty} \frac{x^2 + 3}{1 - x} = \lim_{x \to +\infty} \frac{1 + 3/x^2}{1/x^2 - 1/x} = \frac{1 + 0}{0 - 0} = -\infty$$

This limit is $-\infty$ because the numerator approaches 1 and the denominator approaches 0 through negative values. Thus

$$\lim_{x \to +\infty} \frac{x^2 + 3}{1 - x} \text{ does not exist}$$

and the graph has no horizontal asymptotes. Note also that

$$\lim_{x \to -\infty} \frac{x^2 + 3}{1 - x} \text{ does not exist}$$

The graph is shown in Figure 10.40(b).

More Curve Sketching

We now extend our first- and second-derivative techniques of curve sketching to include functions that have asymptotes.

In general, the following steps are helpful when we sketch the graph of a function.

1. Determine the domain of the function. The domain may be restricted by the nature of the problem or by the equation.
2. Look for vertical asymptotes, especially if the function is a rational function.
3. Look for horizontal asymptotes, especially if the function is a rational function.
4. Find the relative maxima and minima by using the first-derivative test or the second-derivative test.
5. Use the second derivative to find the points of inflection if this derivative is easily found.

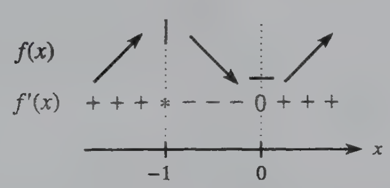
6. Use other information (intercepts, for example) and plot additional points to complete the sketch of the graph.

EXAMPLE 2

Sketch the graph of the function $f(x) = \dfrac{x^2}{(x+1)^2}$.

Solution

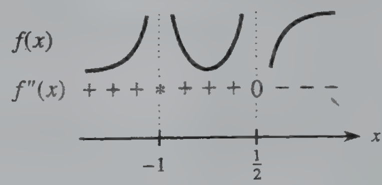
1. The domain is the set of all real numbers except $x = -1$.
2. Because $x = -1$ makes the denominator 0 and does not make the numerator 0, there is a vertical asymptote at $x = -1$.
3. Because
$$\lim_{x\to+\infty} \frac{x^2}{(x+1)^2} = \lim_{x\to+\infty} \frac{x^2}{x^2+2x+1}$$
$$= \lim_{x\to+\infty} \frac{1}{1+\dfrac{2}{x}+\dfrac{1}{x^2}}$$
$$= \frac{1}{1+0+0} = 1$$
there is a horizontal asymptote at $y = 1$.
4. To find any maxima and minima, we first find $f'(x)$.
$$f'(x) = \frac{(x+1)^2(2x) - x^2[2(x+1)]}{(x+1)^4}$$
$$= \frac{2x(x+1)[(x+1)-x]}{(x+1)^4}$$
$$= \frac{2x}{(x+1)^3}$$
Thus $f'(x) = 0$ when $x = 0$ (and $y = 0$), and $f'(x)$ is undefined at $x = -1$ (where the vertical asymptote occurs). Testing $f'(x)$ on either side of $x = 0$ and $x = -1$ gives the following sign diagram. The sign diagram for f' shows that the critical point (0, 0) is a relative minimum and shows how the graph approaches the vertical asymptote at $x = -1$.



* $x = -1$ is a vertical asymptote.

5. The second derivative is
$$f''(x) = \frac{(x+1)^3(2) - 2x[3(x+1)^2]}{(x+1)^6}$$
Factoring $(x+1)^2$ from the numerator and simplifying give
$$f''(x) = \frac{2-4x}{(x+1)^4}$$

We can see that $f''(0) = 2 > 0$, so the second-derivative test also shows that $(0, 0)$ is a relative minimum. We see that $f''(x) = 0$ when $x = \frac{1}{2}$. Checking $f''(x)$ between $x = -1$ (where it is undefined) and $x = \frac{1}{2}$ shows that the graph is concave up on this interval. Note too that $f''(x) < 0$ for $x > \frac{1}{2}$, so the point $(\frac{1}{2}, \frac{1}{9})$ is a point of inflection. Also see the sign diagram for $f''(x)$.



* $x = -1$ is a vertical asymptote.

6. To see how the graph approaches the horizontal asymptote, we check $f(x)$ for large values of $|x|$.

$$f(-100) = \frac{(-100)^2}{(-99)^2} = \frac{10{,}000}{9{,}801} > 1, \quad f(100) = \frac{100^2}{101^2} = \frac{10{,}000}{10{,}201} < 1$$

Thus the graph has the characteristics shown in Figure 10.41(a). The graph is shown in Figure 10.41(b).

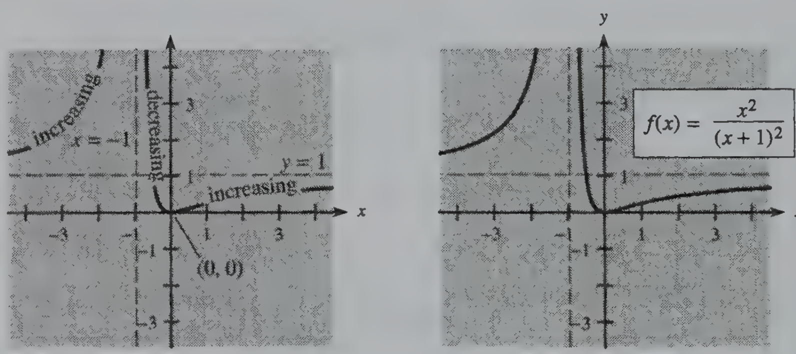


Figure 10.41 (a) (b)

When we wish to learn about a function $f(x)$ or sketch its graph, it is important to understand what information we obtain from $f(x)$, from $f'(x)$, and from $f''(x)$. The following summary may be helpful.

Summary

Source	*Information Provided*
$f(x)$	y-coordinates; horizontal asymptotes, vertical asymptotes; domain restrictions
$f'(x)$	Increasing $[f'(x) > 0]$; decreasing $[f'(x) < 0]$; critical points $[f'(x) = 0$ or $f'(x)$ undefined]; sign-diagram tests for maxima and minima
$f''(x)$	Concave up $[f''(x) > 0]$; concave down $[f''(x) < 0]$; possible points of inflection $[f''(x) = 0$ or $f''(x)$ undefined]; sign-diagram tests for points of inflection; second-derivative test for maxima and minima

CHECKPOINT

1. Let $f(x) = \dfrac{2x + 10}{x - 1}$ and decide whether the following are true or false.
   (a) $f(x)$ has a vertical asymptote at $x = 1$.
   (b) $f(x)$ has $y = 2$ as its horizontal asymptote.
2. Let $f(x) = \dfrac{x^3 - 16}{x} + 1$; then $f'(x) = \dfrac{2x^3 + 16}{x^2}$ and $f''(x) = \dfrac{2x^3 - 32}{x^3}$.
   Use these to determine whether the following are true or false.
   (a) There are no asymptotes.
   (b) $f'(x) = 0$ when $x = -2$
   (c) A partial sign diagram for $f'(x)$ is

   $f'(x)$ − − − − 0 + + + + * + + + +

   x axis: −2, 0

   * means $f'(0)$ is undefined.

   (d) There is a relative minimum at $x = -2$.
   (e) A partial sign diagram for $f''(x)$ is

   $f''(x)$ + + + + * − − − − 0 + + + +

   x axis: 0, $\sqrt[3]{16}$

   * means $f''(0)$ is undefined.

   (f) There are points of inflection at $x = 0$ and $x = \sqrt[3]{16}$.

We now consider the problem introduced in the Application Preview.

EXAMPLE 3

If the total daily cost of producing plastic rafts for swimming pools is given by

$$C = 500 + 8x + 0.05x^2$$

where x is the number of units produced per day, then the average cost per unit produced is given by

$$\overline{C} = \frac{500 + 8x + 0.05x^2}{x} = \frac{500}{x} + 8 + 0.05x \qquad \text{for } x > 0$$

Graph this function.

(a) Discuss what happens to average cost as the number of units decreases, approaching 0.
(b) Find the level of production that minimizes average cost.

Solution

(a) The domain of $\overline{C}(x)$ does not include 0, and $\lim_{x\to 0^+} \overline{C} = +\infty$, so there is a vertical asymptote at $x = 0$. Thus the average cost per unit increases without bound as the number of units produced approaches zero. (b) Finding the derivative of $\overline{C}(x)$ gives

$$\overline{C}' = -\frac{500}{x^2} + 0.05$$

Setting $\overline{C}' = 0$ and solving for x gives the critical values of x.

$$0 = -\frac{500}{x^2} + 0.05$$
$$0.05x^2 = 500$$
$$x = -100 \quad \text{or} \quad x = 100$$

The second derivative, $\overline{C}'' = 1000x^{-3} = \dfrac{1000}{x^3}$, is positive at $x = 100$, so $\overline{C}(100) = 18$ is the minimum possible average daily cost. The graph of this function is shown in Figure 10.42. The graph confirms that average cost is minimized when 100 units are produced and shows the asymptotic behavior of the function at 0.

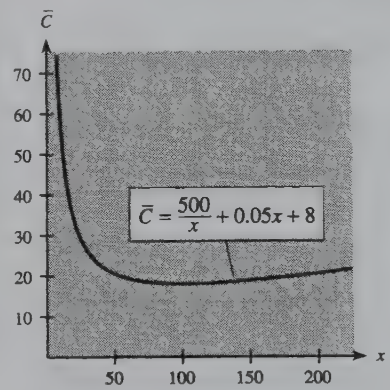


Figure 10.42

Graphing Utilities

If a graphing utility is not available, the procedures previously outlined in this section are necessary to generate a complete and accurate graph. With a graphing utility, the graph of a function is easily generated as long as the viewing window dimensions are appropriate. However, although a graphing utility may reveal the existence of asymptotes, it cannot always precisely locate them. Also, we sometimes need information provided by derivatives to obtain a window that shows all features of a graph.

EXAMPLE 4

Figure 10.43 shows the graph of $f(x) = \dfrac{71x^2}{28(3 - 2x^2)}$.

(a) Determine whether the function has horizontal or vertical asymptotes, and estimate where they occur.
(b) Check your conclusions to (a) analytically.
(c) Discuss which method is more accurate.

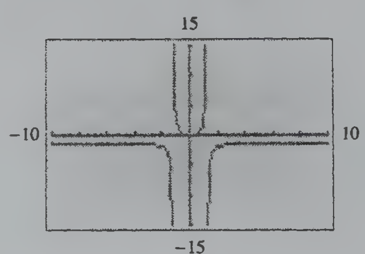


Figure 10.43

Solution

(a) The graph appears to have a horizontal asymptote somewhere between $y = -1$ and $y = -2$, perhaps near $y = -1.5$. Also, there are two vertical asymptotes located approximately at $x = 1.25$ and $x = -1.25$.

(b) We locate the horizontal asymptote by finding the limit of the function as x approaches infinity.

$$\lim_{x\to\infty} \frac{71x^2}{28(3-2x^2)} = \lim_{x\to\infty} \frac{\dfrac{71x^2}{x^2}}{\dfrac{84}{x^2} - \dfrac{56x^2}{x^2}} = \frac{71}{-56} \approx -1.268$$

Vertical asymptotes occur at x-values where $28(3 - 2x^2) = 0$, or

$$\begin{aligned} 3 - 2x^2 &= 0 \\ 3 &= 2x^2 \\ \frac{3}{2} &= x^2 \\ \pm\sqrt{\frac{3}{2}} &= x \quad \text{or} \quad x \approx \pm 1.225 \end{aligned}$$

(c) The analytic method is more accurate, of course, because asymptotes reveal extreme behavior of the function either vertically or horizontally. An accurate graph shows all features, but not necessarily the details of any feature. Despite this, our estimates from the graph were not too bad.

Even with a graphing utility, sometimes analytic methods are needed to determine an appropriate viewing window.

EXAMPLE 5

The standard viewing window of the graph of $f(x) = \dfrac{x+10}{x^2+300}$ appears blank (check and see). Find any asymptotes, maxima, and minima, and determine an appropriate viewing window. Sketch the graph.

Solution

Because $x^2 + 300 = 0$ has no real solution, there are no vertical asymptotes. We locate the horizontal asymptotes by finding the limit of the function as x approaches infinity.

$$\lim_{x\to\infty} \frac{x+10}{x^2+300} = \lim_{x\to\infty} \frac{\dfrac{x}{x^2} + \dfrac{10}{x^2}}{\dfrac{x^2}{x^2} + \dfrac{300}{x^2}} = \frac{0+0}{1+0} = 0$$

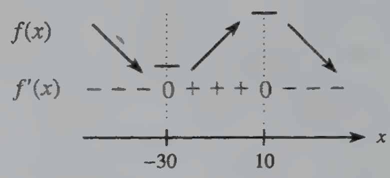
Hence $y = 0$ (the x-axis) is a horizontal asymptote. We then find an appropriate viewing window by locating the critical points.

$$\begin{aligned} f'(x) &= \frac{(x^2+300)(1) - (x+10)(2x)}{(x^2+300)^2} \\ &= \frac{x^2 + 300 - 2x^2 - 20x}{(x^2+300)^2} = \frac{300 - 20x - x^2}{(x^2+300)^2} \end{aligned}$$

$f'(x) = 0$ when the numerator is zero. Thus

$$300 - 20x - x^2 = 0$$
$$0 = x^2 + 20x - 300$$
$$0 = (x + 30)(x - 10)$$
$$x + 30 = 0 \qquad x - 10 = 0$$
$$x = -30 \qquad x = 10$$

The critical points are $x = -30$, $y = -\frac{1}{60} \approx -0.01666667$ and $x = 10$, $y = \frac{1}{20} = 0.05$. A sign diagram for $f'(x)$ is shown at the right.



Without using the information above, a graphing utility may not give a useful graph. An x-range that includes -30 and 10 is needed. Because $y = 0$ is a horizontal asymptote, these relative extrema are absolute, and the y-range must be quite small for the shape of the graph to be seen clearly. Figure 10.44 shows the graph.

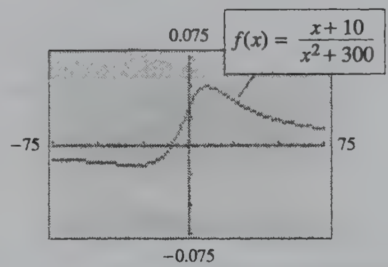


Figure 10.44

EXAMPLE 6

A profit function for a product is given by

$$P(x) = \frac{-x^2 + 16x - 4}{4x^2 + 16}, \qquad \text{for } x \geq 0$$

where x is in thousands of units and $P(x)$ is in billions of dollars. Because of fixed costs, profit is negative when fewer than 254 units are produced and sold. Will a loss occur at any other level of production and sales?

Solution

Looking at the graph of this function over the range $0 \leq x \leq 10$ (see Figure 10.45a), it is not clear whether the graph will eventually cross the x-axis. To see whether a loss ever occurs and to see what profit is approached as the number of units produced and sold becomes large, we evaluate the limit of $P(x)$ as $x \to \infty$.

$$\lim_{x\to\infty} \frac{-x^2 + 16x - 4}{4x^2 + 16} = \lim_{x\to\infty} \frac{-1 + \frac{16}{x} - \frac{4}{x^2}}{4 + \frac{16}{x^2}} = -\frac{1}{4}$$

Thus a loss of $\$\frac{1}{4}$ billion is approached as the number of units increases without bound. Figure 10.45(b) shows that the graph does cross the x-axis, where $-x^2 + 16x - 4 = 0$. Using TRACE, SOLVER, the Quadratic Formula, or a program gives $P(x) = 0$ at $x \approx 0.254$ and $x \approx 15.7459$. Thus if 15,746 units or more are produced and sold, the profit is negative.

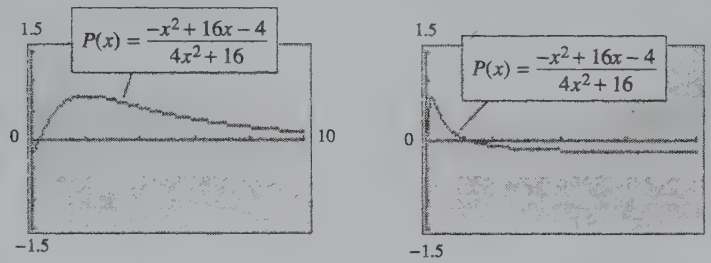


Figure 10.45 (a) (b)

CHECKPOINT SOLUTIONS

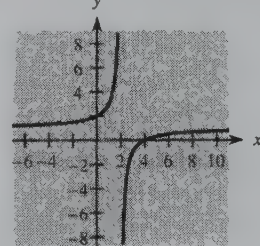
1. (a) True, $x = 1$ makes the denominator of $f(x)$ equal to zero, whereas the numerator is nonzero.
   (b) True, $\lim_{x\to\infty} \frac{2x + 10}{x - 1} = 2$ and $\lim_{x\to\infty} \frac{2x + 10}{x - 1} = 2$.
2. (a) False. There are no horizontal asymptotes, but $x = 0$ is a vertical asymptote.
   (b) True
   (c) True
   (d) True. The relative minimum point is $(-2, f(-2)) = (-2, 13)$.
   (e) True
   (f) False. There is a point of inflection only at $(\sqrt[3]{16}, 1)$. At $x = 0$ the vertical asymptote occurs, so there is no point on the graph and hence no point of inflection.

EXERCISE 10.5

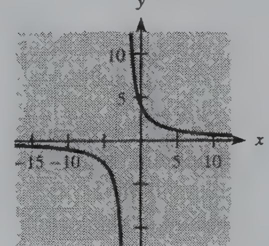
In Problems 1–4, a function and its graph are given. Use the graph to find each of the following, if they exist. Then confirm your results analytically.

(a) vertical asymptotes
(b) $\lim_{x\to\infty} f(x)$
(c) horizontal asymptotes
(d) $\lim_{x\to-\infty} f(x)$

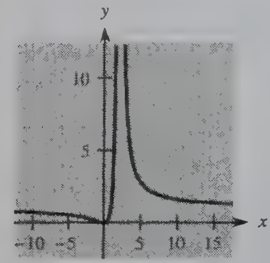
1. $f(x) = \frac{x - 4}{x - 2}$



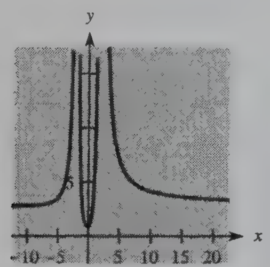
2. $f(x) = \frac{8}{x + 2}$



3. $f(x) = \dfrac{x^2}{(x-2)^2}$



4. $f(x) = \dfrac{3(x^4 + 2x^3 + 6x^2 + 2x + 5)}{(x^2 - 4)^2}$



In Problems 5–10, find any horizontal and vertical asymptotes for each function.

5. $y = \dfrac{2x}{x-3}$ 6. $y = \dfrac{3x-1}{x+5}$

7. $y = \dfrac{x+1}{x^2-4}$ 8. $y = \dfrac{4x}{9-x^2}$

9. $y = \dfrac{3x^3-6}{x^2+4}$ 10. $y = \dfrac{6x^3}{4x^2+9}$

For each function in Problems 11–18, find any horizontal and vertical asymptotes, and use information from the first derivative to sketch the graph.

11. $f(x) = \dfrac{2x+2}{x-3}$ 12. $f(x) = \dfrac{5x-15}{x+2}$

13. $y = \dfrac{x^2+4}{x}$ 14. $y = \dfrac{x^2+4}{x^2}$

15. $y = \dfrac{27x^2}{(x+1)^3}$ 16. $y = \left(\dfrac{x+2}{x-3}\right)^2$

17. $f(x) = \dfrac{16x}{x^2+1}$ 18. $f(x) = \dfrac{4x^2}{x^4+1}$

In Problems 19–24, a function and its first and second derivatives are given. Use these to find any horizontal and vertical asymptotes, critical points, relative maxima, relative minima, and points of inflection. Then sketch the graph of each function.

19. $y = \dfrac{x}{(x-1)^2}$

$y' = -\dfrac{x+1}{(x-1)^3}$

$y'' = \dfrac{2x+4}{(x-1)^4}$

20. $y = \dfrac{(x-1)^2}{x^2}$

$y' = \dfrac{2(x-1)}{x^3}$

$y'' = \dfrac{6-4x}{x^4}$

21. $y = x + \dfrac{3}{\sqrt[3]{x-3}}$

$y' = \dfrac{(x-3)^{2/3} - 1}{(x-3)^{2/3}}$

$y'' = \dfrac{2}{3(x-3)^{5/3}}$

22. $y = 3\sqrt[3]{x} + \dfrac{1}{x}$

$y' = \dfrac{x^{4/3} - 1}{x^2}$

$y'' = \dfrac{6 - 2x^{4/3}}{3x^3}$

23. $f(x) = \dfrac{9(x-2)^{2/3}}{x^2}$

$f'(x) = \dfrac{12(3-x)}{x^3(x-2)^{1/3}}$

$f''(x) = \dfrac{4(7x^2 - 42x + 54)}{x^4(x-2)^{4/3}}$

24. $f(x) = \dfrac{3x^{2/3}}{x+1}$

$f'(x) = \dfrac{2-x}{x^{1/3}(x+1)^2}$

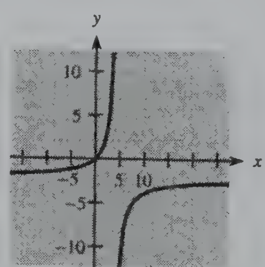
$f''(x) = \dfrac{2(2x^2 - 8x - 1)}{3x^{4/3}(x+1)^3}$

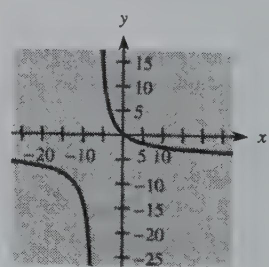
In Problems 25–28, a function and its graph are given.

(a) Use the graph to estimate the location of any horizontal or vertical asymptotes.

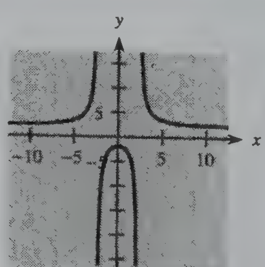
(b) Use the function to precisely determine the location of any asymptotes.

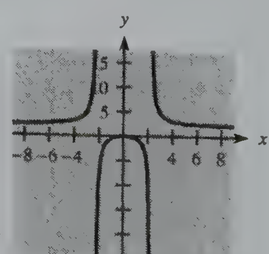
25. $f(x) = \dfrac{9x}{17-4x}$ 26. $f(x) = \dfrac{5-13x}{3x+20}$





27. $f(x) = \dfrac{20x^2 + 98}{9x^2 - 49}$ 28. $f(x) = \dfrac{15x^2 - x}{7x^2 - 35}$





For each function in Problems 29–34, complete the following steps.

(a) Use a graphing utility to graph the function in the standard viewing window.
(b) Analytically determine the location of any asymptotes and extrema.
(c) Graph the function in a viewing window that shows all features of the graph. State the ranges for x-values and y-values for your viewing window.

29. $f(x) = \dfrac{x + 25}{x^2 + 1400}$

30. $f(x) = \dfrac{x - 50}{x^2 + 1100}$

31. $f(x) = \dfrac{100(9 - x^2)}{x^2 + 100}$

32. $f(x) = \dfrac{200x^2}{x^2 + 100}$

33. $f((x) = \dfrac{1000x - 4000}{x^2 - 10x - 2000}$

34. $f(x) = \dfrac{900x + 5400}{x^2 - 30x - 1800}$

Applications

35. ***Cost-benefit*** The percentage p of particulate pollution that can be removed from the smokestacks of an industrial plant by spending C dollars is given by

$$p = \frac{100C}{7300 + C}$$

(a) Find any C-values where the rate of change of p with respect to C does not exist. Make sure that these make sense in the problem.
(b) Find C-values for which p is increasing.
(c) If there is a horizontal asymptote, find it.
(d) Can 100% of the pollution be removed?

36. ***Cost-benefit*** The percentage p of impurities that can be removed from the waste water of a manufacturing process at a cost of C dollars is given by

$$p = \frac{100C}{8100 + C}$$

(a) Find any C-values where the rate of change of p with respect to C does not exist. Make sure that these make sense in the problem.
(b) Find C-values for which p is increasing.
(c) Find any horizontal asymptotes.
(d) Can 100% of the pollution be removed?

37. ***Revenue*** A recently released film has its weekly revenue given by

$$R(t) = \frac{50t}{t^2 + 36}, \qquad t \ge 0$$

where $R(t)$ is in millions of dollars and t is in weeks.

(a) Graph $R(t)$.
(b) When will revenue be maximized?
(c) Suppose that if revenue decreases for 4 consecutive weeks, the film will be removed from theaters and will be released as a video 12 weeks later. When will the video come out?

38. ***Production costs*** Suppose that the total cost of producing a shipment of a certain product is

$$C = 5000x + \frac{125{,}000}{x}, \qquad x > 0$$

where x is the number of machines used in the production process.

(a) Graph this total cost function.
(b) Using how many machines will minimize the total cost?

39. ***Wind chill*** If x is the wind speed in miles per hour and is greater than or equal to 5, then the wind chill (in degrees Fahrenheit) for an air temperature of 0°F can be approximated by the function

$$f(x) = \frac{289.173 - 58.5731x}{x + 1}, \qquad x \ge 5$$

(a) Ignoring the restriction $x \ge 5$, does $f(x)$ have a vertical asymptote? If so, what is it?
(b) Does $f(x)$ have a vertical asymptote within its domain?
(c) Does $f(x)$ have a horizontal asymptote? If so, what is it?
(d) In the context of wind chill, does $\lim\limits_{x\to\infty} f(x)$ have a physical interpretation? If so, what is it, and is it meaningful?

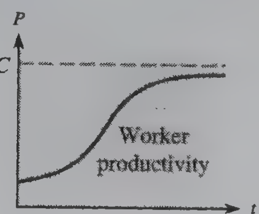
40. ***Profit*** An entrepreneur starts new companies and sells them when their growth is maximized. Suppose that the annual profit for a new company is given by

$$P(x) = 22 - \frac{1}{2}x - \frac{18}{x + 1}$$

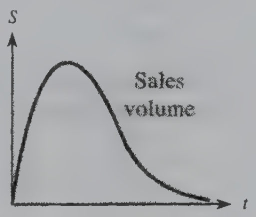
where P is in thousands of dollars and x is the number of years after the company is formed. If she wants to sell the company before profits begin to decline, after how many years should she sell it?

41. ***Productivity*** The figure is a typical graph of worker productivity per hour P as a function of time t on the job.

(a) What is the horizontal asymptote?
(b) What is $\lim\limits_{x\to\infty} P(t)$?
(c) What is the horizontal asymptote for $P'(t)$?
(d) What is $\lim\limits_{x\to\infty} P'(t)$?



42. ***Sales volume*** The figure shows a typical curve that gives the volume of sales S as a function of time t after an ad campaign.
   (a) What is the horizontal asymptote?
   (b) What is $\lim_{t\to\infty} S(t)$?
   (c) What is the horizontal asymptote for $S'(t)$?
   (d) What is $\lim_{t\to\infty} S'(t)$?



Farm workers The percentage of U.S. workers in farm occupations during certain years is shown in the table.

Year	*Percent of All Workers in Farm Occupations*
1820	71.8
1850	63.7
1870	53
1900	37.5
1920	27
1930	21.2
1940	17.4
1950	11.6
1960	6.1
1970	3.6
1980	2.7
1985	2.8
1990	2.4

Source: *The World Almanac and Book of Facts*, 1993

Assume that the percentage of U.S. workers in farm occupations can be modeled with the function

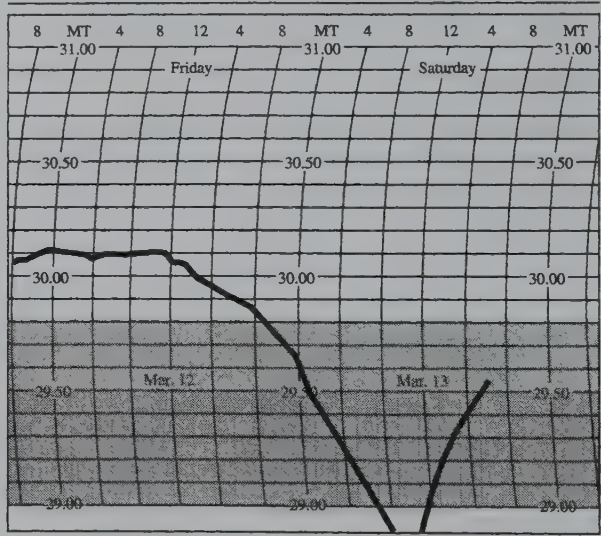
$$f(t) = 1000 \cdot \frac{-8.0912t + 1558.9}{1.09816t^2 - 122.183t + 21472.6}$$

where t is the number of years past 1800. Use this model in Problems 43 and 44.

43. (a) Find $\lim_{t\to\infty} f(t)$.
   (b) Interpret your answer to (a).
   (c) Does $f(t)$ have any vertical asymptotes within its domain $t \geq 0$?

44. (a) Use a graphing utility to graph $f(t)$ for $t = 0$ to $t = 220$.
   (b) From the graph, identify t-values where the model is inappropriate, and explain why it is inappropriate.

45. ***Barometric pressure*** The figure shows a barograph readout of the barometric pressure as recorded by Georgia Southern University's meteorological equipment. The figure shows a tremendous drop in barometric pressure on Saturday morning, March 13, 1993.
   (a) If $B(t)$ is barometric pressure expressed as a function of time, as shown in the figure, does $B(t)$ have a vertical asymptote sometime after 8 A.M. on Saturday, March 13, 1993? Explain why or why not.
   (b) Consult your library or some other resource to find out what happened in Georgia (and in the eastern United States) on March 13, 1993, to cause such a dramatic drop in barometric pressure.



Source: *Statesboro Herald*, March 14, 1993

KEY TERMS AND FORMULAS

Section	Key Terms	Formula
10.1	Relative maxima and minima	
	Increasing	$f'(x) > 0$
	Decreasing	$f'(x) < 0$
	Critical points	$f'(x) = 0$ or $f'(x)$ undefined
	Sign diagram for $f'(x)$	
	First-derivative test	
	Horizontal point of inflection	
10.2	Concave up	$f''(x) > 0$
	Concave down	$f''(x) < 0$
	Point of inflection	May occur where $f''(x) = 0$ or $f''(x)$ undefined
	Sign diagram for $f''(x)$	
	Second-derivative test	
10.3	Absolute extrema	
	Average cost	$\overline{C}(x) = C(x)/x$
	Profit maximization	
	Competitive market	$R(x) = p \cdot x$ where $p =$ equilibrium price
	Monopolistic market	$R(x) = p \cdot x$ where $p = f(x)$ is the demand function
10.4	Inventory cost models	
10.5	Asymptotes	
	Horizontal: $y = b$	$\lim_{x\to+\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$
	Vertical: $x = c$ for rational function $y = f(x)/g(x)$	y unbounded near $x = c$ if $g(c) = 0$ and $f(c) \neq 0$

REVIEW EXERCISES

Section 10.1

In Problems 1–4, find all critical points and determine whether they are relative maxima, relative minima, or horizontal points of inflection.

1. $y = -x^2$
2. $p = q^2 - 4q - 5$
3. $f(x) = 1 - 3x + 3x^2 - x^3$
4. $f(x) = \dfrac{3x}{x^2 + 1}$

In Problems 5–10.

(a) Find all critical values, including those where $f'(x)$ is undefined.

(b) Find the relative maxima and minima, if any exist.

(c) Find the horizontal points of inflection, if any exist.

(d) Sketch the graph.

5. $y = x^3 + x^2 - x - 1$
6. $f(x) = 4x^3 - x^4$

7. $f(x) = x^3 - \frac{15}{2}x^2 - 18x + \frac{3}{2}$
8. $y = 5x^7 - 7x^5 - 1$
9. $y = x^{2/3} - 1$
10. $y = x^{2/3}(x - 4)^2$

Section 10.2

11. Is the graph of $y = x^4 - 3x^3 + 2x - 1$ concave up or concave down at $x = 2$?
12. Find intervals where the graph of $y = x^4 - 2x^3 - 12x^2 + 6$ is concave upward and intervals where it is concave downward, and find points of inflection.
13. Find the relative maxima, relative minima, and points of inflection of the graph of $y = x^3 - 3x^2 - 9x + 10$.

In Problems 14 and 15, find any relative maxima, relative minima, and points of inflection, and sketch each graph.

14. $y = x^3 - 12x$
15. $y = 2 + 5x^3 - 3x^5$

Section 10.3

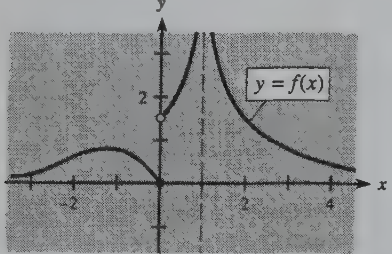
16. Given $R = 280x - x^2$, find the absolute maximum and minimum for R when (a) $0 \le x \le 200$ and (b) $0 \le x \le 100$.
17. Given $y = 6400x - 18x^2 - \frac{x^3}{3}$, find the absolute maximum and minimum for y when (a) $0 \le x \le 50$ and (b) $0 \le x \le 100$.

Section 10.5

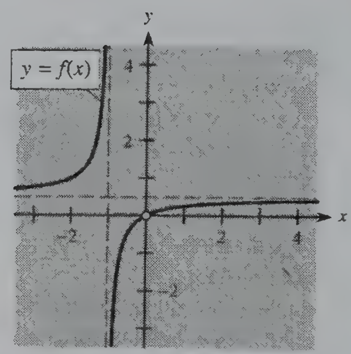
In Problems 18 and 19, use the graphs to find the following items.

(a) vertical asymptotes
(b) horizontal asymptotes
(c) $\lim_{x\to+\infty} f(x)$
(d) $\lim_{x\to-\infty} f(x)$

18.



19.



In Problems 20 and 21, find any horizontal asymptotes and any vertical asymptotes.

20. $y = \frac{3x + 2}{2x - 4}$
21. $y = \frac{x^2}{1 - x^2}$

In Problems 22–24:

(a) Find any horizontal and vertical asymptotes.
(b) Find any relative maxima and minima.
(c) Sketch each graph.

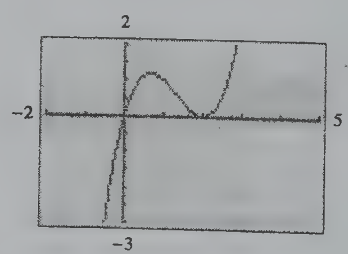
22. $y = \frac{3x}{x + 2}$
23. $y = \frac{8(x - 2)}{x^2}$
24. $y = \frac{x^2}{x - 1}$

Sections 10.1 and 10.2

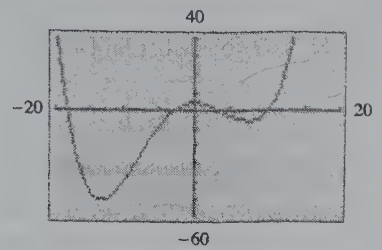
In Problems 25 and 26, a function and its graph are given.

(a) Use the graph to determine (estimate) x-values where $f'(x) > 0$, where $f'(x) < 0$, and where $f'(x) = 0$.
(b) Use the graph to determine x-values where $f''(x) > 0$, where $f''(x) < 0$, and where $f''(x) = 0$.
(c) Check your conclusions to (a) by finding $f'(x)$ and graphing it with a graphing utility.
(d) Check your conclusions to (b) by finding $f''(x)$ and graphing it with a graphing utility.

25. $f(x) = x^3 - 4x^2 + 4x$

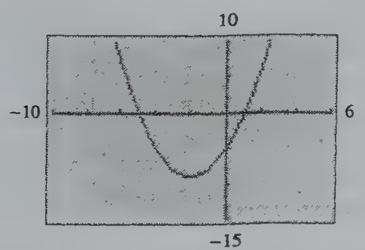


26. $f(x) = 0.0025x^4 + 0.02x^3 - 0.48x^2 + 0.08x + 4$

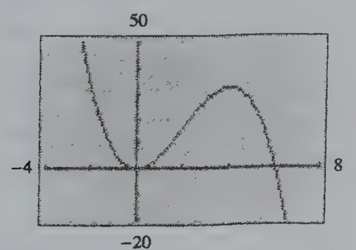


In Problems 27 and 28, $f'(x)$ and its graph are given.
(a) Use the graph of $f'(x)$ to determine (estimate) where the graph of $f(x)$ is increasing, where it is decreasing, and where it has relative extrema.
(b) Use the graph of $f'(x)$ to determine where $f''(x) > 0$, where $f''(x) < 0$, and where $f''(x) = 0$.
(c) Verify that the given $f(x)$ has $f'(x)$ as its derivative, and graph $f(x)$ to check your conclusions in (a).
(d) Calculate $f''(x)$ and graph it to check your conclusions in (b).

27. $f'(x) = x^2 + 4x - 5 \quad \left(\text{for } f(x) = \frac{x^3}{3} + 2x^2 - 5x\right)$

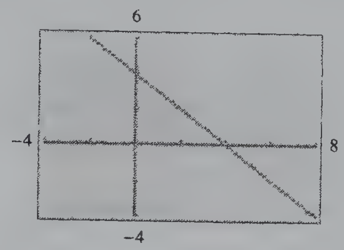


28. $f'(x) = 6x^2 - x^3 \quad \left(\text{for } f(x) = 2x^3 - \frac{x^4}{4}\right)$

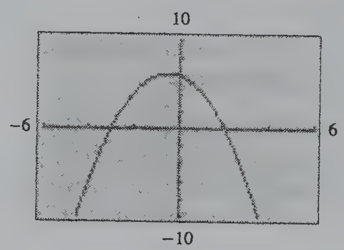


In Problems 29 and 30, $f''(x)$ and its graph are given.
(a) Use the graph to determine (estimate) where the graph of $f(x)$ is concave upward, where it is concave downward, and where it has points of inflection.
(b) Verify that the given $f(x)$ has $f''(x)$ as its second derivative, and graph $f(x)$ to check your conclusions in (a).

29. $f''(x) = 4 - x \quad \left(\text{for } f(x) = 2x^2 - \frac{x^3}{6}\right)$



30. $f''(x) = 6 - x - x^2 \quad \left(\text{for } f(x) = 3x^2 - \frac{x^3}{6} - \frac{x^4}{12}\right)$



Applications

Section 10.3

31. ***Cost*** Suppose the total cost function for a product is

$$C(x) = 3x^2 + 15x + 75$$

How many units minimize the average cost? Find the minimum average cost.

32. ***Revenue*** Suppose the total revenue function for a product is given by

$$R(x) = 32x - 0.01x^2$$

(a) How many units will maximize the total revenue? Find the maximum revenue.
(b) If production is limited to 2500 units, how many units will maximize the total revenue? Find the maximum revenue.

33. ***Profit*** Suppose the profit function for a product is

$$P(x) = 1080x + 9.6x^2 - 0.1x^3 - 50{,}000$$

Find the maximum profit.

34. ***Profit*** How many units (x) will maximize profit if $R(x) = 46x - x^2$ and $C(x) = 5x^2 + 10x + 3$?

35. ***Profit*** A product can be produced at a total cost $C(x) = 800 + 4x$, where x is the number produced and is limited to at most 150 units. If the total revenue is given by $R(x) = 80x - \frac{1}{4}x^2$, determine the level of production that will maximize the profit.

36. ***Average cost*** The total cost function for a product is $C = 2x^2 + 54x + 98$. Producing how many units will minimize average cost?

37. ***Revenue*** McRobert's TV Shop sells 200 sets per month at a price of \$400 per unit. Market research indicates that the shop can sell one additional set for each \$1 that it reduces the price. At what selling price will the shop maximize revenue?

38. ***Profit*** If, in Problem 37, the sets cost the shop \$250 each, when will profit be maximized?

39. ***Profit*** Suppose that for a product in a competitive market, the demand function is $p = 1200 - 2x$ and the supply function is $p = 200 + 2x$, where x is the number of units. A firm's average cost function for this product is

$$\overline{C}(x) = \frac{12{,}000}{x} + 50 + x$$

Find the maximum profit. *Hint:* First find the equilibrium price.

40. ***Profit*** The monthly demand function for a product sold by a monopoly is $p = 800 - x$, and its average cost is $\overline{C} = 200 + x$.
    (a) Determine the quantity that will maximize profit.
    (b) Find the selling price at the optimal quantity.

41. ***Profit*** Suppose that in a monopolistic market, the demand function for a commodity is

$$p = 7000 - 10x - \frac{x^2}{3}$$

If a company's average cost function for this commodity is

$$\overline{C}(x) = \frac{40{,}000}{x} + 600 + 8x$$

find the maximum profit.

Section 10.4

42. ***Reaction to a drug*** The reaction R to an injection of a drug is related to the dosage x according to

$$R(x) = x^2\left(500 - \frac{x}{3}\right)$$

Find the dosage that yields the maximum reaction.

43. ***Productivity*** The number of parts produced per hour by a worker is given by

$$N = 4 + 3t^2 - t^3$$

where t is the number of hours on the job without a break. If the worker starts at 8 A.M., when will she be at maximum productivity during the morning?

44. ***Population*** Population estimates show that the equation $P = 300 + 10t - t^2$ represents the size of the graduating class of a high school, where t represents the number of years after 1990, $0 \le t \le 10$. What will be the largest graduating class in the decade?

45. ***Night brightness*** Suppose that an observatory is to be built between cities A and B, which are 30 miles apart. For the best viewing, the observatory should be located where the night brightness from these cities is minimum. If the night brightness of city A is 8 times that of city B, then the night brightness b between the two cities and x miles from A is given by

$$b = \frac{8k}{x^2} + \frac{k}{(30 - x)^2}$$

where k is a constant. Find the best location for the observatory; that is, find x that minimizes b.

46. ***Product design*** A playpen manufacturer wants to make a rectangular enclosure with maximum play area. To remain competitive, he wants the perimeter of the base to be only 16 feet. What dimensions should the playpen have?

47. ***Printing design*** A printed page is to contain 56 square inches and have a $\frac{3}{4}$-inch margin at the bottom and 1-inch margins at the top and on both sides. Find the dimensions that minimize the size of the page (and hence the costs for paper).

48. ***Drug sensitivity*** The reaction R to an injection of a drug is related to the dosage x according to

$$R(x) = x^2\left(500 - \frac{x}{3}\right)$$

The sensitivity to the drug is defined by dR/dx. Find the dosage that maximizes sensitivity.

49. ***Photosynthesis*** The amount of photosynthesis that takes place in a certain plant depends on the intensity of light x according to the equation

$$f(x) = 145x^2 - 30x^3$$

The rate of change of the amount of photosynthesis with respect to the intensity is $f'(x)$. Find the intensity that maximizes the rate of change.

50. ***Inventory cost model*** A company needs 288,000 items per year. Production costs are \$1500 to prepare for a production run and \$30 for each item produced. Inventory costs are \$1.50 per year for each item stored. Find the number of items that should be produced in each run so that the total costs of production and storage are minimum.

CHAPTER TEST

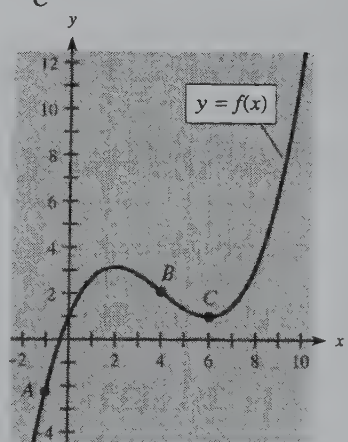
Find the local maxima, local minima, horizontal points of inflection, and asymptotes, if they exist, for each of the functions in Problems 1–3. Graph each function.

1. $f(x) = x^3 + 6x^2 + 9x + 3$
2. $y = x^3 - 3x^2 + 3x + 4$
3. $y = \dfrac{x^2 - 3x + 6}{x - 2}$

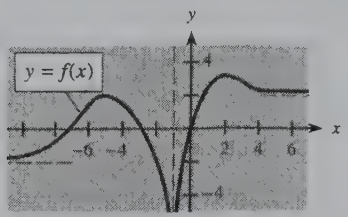
In Problems 4–6, use the function $y = 3x^5 - 5x^3 + 2$.

4. Over what intervals is the graph of this function concave upward?
5. Find the points of inflection of this function.
6. Find the relative maxima and minima of this function.
7. Find the absolute maximum and minimum for $f(x) = 2x^3 - 15x^2 + 3$ on the interval $[-2, 8]$.
8. Find all horizontal and vertical asymptotes for the function $f(x) = \dfrac{200x - 500}{x + 300}$.
9. Use the graph of $y = f(x)$ and the indicated points to complete the following chart. Enter +, −, or 0, according to whether f, f', and f'' are positive, negative, or zero at each point.

Point	f	f'	f''
A			
B			
C			



10. Use the figure to complete the following.
    (a) $\lim_{x \to -\infty} f(x) = ?$
    (b) What is the vertical asymptote?



11. If $f(6) = 10$, $f'(6) = 0$, and $f''(6) = -3$, what can we conclude about the point on the graph of $y = f(x)$ where $x = 6$? Explain.
12. If x represents the number of years past 1900, the equation that models the number of Catholics per priest in the United States is

$$y = 0.19x^2 - 16.59x + 1038.29$$

(Source: Dr. Robert G. Kennedy, University of St. Thomas)
    (a) During which year does this model indicate that the number of Catholics per priest is a minimum?
    (b) What does this mean about the number of priests relative to the number of Catholics in the United States?
13. The revenue function for a product is $R(x) = 164x$ and the cost function for the product is

$$C(x) = 0.01x^2 + 20x + 300$$

where x is the number of units produced and sold.
    (a) Selling how many units of the product gives maximum profit?
    (b) What is the maximum possible profit?
14. The cost of producing x units of a product is given by

$$C(x) = 100 + 20x + 0.01x^2$$

Producing how many units will give a minimum average cost?
15. A firm sells 100 TV sets per month at \$300 each, but market research indicates that it can sell 1 more set per month for each \$2 reduction of the price. At what price will the revenue be maximized?
16. An open-top box is made by cutting squares from the corners of a piece of tin and folding up the sides. If the piece of tin was originally 20 centimeters on a side, how long should the sides of the removed squares be to maximize the resulting volume?
17. A company estimates that it will need 784,000 items during the coming year. It costs \$420 to manufacture each item, \$2500 to prepare for each production run, and \$5 per year for each item stored. How many units should be in each production run so that the total costs of production and storage are minimized?
18. The table below gives the national debt, in trillions of dollars, for the years 1990–1997, with White House estimates for 1998 and 1999.
    (a) Use x as the number of years past 1900 to create a model using the data from 1990 to 1997.
    (b) Find what year the model indicates that the national debt reaches its maximum.

Year	*National Debt (\$ Trillions)*
1990	3.2
1991	3.6
1992	4.0
1993	4.4
1994	4.6
1995	4.9
1996	5.2
1997	5.4
1998	5.5
1999	5.7

Source: *USA Today*, Jan. 7, 1998

Extended Applications / Group Projects

I. Production Management

Metal Containers, Inc., is reviewing the way it submits bids on U.S. Army contracts. The army often requests open-top boxes, with square bases and of specified volumes. The army also specifies the materials for the boxes, and the base is usually made of a different material than the sides. The box is put together by riveting a bracket at each of the eight corners. For Metal Containers, the total cost of producing a box is the sum of the cost of the materials for the box and the labor costs associated with affixing each bracket.

Instead of estimating each job separately, the company wants to develop an overall approach that will allow it to cost out proposals more easily. To accomplish this, company managers need you to devise a formula for the total cost of producing each box and determine the dimensions that allow a box of specified volume to be produced at minimum cost. Use the following notation to help you solve this problem.

Cost of the material for the base $= A$ per square unit

Cost of the material for the sides $= B$ per square unit

Cost of each bracket $= C$

Cost to affix each bracket $= D$

Length of the sides of the base $= x$

Height of the box $= h$

Volume specified by the army $= V$

1. Write an expression for the company's total cost in terms of these quantities.
2. At the time an order is received for boxes of a specified volume, the costs of the materials and labor will be fixed and only the dimensions will vary. Find a formula for each dimension of the box so that the total cost is a minimum.
3. The army requests bids on boxes of 48 cubic feet with base material costing the container company \$12 per square foot and side material costing \$8 per square foot. Each bracket costs \$5, and the associated labor cost is \$1 per bracket. Use your formulas to find the dimensions of the box that meet the army's requirements at a minimum cost. What is this cost?

Metal Containers asks you to determine how best to order the brackets it uses on its boxes. You are able to obtain the following information: The company uses approximately 100,000 brackets a year, and the purchase price of each is \$5. It buys the same number of brackets (say, n) each time it places an order with the supplier, and it costs \$60 to process each order. Metal Containers also has additional costs associated with storing, insuring, and financing its inventory of brackets. These carrying costs amount to 15% of the average value of inventory annually. The brackets are used steadily and deliveries are made just as inventory reaches zero, so that inventory fluctuates between zero and n brackets.

4. If the total annual cost associated with the bracket supply is the sum of the annual purchasing cost and the annual carrying costs, what order size n would minimize the total cost?
5. In the general case of the bracket-ordering problem, the order size n that minimizes the total cost of the bracket supply is called the economic order quantity, or EOQ. Use the following notations to determine a general formula for the EOQ.

Fixed cost per order $= F$	Unit cost $= C$
Quantity purchased per year $= P$	Carrying cost (as a decimal rate) $= r$

II. Room Pricing in the Off Season

The data in the table below, from a survey of resort hotels with comparable rates on Hilton Head Island, show that room occupancy during the off season (November through February) is related to the price charged for a basic room.

Price per Day	*Occupancy Rate, %*
\$ 69	53
89	47
95	46
99	45
109	40
129	32

The goal is to use these data to help answer the following questions.

A. What price per day will maximize the daily off season revenue for a typical hotel in this group if it has 200 rooms available?
B. Suppose that for this typical hotel the daily cost is \$4592 plus \$30 per occupied room. What price will maximize the profit for this hotel in the off season?

The price per day that will maximize the off season profit for this typical hotel applies to this group of hotels. To find the room price per day that will maximize the daily revenue and the room price per day that will maximize the profit for this hotel (and thus the group of hotels) in the off season, complete the following.

1. Multiply each occupancy rate by 200 to get the hypothetical room occupancy. Create the revenue data points that compare the price with the revenue, R, which is equal to price times the room occupancy.
2. Use technology to create an equation that models the revenue, R, as a function of the price per day, x.
3. Use maximization techniques to find the price that these hotels should charge to maximize the daily revenue.
4. Use technology to get the occupancy as a function of the price, and use the occupancy function to create a daily cost function.
5. Form the profit function.
6. Use maximization techniques to find the price that will maximize the profit.

Warm-up

Prerequisite Problem Type	For Section	Answer	Section for Review
(a) Simplify: $\frac{1}{(3x)^{1/2}} \cdot \frac{1}{2}(3x)^{-1/2} \cdot 3$ (b) Write with a positive exponent: $\sqrt{x^2 - 1}$	**11.1**	(a) $\frac{1}{2x}$ (b) $(x^2 - 1)^{1/2}$	0.4 Rational exponents
(a) Vertical lines have ______ slopes. (b) Horizontal lines have ______ slopes.	**11.3**	(a) Undefined (b) 0	1.3 Slopes
Write the equation of the line passing through $(-2, -2)$ with slope 5.	**11.3**	$y = 5x + 8$	1.3 Equations of lines
Solve: $x^2 + y^2 - 9 = 0$, for y.	**11.3**	$y = \pm\sqrt{9 - x^2}$	2.1 Quadratic equations
(a) Write $\log_a(x + h) - \log_a x$ as an expression involving one logarithm. (b) Does $\ln x^4 = 4 \ln x$? (c) Does $\frac{x}{h}\log_a\left(\frac{x+h}{x}\right) = \log_a\left(1 + \frac{h}{x}\right)^{x/h}$? (d) Expand $\ln(xy)$ to separate x and y. (e) If $y = a^x$, then $x =$ ______. (f) Simplify $\ln e^x$.	**11.1** **11.2**	(a) $\log_a\left(\frac{x+h}{x}\right)$ (b) Yes (c) Yes (d) $\ln x + \ln y$ (e) $\log_a y$ (f) x	5.2 Logarithms
Find the derivative of (a) $y = x^2 - 2x - 2$ (b) $T(q) = 400q - \frac{4}{3}q^2$ (c) $y = \sqrt{9 - x^2}$	**11.1–11.5**	(a) $y' = 2x - 2$ (b) $T'(q) = 400 - \frac{8}{3}q$ (c) $y' = -x(9 - x^2)^{-1/2}$	9.4–9.6 Derivatives
If $\frac{dy}{dx} = \frac{-x}{\sqrt{9 - x^2}}$, find the slope of the tangent to $y = f(x)$ at $x = \sqrt{5}$.	**11.3**	Slope $= -\frac{\sqrt{5}}{2}$	9.3–9.7 Derivatives

Chapter 11

Derivatives Continued

In this chapter we will develop derivative formulas for logarithmic and exponential functions and apply them to problems in the management, life, and social sciences. We will also develop methods for finding derivatives of one variable with respect to another even though the relationship between them may not be functional. This method is called ***implicit differentiation.***

We will use implicit derivatives with respect to time to solve problems involving the rates of change of two or more variables. These problems are called ***related-rates problems.***

The special business and economics applications include elasticity of demand and maximization of taxation revenue.

11.1 Derivatives of Logarithmic Functions

OBJECTIVE

- *To find derivatives of logarithmic functions*

APPLICATION PREVIEW

The table below shows the expected life span at birth for people born in certain years in the United States. Assume that for these years, the expected life span at birth can be modeled with the function $l(x) = 11.64 + 14.14 \ln x$, where x is the number of years past 1900. The graph of this function is shown in Figure 11.1. If we wanted to use this model to find the rate of change of life span with respect to the number of years past 1900, we would need the derivative of this function and hence the derivative of the logarithmic function $\ln x$.

Year	Life Span (years)	Year	Life Span (years)
1920	54.1	1987	75.0
1930	59.7	1988	74.9
1940	62.9	1989	75.2
1950	68.2	1990	75.4
1960	69.7	1991	75.5
1970	70.8	1993	75.5
1975	72.6	1994	75.7
1980	73.7	1996	76.1

Source: National Center for Health Statistics, 1997

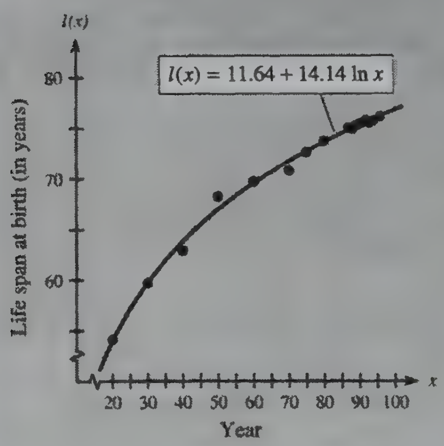


Figure 11.1

In Chapter 5, "Exponential and Logarithmic Functions," we defined the logarithmic function $y = \log_a x$ and developed properties of logarithms that will be important in this section. Although logarithmic functions can involve logarithms of any base, including base 10 (called **common logarithms**), most of the problems in calculus and many of the applications to the management, life, and social sciences involve logarithms with base e, called **natural logarithms.** We state the properties of logarithms for natural logarithms next.

Properties of Natural Logarithms

I. $\ln e^x = x$, for any real number x
II. $e^{\ln x} = x$, for $x > 0$
III. $\ln(MN) = \ln M + \ln N$, for M and N positive real numbers
IV. $\ln(M/N) = \ln M - \ln N$, for M and N positive real numbers
V. $\ln(M^N) = N \ln M$, for M a positive real number and N any real number

The formula for the derivative of $y = \ln x$ follows.

Derivative of $y = \ln x$

If $y = \ln x$, then $\dfrac{dy}{dx} = \dfrac{1}{x}$.

The proof follows.

If $y = f(x) = \ln x$, then

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to0} \frac{\ln(x+h) - \ln x}{h} \\
&= \lim_{h\to0} \frac{\ln\left(\dfrac{x+h}{x}\right)}{h} && \text{Property IV} \\
&= \lim_{h\to0} \frac{x}{x} \cdot \frac{1}{h} \ln\left(\frac{x+h}{x}\right) && \text{Introduce } \tfrac{x}{x}. \\
&= \lim_{h\to0} \frac{1}{x} \cdot \frac{x}{h} \ln\left(1 + \frac{h}{x}\right) \\
&= \lim_{h\to0} \frac{1}{x} \ln\left(1 + \frac{h}{x}\right)^{x/h} && \text{Property V}
\end{aligned}$$

The natural logarithmic function is continuous when it is defined, so

$$\frac{dy}{dx} = \frac{1}{x} \ln\left[\lim_{h\to0}\left(1 + \frac{h}{x}\right)^{x/h}\right]$$

We can evaluate

$$\lim_{h\to0}\left(1 + \frac{h}{x}\right)^{x/h} \tag{1}$$

by recalling from Problem 51 of Exercise 9.1 that

$$\lim_{a\to0}(1 + a)^{1/a} = e$$

and noting that equation (1) has this form. Thus

$$\lim_{h\to0}\left(1 + \frac{h}{x}\right)^{x/h} = e$$

and

$$\frac{dy}{dx} = \frac{1}{x} \ln e = \frac{1}{x}$$

EXAMPLE 1

If $y = x^3 + 3\ln x$, find dy/dx.

Solution

$$\frac{dy}{dx} = 3x^2 + 3\left(\frac{1}{x}\right) = 3x^2 + \frac{3}{x}$$

EXAMPLE 2

If $y = x^2 \ln x$, find y'.

Solution

By the Product Rule,

$$y' = x^2 \cdot \frac{1}{x} + (\ln x)(2x) = x + 2x \ln x$$

We can use the Chain Rule to find the formula for the derivative of $y = \ln u$, where $u = f(x)$.

Derivatives of the Natural Logarithmic Functions

If $y = \ln u$, where u is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$$

EXAMPLE 3

Find $f'(x)$ for each of the following.

(a) $f(x) = \ln(x^2)$ (b) $f(x) = \ln(x^4 - 3x + 7)$

Solution

(a) $f'(x) = \dfrac{1}{x^2}(2x) = \dfrac{2x}{x^2} = \dfrac{2}{x}$

(b) $f'(x) = \dfrac{1}{x^4 - 3x + 7}(4x^3 - 3) = \dfrac{4x^3 - 3}{x^4 - 3x + 7}$

EXAMPLE 4

(a) Find $f'(x)$ if $f(x) = \frac{1}{3}\ln(2x^6 - 3x + 2)$.

(b) Find $g'(x)$ if $g(x) = \dfrac{\ln(2x + 1)}{2x + 1}$.

Solution

(a) $f'(x)$ is $\frac{1}{3}$ of the derivative of $\ln(2x^6 - 3x + 2)$.

$$\begin{aligned} f'(x) &= \frac{1}{3} \cdot \frac{1}{2x^6 - 3x + 2}(12x^5 - 3) \\ &= \frac{4x^5 - 1}{2x^6 - 3x + 2} \end{aligned}$$

(b) We begin with the Quotient Rule.

$$g'(x) = \frac{(2x+1)\frac{1}{2x+1}(2) - [\ln(2x+1)]2}{(2x+1)^2}$$
$$= \frac{2 - 2\ln(2x+1)}{(2x+1)^2}$$

EXAMPLE 5

Use logarithm properties to find dy/dx when

$$y = \ln[x(x^5 - 2)^{10}]$$

Solution

We use logarithm Properties III and V to rewrite the function.

$$y = \ln x + \ln(x^5 - 2)^{10} \qquad \text{Property III}$$
$$y = \ln x + 10\ln(x^5 - 2) \qquad \text{Property V}$$

We now take the derivative.

$$\frac{dy}{dx} = \frac{1}{x} + 10 \cdot \frac{1}{x^5 - 2} \cdot 5x^4$$
$$= \frac{1}{x} + \frac{50x^4}{x^5 - 2}$$

CHECKPOINT

1. If $y = \ln(3x^2 + 2)$, find y'.
2. If $y = \ln x^6$, find y'.
3. If $y = \ln \sqrt[3]{x^2 + 1}$, find y'.

We now return to the life span model in the Application Preview.

EXAMPLE 6

Assume that the average life span (in years) for people born in 1920–1989 can be modeled by

$$l(x) = 11.64 + 14.14\ln x$$

where x is the number of years past 1900.

(a) Find the function that models the rate of change of life span.
(b) Does $l(x)$ have a maximum value for $x > 0$?
(c) Evaluate $\lim_{x\to\infty} l'(x)$.
(d) What do (b) and (c) tell us about the average life span?

Solution

(a) The rate of change of life span is given by the derivative.

$$l'(x) = 0 + 14.14\left(\frac{1}{x}\right) = \frac{14.14}{x}$$

(b) For $x > 0$, we see that $l'(x) > 0$. Hence $l(x)$ is increasing for all values of $x > 0$, so $l(x)$ never achieves a maximum value. That is, there is no maximum life span.

(c) $\lim_{x\to\infty} l'(x) = \lim_{x\to\infty} \frac{14.14}{x} = 0$

(d) If this model is accurate, life span will continue to increase, but at an ever slower rate.

EXAMPLE 7

Suppose the cost function for x units of a product is given by

$$C(x) = 18{,}250 + 615 \ln(4x + 10)$$

where $C(x)$ is in dollars. Find the marginal cost when 100 units are produced, and explain what it means.

Solution

Marginal cost is given by $C'(x)$.

$$\overline{MC} = C'(x) = 615\left(\frac{1}{4x+10}\right)(4) = \frac{2460}{4x+10}$$

$$\overline{MC}(100) = \frac{2460}{4(100)+10} = \frac{2460}{410} = 6$$

When 100 units are produced, the marginal cost is 6. This means that the approximate cost of producing the 101st item is \$6.

The **change-of-base formula,** introduced in Section 5.2, "Logarithmic Functions and Their Properties," can be used to express logarithms with base a as natural logarithms (that is, logarithms with base e):

$$\log_a x = \frac{\ln x}{\ln a}$$

We can apply this change-of-base formula to find the derivative of a logarithm with any base, as the following example illustrates.

EXAMPLE 8

If $y = \log_4(x^3 + 1)$, find dy/dx.

Solution

By using the change-of-base formula, we have

$$y = \log_4(x^3+1) = \frac{\ln(x^3+1)}{\ln 4} = \frac{1}{\ln 4} \cdot \ln(x^3+1)$$

Thus

$$\frac{dy}{dx} = \frac{1}{\ln 4} \cdot \frac{1}{x^3+1} \cdot 3x^2 = \frac{3x^2}{(x^3+1)\ln 4}$$

EXAMPLE 9

Let $f(x) = x \ln x - x$. Use the graph of the derivative of $f(x)$ for $x > 0$ to answer the following questions.

(a) At what value a does the graph of $f'(x)$ cross the x-axis (that is, where is $f'(x) = 0$)?
(b) What value a is a critical value for $y = f(x)$?
(c) Does $f(x)$ have a relative maximum or a relative minimum at $x = a$?

Solution

$f'(x) = x \cdot \dfrac{1}{x} + \ln x - 1 = \ln x$. The graph of $f'(x) = \ln x$ is shown in Figure 11.2.

(a) The graph crosses the x-axis at $x = 1$, so $f'(a) = 0$ if $a = 1$.
(b) $a = 1$ is a critical value of $f(x)$.
(c) Because $f'(x)$ is negative for $x < 1$, $f(x)$ is decreasing for $x < 1$.
Because $f'(x)$ is positive for $x > 1$, $f(x)$ is increasing for $x > 1$.
Therefore, $f(x)$ has a relative minimum at $x = 1$.
The graph of $y = x \ln x - x$ is shown in Figure 11.3. It has a relative minimum at $x = 1$.

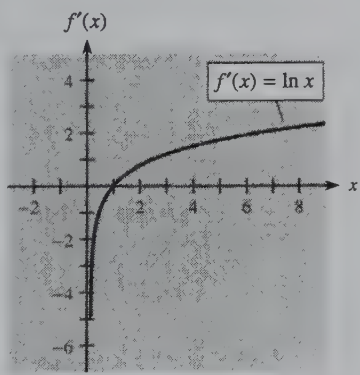


Figure 11.2

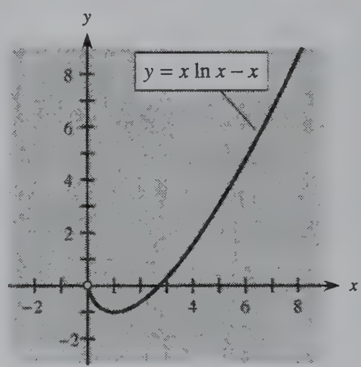


Figure 11.3

CHECKPOINT SOLUTIONS

1. $y' = \dfrac{6x}{3x^2 + 2}$

2. $y' = \dfrac{1}{x^6} \cdot 6x^5 = \dfrac{6}{x}$; or, by first using logarithm Property V, we get $y = 6 \ln x$, so $y' = 6\left(\dfrac{1}{x}\right) = \dfrac{6}{x}$.

3. By first using logarithm Property V, we get $y = \ln(x^2 + 1)^{1/3} = \frac{1}{3}\ln(x^2 + 1)$. Then

$$y' = \frac{1}{3} \cdot \frac{2x}{x^2 + 1} = \frac{2x}{3(x^2 + 1)}$$

EXERCISE 11.1

Find the derivatives of the functions in Problems 1–10.

1. $f(x) = 4 \ln x$
2. $y = 3 \ln x$
3. $y = \ln 8x$
4. $y = \ln 5x$
5. $y = \ln x^4$
6. $f(x) = \ln x^3$
7. $f(x) = \ln(4x + 9)$
8. $y = \ln(6x + 1)$
9. $y = \ln(2x^2 - x) + 3x$
10. $y = \ln(8x^3 - 2x) - 2x$
11. Find dp/dq if $p = \ln(q^2 + 1)$.
12. Find $\dfrac{ds}{dq}$ if $s = \ln\left(\dfrac{q^2}{4} + 1\right)$.

In each of Problems 13–20, find the derivative of the function in (a). Then find the derivative of the function in (b) or show that the function in (b) is the same function as that in (a).

13. (a) $y = \ln x - \ln(x - 1)$
    (b) $y = \ln \dfrac{x}{x - 1}$
14. (a) $y = \ln(x - 1) + \ln(2x + 1)$
    (b) $y = \ln[(x - 1)(2x + 1)]$
15. (a) $y = \frac{1}{3} \ln(x^2 - 1)$
    (b) $y = \ln \sqrt[3]{x^2 - 1}$
16. (a) $y = 3 \ln(x^4 - 1)$
    (b) $y = \ln(x^4 - 1)^3$
17. (a) $y = \ln(4x - 1) - 3 \ln x$
    (b) $y = \ln\left(\dfrac{4x - 1}{x^3}\right)$
18. (a) $y = 3 \ln x - \ln(x + 1)$
    (b) $y = \ln\left(\dfrac{x^3}{x + 1}\right)$
19. (a) $y = 3 \ln x + \frac{1}{2} \ln(x + 1)$
    (b) $y = \ln(x^3 \sqrt{x + 1})$
20. (a) $y = 2 \ln x + \ln(x^4 - x + 1)$
    (b) $y = \ln[x^2(x^4 - x + 1)]$
21. Find $\dfrac{dp}{dq}$ if $p = \ln\left(\dfrac{q^2 - 1}{q}\right)$.
22. Find $\dfrac{ds}{dt}$ if $s = \ln[t^3(t^2 - 1)]$.
23. Find $\dfrac{dy}{dt}$ if $y = \ln\left(\dfrac{t^2 + 3}{\sqrt{1 - t}}\right)$.
24. Find $\dfrac{dy}{dx}$ if $y = \ln\left(\dfrac{3x + 2}{x^2 - 5}\right)^{1/4}$.

In Problems 25–38, find dy/dx.

25. $y = x - \ln x$
26. $y = x^2 \ln(2x + 3)$
27. $y = \dfrac{\ln x}{x}$
28. $y = \dfrac{1 + \ln x}{x^2}$
29. $y = \ln(x^4 + 3)^2$
30. $y = \ln(3x + 1)^{1/2}$
31. $y = (\ln x)^4$
32. $y = (\ln x)^{-1}$
33. $y = [\ln(x^4 + 3)]^2$
34. $y = \sqrt{\ln(3x + 1)}$
35. $y = \log_4 x$
36. $y = \log_5 x$
37. $y = \log_6(x^4 - 4x^3 + 1)$
38. $y = \log_2(1 - x - x^2)$

In Problems 39–42, find the relative maxima and relative minima, and sketch the graph with a graphing utility to check your results.

39. $y = x \ln x$
40. $y = x^2 \ln x$
41. $y = x^2 - 8 \ln x$
42. $y = \ln x - x$

Applications

43. ***Marginal cost*** Suppose that the total cost (in dollars) for a product is given by

$$C(x) = 1500 + 200 \ln(2x + 1)$$

where x is the number of units produced.
(a) Find the marginal cost function.
(b) Find the marginal cost when 200 units are produced, and interpret your result.

44. ***Investing*** The number of years t that it takes for an investment to double is a function of the interest rate r, compounded continuously, according to

$$t = \frac{\ln 2}{r}$$

At what rate is the required time changing with respect to r if $r = 10\%$, compounded continuously?

45. ***Marginal revenue*** The total revenue from the sale of x units of a product is given by

$$R(x) = \frac{2500x}{\ln(10x + 10)}$$

(a) Find the marginal revenue function.
(b) Find the marginal revenue when 100 units are sold, and interpret your result.

46. ***Supply*** Suppose that the supply of q units of a product at price x dollars is given by

$$q = 10 + 50 \ln(3x + 1)$$

Find the rate of change of supply with respect to price.

47. ***Demand*** The demand function for a product is given by $p = 4000/\ln(x + 10)$, where p is the price per unit when x units are demanded.
(a) Find the rate of change of price with respect to the number of units sold when 40 units are sold.
(b) Find the rate of change of price with respect to the number of units sold when 90 units are sold.
(c) Find the second derivative to see whether the rate at which the price is changing at 40 units is increasing or decreasing.

48. ***pH level*** The pH of a solution is given by

$$pH = -\log[H^+]$$

where $[H^+]$ is the concentration of hydrogen ions (in gram atoms per liter). What is the rate of change of pH with respect to $[H^+]$?

49. ***Reynolds number*** If the Reynolds number relating to the flow of blood exceeds R, where

$$R = A\ln(r) - Br$$

and r is the radius of the aorta and A and B are positive constants, the blood flow becomes turbulent. What is the radius r that makes R a maximum?

50. ***Decibels*** The loudness of sound (L, measured in decibels) perceived by the human ear depends on intensity levels (I) according to

$$L = 10\log(I/I_0)$$

where I_0 is the standard threshold of audibility. Using the change-of-base formula, we get

$$L = \frac{10\ln(I/I_0)}{\ln 10}$$

At what rate is the loudness changing with respect to the intensity when the intensity is 100 times the standard threshold of audibility?

51. ***Richter scale*** The Richter scale reading, R, used for measuring the magnitude of an earthquake with intensity I is determined by

$$R = \frac{\ln(I/I_0)}{\ln 10}$$

where I_0 is a standard minimum threshold of intensity. If $I_0 = 1$, what is the rate of change of the Richter scale reading with respect to intensity?

52. ***Violent crime*** The following data represent the number of violent crimes (murders, rapes, and armed robberies) per 100,000 people for the years 1987–1992.

Year	*Violent Crimes (per 100,000)*
1987	610
1988	637
1989	663
1990	732
1991	758
1992	765

Source: FBI Crime Report

Letting $t = 0$ in 1980, these data can be modeled by $y = -22.9 + 321\ln t$. If this model is accurate, at what rate did violent crimes change in 1990?

53. ***Poverty threshold*** The table below gives the average poverty thresholds for individuals for 1987–1994.
(a) Use a logarithmic equation to model these data, with x equal to the number of years past 1980.
(b) Use this model to predict the rate at which the poverty threshold will be growing in 2003.

Year	*Poverty Threshold Income*
1987	$5778
1988	6022
1989	6310
1990	6652
1991	6932
1992	7143
1993	7363
1994	7547

Source: U.S. Bureau of the Census

54. ***Grade point averages*** The core grade point averages (based on 15 college-preparatory high school courses) of University of South Carolina freshman classes from 1992 to 1997 are shown in the table below.
(a) Find a logarithmic equation that models GPA as a function of the year of entering USC. (That is, use $x = 0$ as 1900.)
(b) Using this model, find the rate at which the grade point average was changing in 1997.

Year	*Core H.S. Grade Point Average*
1992	2.81
1993	2.91
1994	2.93
1995	3.02
1996	3.10
1997	3.11

Source: University of South Carolina

11.2 Derivatives of Exponential Functions

OBJECTIVE

- *To find derivatives of exponential functions*

APPLICATION PREVIEW

We saw in Chapter 6, "Mathematics of Finance," that the amount that accrues when \$100 is invested at 8%, compounded continuously, is

$$S(t) = 100e^{0.08t}$$

where t is the number of years. If we want to find the rate at which the money in this account is growing at the end of 1 year, then we need to find the derivative of this function, which is an exponential function.

In the previous section we found derivatives of logarithmic functions. In this section we turn our attention to exponential functions. The formula for the derivative of $y = e^x$ is developed as follows.

From Property I of logarithms, we know that

$$\ln e^x = x$$

Taking the derivative, with respect to x, of both sides of this equation, we have

$$\frac{d}{dx} \ln e^x = \frac{d}{dx} x$$

Using the Chain Rule for logarithms gives

$$\frac{1}{e^x} \cdot \frac{d}{dx} e^x = 1$$

and solving for $\frac{d}{dx} e^x$ yields

$$\frac{d}{dx} e^x = e^x$$

Thus we can conclude the following.

Derivative of $y = e^x$

If $y = e^x$, then $\frac{dy}{dx} = e^x$.

EXAMPLE 1

If $p = e^q$, find dp/dq.

Solution

$$\frac{dp}{dq} = e^q$$

As with logarithmic functions, the Chain Rule permits us to expand our derivative formulas.

Derivatives of Exponential Functions

If $y = e^u$, where u is a differentiable function of x, then

$$\frac{dy}{dx} = e^u \cdot \frac{du}{dx}$$

EXAMPLE 2

If $f(x) = e^{4x^3}$, find $f'(x)$.

Solution

$$f'(x) = e^{4x^3} \cdot 12x^2 = 12x^2e^{4x^3}$$

EXAMPLE 3

If $p = qe^{5q}$, find dp/dq.

Solution

Using the Product Rule, we get

$$\frac{dp}{dq} = (q)(e^{5q} \cdot 5) + (e^{5q})(1) = 5qe^{5q} + e^{5q}$$

EXAMPLE 4

If $s = 3te^{3t^2 + 5t}$, find ds/dt.

Solution

$$\begin{aligned}\frac{ds}{dt} &= 3t \cdot e^{3t^2+5t}(6t + 5) + e^{3t^2+5t} \cdot 3 \\ &= (18t^2 + 15t)e^{3t^2 + 5t} + 3e^{3t^2 + 5t}\end{aligned}$$

CHECKPOINT

1. If $y = 2e^{4x}$, find y'.
2. If $y = e^{x^2+6x}$, find y'.
3. If $s = te^{t^2}$, find ds/dt.

EXAMPLE 5

If $y = e^{\ln x^2}$, find y'.

Solution

$$y' = e^{\ln x^2} \cdot \frac{1}{x^2} \cdot 2x = \frac{2}{x}e^{\ln x^2}$$

By Property II of logarithms (see the previous section and Section 5.2, "Logarithmic Functions and Their Properties"), $e^{\ln u} = u$, and we can simplify the derivative to

$$y' = \frac{2}{x} \cdot x^2 = 2x$$

Note that if we had used this property *before* taking the derivative, we would have had

$$y = e^{\ln x^2} = x^2$$

Then the derivative is $y' = 2x$.

Let us now return to the problem described in the Application Preview.

EXAMPLE 6

The future value when \$100 is invested at 8%, compounded continuously, is $S(t) = 100e^{0.08t}$, where t is the number of years. At what rate is the money in this account growing

(a) at the end of 1 year?
(b) at the end of 10 years?

Solution

The rate of growth of the money is given by

$$S'(t) = 100e^{0.08t}(0.08) = 8e^{0.08t}$$

(a) The rate of growth of the money at the end of 1 year is

$$S'(1) = 8e^{0.08} = 8.666$$

Thus the future value will change by about \$8.67 during the next year.

(b) The rate of growth of the money at the end of 10 years is

$$S'(10) = 8e^{0.08(10)} = 17.804$$

Thus the future value will change by about \$17.80 during the next year.

EXAMPLE 7

If $u = w/e^{3w}$, find u'.

Solution

The function is a quotient, with the denominator equal to e^{3w}. Using the Quotient Rule gives

$$\begin{aligned} u' &= \frac{e^{3w} \cdot 1 - w \cdot e^{3w} \cdot 3}{(e^{3w})^2} \\ &= \frac{e^{3w} - 3we^{3w}}{e^{6w}} \\ &= \frac{1 - 3w}{e^{3w}} \end{aligned}$$

EXAMPLE 8

North Forty, Inc. is a wilderness camping equipment manufacturer. The revenue function for its best-selling backpack, the Sierra, can be modeled by the function

$$R(x) = 25xe^{(1 - 0.01x)}$$

where $R(x)$ is the revenue in thousands of dollars from the sale of x thousand Sierra backpacks. Find the marginal revenue when 75,000 packs are sold, and explain what it means.

Solution

The marginal revenue function is given by $R'(x)$, and to find this derivative we use the Product Rule.

$$R'(x) = \overline{MR} = 25x[e^{(1-0.01x)} \cdot (-0.01)] + e^{(1-0.01x)}(25)$$
$$\overline{MR} = 25e^{(1-0.01x)}(1 - 0.01x)$$

To find the marginal revenue when 75,000 packs are sold, we use $x = 75$.

$$\overline{MR}\,(75) = 25e^{(1-0.75)}(1 - 0.75) \approx 8.025$$

This means that the sale of one (thousand) more Sierra backpacks will yield approximately \$8.025 (thousand) in additional revenue.

CHECKPOINT

4. If the sales of a product are given by $S = 1000e^{-0.2x}$, where x is the number of days after the end of an advertising campaign, what is the rate of decline in sales 20 days after the end of the campaign?

In a manner similar to that used to find the derivative of $y = e^x$, we can develop a formula for the derivative of $y = a^x$ for any base a.

Derivative of $y = a^u$

If $y = a^x$, then

$$\frac{dy}{dx} = a^x \ln a$$

If $y = a^u$, where u is a differentiable function of x, then

$$\frac{dy}{dx} = a^u \frac{du}{dx} \ln a$$

EXAMPLE 9

If $y = 4^x$, find dy/dx.

Solution

$$\frac{dy}{dx} = 4^x \ln 4$$

Graphing Utilities

We can make use of a graphing utility to study the behavior of an exponential function and its derivative.

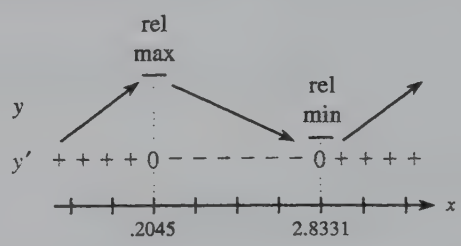
EXAMPLE 10

For the function $y = e^x - 3x^2$, complete the following.

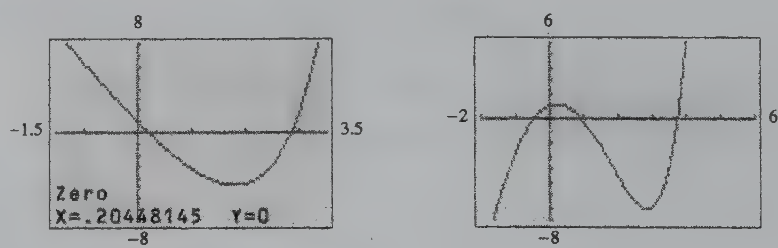
(a) Approximate the critical values of the function to four decimal places.
(b) Determine whether relative maxima or relative minima occur at the critical values.

Solution

(a) The derivative is $y' = e^x - 6x$. Using the built-in features of a graphing utility, we find that $y' = 0$ at $x = 0.2045$ (see Figure 11.4a) and at $x = 2.8331$.
(b) From the graph of $y' = e^x - 6x$ in Figure 11.4(a), we can observe where $y' > 0$ and where $y' < 0$. From this we can make a sign diagram to determine relative maxima and relative minima.



The graph of $y = e^x - 3x^2$ in Figure 11.4(b) shows that the relative maximum point is (0.2045, 1.1015) and that the relative minimum point is (2.8331, −7.0813).



Graph of $y' = e^x - 6x$
(a)

Graph of $y = e^x - 3x^2$
(b)

Figure 11.4

CHECKPOINT SOLUTIONS

1. $y' = 2e^{4x}(4) = 8e^{4x}$
2. $y' = (2x + 6)e^{x^2 + 6x}$
3. By the Product Rule, $\dfrac{ds}{dt} = e^{t^2}(1) + t[e^{t^2}(2t)] = e^{t^2} + 2t^2e^{t^2}$.
4. The rate of decline is given by dS/dx.

$$\frac{dS}{dx} = 1000e^{-0.2x}(-0.2) = -200e^{-0.2x}$$

$$\left.\frac{dS}{dx}\right|_{x=20} = -200e^{(-0.2)(20)}$$

$$= -200e^{-4} \approx -3.663 \text{ sales/day}$$

EXERCISE 11.2

Find the derivatives of the functions in Problems 1–32.

1. $y = 5e^x - x$
2. $y = x^2 - 3e^x$
3. $f(x) = e^x - x^e$
4. $f(x) = 4e^x - \ln x$
5. $y = e^{x^3}$
6. $y = e^{x^2-1}$
7. $y = 6e^{3x^2}$
8. $y = 1 - 2e^{-x^3}$
9. $y = 2e^{(x^2+1)^3}$
10. $y = e^{\sqrt{x^2-9}}$
11. $y = e^{\ln x^3}$
12. $y = e^3 + e^{\ln x}$
13. $y = e^{-1/x}$
14. $y = 2e^{\sqrt{x}}$
15. $y = e^{-1/x^2} + e^{-x^2}$
16. $y = \dfrac{2}{e^{2x}} + \dfrac{e^{2x}}{2}$
17. $s = t^2e^t$
18. $p = 4qe^{q^3}$
19. $y = e^{x^4} - (e^x)^4$
20. $y = 4(e^x)^3 - 4e^{x^3}$
21. $y = \ln(e^{4x} + 2)$
22. $y = \ln(e^{2x} + 1)$
23. $y = e^{-3x}\ln(2x)$
24. $y = e^{2x^2}\ln(4x)$
25. $y = \dfrac{1 + e^{5x}}{e^{3x}}$
26. $y = \dfrac{x}{1 + e^{2x}}$
27. $y = (e^{3x} + 4)^{10}$
28. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
29. $y = 6^x$
30. $y = 3^x$
31. $y = 4^{x^2}$
32. $y = 5^{x-1}$
33. (a) What is the slope of the line tangent to $y = xe^{-x}$ at $x = 1$?
    (b) Write the equation of the line tangent to the graph of $y = xe^{-x}$ at $x = 1$.
34. (a) What is the slope of the line tangent to $y = e^{-x}/(1 + e^{-x})$ at $x = 0$?
    (b) Write the equation of the line tangent to the graph of $y = e^{-x}/(1 + e^{-x})$ at $x = 0$.
35. The equation for the standard normal probability distribution is

$$y = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$$

    (a) At what value of z will the curve be at its highest point?
    (b) Graph this function with a graphing utility to verify your answer.
36. (a) Find the mode of the normal distribution* given by

$$y = \frac{1}{\sqrt{2\pi}}e^{-(x-10)^2/2}$$

    (b) What is the mean of this normal distribution?
    (c) Use a graphing utility to verify your answers.

*The mode occurs at the highest point on normal curves and equals the mean.

In Problems 37–40, find any relative maxima and minima. Use a graphing utility to check your results.

37. $y = \dfrac{e^x}{x}$
38. $y = \dfrac{x}{e^x}$
39. $y = x - e^x$
40. $y = \dfrac{x^2}{e^x}$

Applications

41. ***Future value*** If \$$P$ is invested for n years at 10%, compounded continuously, the future value is given by the function

$$S = Pe^{0.1n}$$

    (a) At what rate is the future value growing at any time (for any n)?
    (b) At what rate is the future value growing after 1 year ($n = 1$)?
    (c) Is the rate of growth of the future value after 1 year greater than 10%? Why?
42. ***Future value*** The future value that accrues when \$700 is invested at 9%, compounded continuously, is

$$S(t) = 700e^{0.09t}$$

    where t is the number of years.
    (a) At what rate is the money in this account growing when $t = 4$?
    (b) At what rate is it growing when $t = 10$?
43. ***Sales decay*** After the end of an advertising campaign, the sales of a product are given by

$$S = 100{,}000e^{-0.5t}$$

    where S is weekly sales and t is the number of weeks since the end of the campaign. Find the rate of change of S (that is, the rate of *sales decay*).
44. ***Sales decay*** The sales decay for a product is given by

$$S = 50{,}000e^{-0.8t}$$

    where S is the daily sales and t is the number of days since the end of a promotional campaign. Find the rate of sales decay.
45. ***Marginal cost*** Suppose that the total cost in dollars of producing x units of a product is given by

$$C(x) = 10{,}000 + 600xe^{x/600}$$

    Find the marginal cost when 600 units are produced.

46. ***Marginal revenue*** Suppose that the revenue in dollars from the sale of x units of a product is given by

$$R(x) = 1000xe^{-x/50}$$

Find the marginal revenue function.

47. ***Drugs in a bloodstream*** The concentration y of a certain drug in the bloodstream at any time t (in hours) is given by

$$y = 100(1 - e^{-0.462t})$$

Find the rate of change of the concentration after 1 hour. Give your answer to three decimal places.

48. ***Radioactive decay*** The amount of the radioactive isotope thorium-234 present at time t is given by

$$Q(t) = 100e^{-0.02828t}$$

Find the rate of radioactive decay of the isotope.

49. ***Spread of disease*** Suppose that the spread of a disease through the student body at an isolated college campus can be modeled by

$$y = \frac{10{,}000}{1 + 9999e^{-0.99t}}$$

where y is the total number affected at time t (in days). Find the rate of change of y.

50. ***Spread of a rumor*** The number of people $N(t)$ in a community who are reached by a particular rumor at time t (in days) is given by

$$N(t) = \frac{50{,}500}{1 + 100e^{-0.7t}}$$

Find the rate of change of $N(t)$.

51. ***Chemical reaction*** The number of molecules of a certain substance that have enough energy to activate a reaction is given by

$$y = 100{,}000e^{-1/x}$$

where y is the number of molecules and x is the (absolute) temperature of the substance. What is the rate of change of y with respect to temperature?

52. ***Newton's law of cooling*** When a body is moved from one medium to another, its temperature T will change according to the equation

$$T = T_0 + Ce^{kt}$$

where T_0 is the temperature of the new medium, C is the temperature difference between the mediums (old − new), t is the time in the new medium, and k is a constant. If T_0, C, and k are held constant, what is the rate of change of T with respect to time?

53. ***World population*** Suppose that world population can be considered to be growing according to the equation

$$N = N_0(1 + r)^t$$

where N_0 and r are constants. Find the rate of change of N with respect to t.

54. ***Blood pressure*** Medical research has shown that between heartbeats, the pressure in the aorta of a normal adult is a function of time and can be modeled by the equation

$$P = 95e^{-0.491t}$$

(a) Use the derivative to find the rate at which the pressure changes at any time t.
(b) Use the derivative to find the rate at which the pressure changes after 0.1 second.
(c) Is the pressure increasing or decreasing?

55. ***Richter scale*** The intensity of an earthquake is related to the Richter scale reading R by

$$\frac{I}{I_0} = 10^R$$

where I_0 is a standard minimum intensity. If $I_0 = 1$, what is the rate of change of the intensity I with respect to the Richter scale reading?

56. ***Decibel readings*** The intensity level of sound, I, is given by

$$\frac{I}{I_0} = 10^{L/10}$$

where L is the decibel reading and I_0 is the standard threshold of audibility. At what rate is I/I_0 changing with respect to L when $L = 20$?

57. ***National health care*** Using data from the Congressional Budget Office (reported in *Newsweek*, October 4, 1993), the national health expenditure H can be modeled by

$$H = 45e^{0.0898t}$$

where t is the number of years past 1960 and H is in billions of dollars. If this model is accurate, at what rate did health care expenditures change in 1997?

58. ***Revenue*** According to data published in *USA Today* (March 1, 1994), the revenue R from wireless technology can be modeled by

$$R = 0.572e^{0.3860t}$$

where t is the number of years past 1985 and R is in billions of dollars. If this model is accurate, at what rate did revenue change in 1999?

59. ***U.S. debt*** The following table shows the U.S. national debt and the percent of federal expenditures devoted to payment of the interest on this debt for selected years from 1900 to 1989.

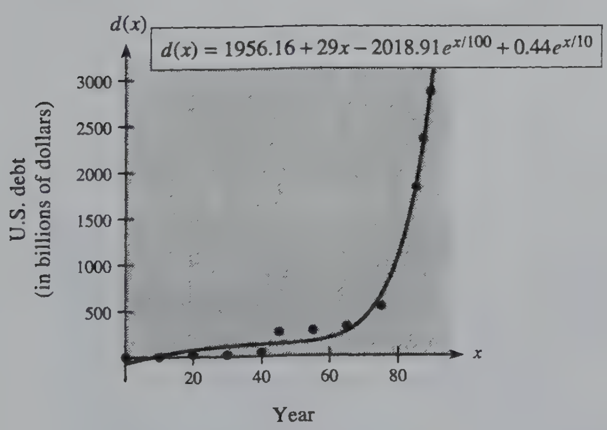
Year	*U.S. Debt (billions)*	*Percent to Interest*
1900	$1.2	0.0
1910	1.1	0.0
1920	24.2	0.0
1930	16.1	0.0
1940	43.0	10.5
1945	258.7	4.1
1955	272.8	9.4
1965	313.8	9.6
1975	533.2	9.8
1985	1823.1	18.9
1987	2350.3	19.5
1989	2857.4	25.0

Source: *World Almanac,* 1991

Assume that the amount of the national debt (in billions of dollars) can be modeled by the function

$$d(x) = 1956.16 + 29x - 2018.91e^{x/100} + 0.44e^{x/10}$$

where x is the number of years past 1900. This function and the data points are graphed in the following figure.



(a) What function describes how fast the national debt is changing?

(b) Find the instantaneous rate of change of the national debt model $d(x)$ in 1940 and 1985.

(c) The Gramm-Rudman Balanced Budget Act (passed in 1985) was designed to limit the growth of the national debt and hence change the function that modeled it. Why was this legislation necessary?

60. ***Social Security*** The number of workers available to support Social Security beneficiaries is declining, as the following table indicates.

Year	*No. of Workers per Retiree*
1950	16.5
1960	5.0
1980	3.7
1990	3.4

Source: Social Security Administration

These data can be modeled by the equation

$$y = 16.03e^{-0.0346t}$$

where $t = 0$ in 1940. Using this model, find the rate at which the number of workers was changing in 1990 (when $t = 50$).

61. ***Purchasing power*** The following table gives the purchasing power of \$1 based on consumer prices for 1963–1995. Using these data with $x = 0$ in 1960, the dollar's purchasing power, P, can be modeled by

$$P = 4.2885e^{-0.05751x}$$

Use the model to find the rate of decay of the purchasing power of \$1 in 1990.

Year	*Purchasing Power of \$1*	*Year*	*Purchasing Power of \$1*	*Year*	*Purchasing Power of \$1*
1963	3.265	1974	2.029	1985	0.928
1964	3.22	1975	1.859	1986	0.913
1965	3.166	1976	1.757	1987	0.880
1966	3.08	1977	1.649	1988	0.846
1967	2.993	1978	1.532	1989	0.807
1968	2.873	1979	1.38	1990	0.766
1969	2.726	1980	1.215	1991	0.734
1970	2.574	1981	1.098	1992	0.713
1971	2.466	1982	1.035	1993	0.692
1972	2.391	1983	1.003	1994	0.675
1973	2.251	1984	0.961	1995	0.656

Source: U.S. Bureau of Labor Statistics

62. ***Postal rates*** Using U.S. Postal Service data, the cost of first-class postage, y, for a letter weighing 1 ounce or less can be modeled by

$$y = 2.5206e^{0.05789x}$$

where x is the number of years past 1950. Use the model to find the rate of change of postal costs in 1974 and in 1995, with correct units.

63. ***TV cable rates*** The average monthly rates for basic cable, without premium channels, for the years 1982–1997 are given in the table.
   (a) These data can be modeled by an exponential function. Write the equation of the function, using x as the number of years past 1980.
   (b) At what rate does this model indicate that the basic cable rate will be growing in 2003? Give the correct units.

Year	*Basic Cable Rate*	*Year*	*Basic Cable Rate*
1982	\$8.30	1990	\$16.78
1983	8.61	1991	18.10
1984	8.98	1992	19.08
1985	9.73	1993	19.39
1986	10.67	1994	21.62
1987	12.18	1995	23.07
1988	13.86	1996	24.41
1989	15.21	1997	26.00

Source: Nielsen Media Research and Paul Kagan Associates, *The Island Packet*, April 19, 1998

64. ***Consumer price index*** The consumer price index (CPI) is calculated by finding the total price of various items that have been averaged according to a prescribed formula. The following table gives the consumer price indexes of all urban consumers (CPI-U) for selected years from 1940 to 1995.
   (a) With x representing years past 1900, find an exponential equation that models these data.
   (b) Use your model to predict the rate of growth in this price index in 2005.

Year	*Consumer Price Index*
1940	14
1950	24.1
1960	29.6
1970	38.8
1980	82.4
1990	130.7
1995	152.4

Source: Bureau of Labor Statistics

65. ***Prison population*** The prison population has grown exponentially from 1980 to 1995.
   (a) Use $x = 0$ in 1900 to find an exponential function that models the data given in the table below.
   (b) What does your model predict as the rate of growth of the prison population in the year 2000?

Year	*Prison Population*	*Year*	*Prison Population*
1980	319,598	1988	606,810
1981	360,029	1989	683,382
1982	402,914	1990	743,382
1983	423,898	1991	792,535
1984	448,264	1992	850,566
1985	487,593	1993	909,381
1986	526,436	1994	990,147
1987	562,814	1995	1,078,545

Source: Bureau of Justice Statistics, U.S. Dept. of Justice

11.3 Implicit Differentiation

OBJECTIVES

- *To find derivatives by using implicit differentiation*
- *To find slopes of tangents by using implicit differentiation*

APPLICATION PREVIEW

In the retail electronics industry, suppose the monthly demand for Precision, Inc., stereo headphones is given by

$$p = \frac{10,000}{(x+1)^2}$$

where p is the price in dollars per set of headphones and x is demand in hundreds of sets of headphones. If we want to find the rate of change of the quantity demanded with respect to the price, then we need to find dx/dp. Although we can (with some difficulty) solve this equation for x so that dx/dp can be found, the resulting equation does not define x as a function of p. In this case, and in other cases where we cannot solve equations for the variable we need, we can find derivatives with a technique called **implicit differentiation.**

When an equation has the form $F(x, y) = 0$, we can say that y is defined implicitly as a function of x whether or not we can solve for y. For example, the equation $xy - 4x + 1 = 0$ is in the form $F(x, y) = 0$, but we can solve for y to write the equation in the form $y = (4x - 1)/x$. We can say that $xy - 4x + 1 = 0$ defines y **implicitly** as a function of x, whereas $y = (4x - 1)/x$ defines the function explicitly.

The equation $x^2 + y^2 - 9 = 0$ has a circle as its graph. If we solve the equation for y, we get $y = \pm\sqrt{9 - x^2}$, which indicates that y is not a function of x. We can, however, consider the equation as defining *two* functions, $y = \sqrt{9 - x^2}$ and $y = -\sqrt{9 - x^2}$ (see Figure 11.5). We say that the equation $x^2 + y^2 - 9 = 0$ defines the two functions implicitly.

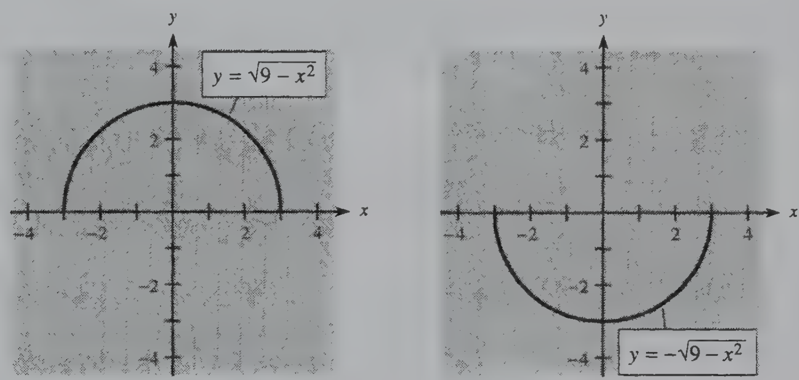


Figure 11.5 (a) (b)

Even though an equation such as

$$\ln xy + xe^y + x - 3 = 0$$

may be difficult or even impossible to solve for y, and even though the equation may not represent y as a single function of x, we can use the technique of **implicit differentiation** to find the derivative of y with respect to x. The word *implicit* means we are implying that y is a function of x without verifying it. We simply take the derivative of both sides of $f(x, y) = 0$ and then solve algebraically for dy/dx.

For example, we can find the derivative dy/dx from $x^2 + y^2 - 9 = 0$ by taking the derivative of both sides of the equation.

$$\frac{d}{dx}(x^2 + y^2 - 9) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) + \frac{d}{dx}(-9) = \frac{d}{dx}(0)$$

We have assumed that y is a function of x; the derivative of y^2 is treated like the derivative of u^n, where u is a function of x. Thus the derivative is

$$2x + 2y^1 \cdot \frac{dy}{dx} + 0 = 0$$

Solving for dy/dx gives

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

Let us now compare this derivative with the derivative of the two functions $y = \sqrt{9 - x^2}$ and $y = -\sqrt{9 - x^2}$. The derivative of $y = \sqrt{9 - x^2}$ is

$$\frac{dy}{dx} = \frac{1}{2}(9 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{9 - x^2}}$$

and the derivative of $y = -\sqrt{9 - x^2}$ is

$$\frac{dy}{dx} = -\frac{1}{2}(9 - x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{9 - x^2}}$$

Note that if we substituted $\pm\sqrt{9 - x^2}$ for y in our "implicit" derivative, we would get the two derivatives that were obtained from the "explicit" functions.

EXAMPLE 1

Find the slope of the tangent to the graph of $x^2 + y^2 - 9 = 0$ at $(\sqrt{5}, 2)$.

Solution

The slope of the tangent to the curve is the derivative of the equation, evaluated at the given point. Taking the derivative implicitly gives us $dy/dx = -x/y$. Evaluating the derivative at $(\sqrt{5}, 2)$ gives the slope of the tangent as $-\sqrt{5}/2$.

We also found the derivative of $x^2 + y^2 - 9 = 0$ by solving for y explicitly. The function whose graph contains $(\sqrt{5}, 2)$ is $y = \sqrt{9 - x^2}$, and its derivative is

$$\frac{dy}{dx} = \frac{-x}{\sqrt{9 - x^2}}$$

Evaluating the derivative at $(\sqrt{5}, 2)$, we get the slope of the tangent: $-\sqrt{5}/2$. Thus we see that both methods give us the same slope for the tangent but that the implicit method is easier to use.

EXAMPLE 2

Find dy/dx if $x^2 + 4x - 3y^2 + 4y = 0$.

Solution

Taking the derivative implicitly gives the following.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(4x) + \frac{d}{dx}(-3y^2) + \frac{d}{dx}(4y) = \frac{d}{dx}(0)$$

$$2x + 4 - 6y\frac{dy}{dx} + 4\frac{dy}{dx} = 0$$

Next we solve for $\dfrac{dy}{dx}$.

$$(-6y+4)\frac{dy}{dx} = -2x-4$$

$$\frac{dy}{dx} = \frac{-2x-4}{-6y+4}$$

$$\frac{dy}{dx} = \frac{x+2}{3y-2}$$

EXAMPLE 3

Write the equation of the tangent to the graph of $x^3 + xy + 4 = 0$ at the point $(2, -6)$.

Solution

Taking the derivative implicitly gives

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(xy) + \frac{d}{dx}(4) = \frac{d}{dx}(0)$$

The $\dfrac{d}{dx}(xy)$ indicates that we should take the derivative of the *product* of x and y. Because we are assuming that y is a function of x, and because x is a function of x, we must use the Product Rule to find $\dfrac{d}{dx}(xy)$.

$$\frac{d}{dx}(xy) = x \cdot 1\frac{dy}{dx} + y \cdot 1 = x\frac{dy}{dx} + y$$

Thus we have $$3x^2 + \left(x\frac{dy}{dx} + y\right) + 0 = 0$$

Solving for dy/dx gives $$\frac{dy}{dx} = \frac{-3x^2 - y}{x}$$

The slope of the tangent to the curve at $x = 2$, $y = -6$ is

$$m = \frac{-3(2)^2 - (-6)}{2} = -3$$

The equation of the tangent line is

$$y - (-6) = -3[x - (2)], \quad \text{or} \quad y = -3x$$

A graphing utility can be used to graph the function of Example 3 and the line that is tangent to the curve at $(2, -6)$. To graph the equation, we solve the equation for y, getting

$$y = \frac{-x^3 - 4}{x}$$

The graph of the equation and the line that is tangent to the curve at (2, −6) are shown in Figure 11.6.

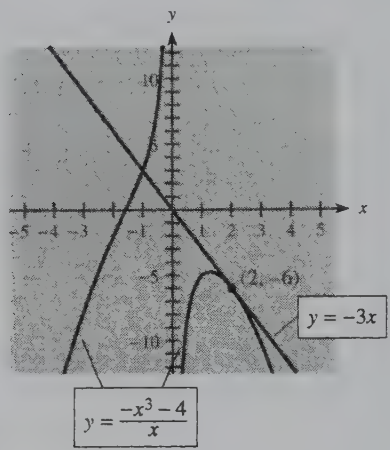


Figure 11.6

CHECKPOINT

1. Find the following:
   (a) $\frac{d}{dx}(x^3)$ (b) $\frac{d}{dx}(y^4)$ (c) $\frac{d}{dx}(x^2y^5)$
2. Find $\frac{dy}{dx}$ for $x^3 + y^4 = x^2y^5$.

EXAMPLE 4

At what point(s) does $x^2 + 4y^2 - 2x + 4y - 2 = 0$ have a horizontal tangent? At what point(s) does it have a vertical tangent?

Solution

First we find the derivative implicitly.

$$2x + 8y \cdot y' - 2 + 4y' - 0 = 0$$

$$y' = \frac{2-2x}{8y+4} = \frac{1-x}{4y+2}$$

Horizontal tangents will occur where $y' = 0$—that is, where $x = 1$. We can now find the corresponding y-value(s) by substituting 1 for x in the original equation and solving.

$$1 + 4y^2 - 2 + 4y - 2 = 0$$

$$4y^2 + 4y - 3 = 0$$

$$(2y - 1)(2y + 3) = 0$$

$$y = \frac{1}{2} \quad \text{or} \quad y = -\frac{3}{2}$$

Thus horizontal tangents occur at $(1, \frac{1}{2})$, and $(1, -\frac{3}{2})$; see Figure 11.7.

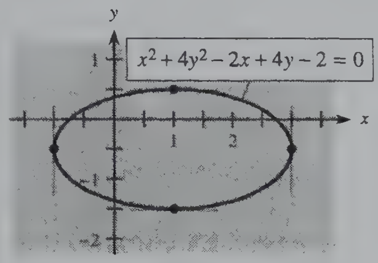


Figure 11.7

Vertical tangents will occur where the derivative is undefined—that is, where $y = -\frac{1}{2}$. To find the corresponding x-value(s), we substitute $-\frac{1}{2}$ in the equation for y and solve for x.

$$x^2 + 4\left(-\frac{1}{2}\right)^2 - 2x + 4\left(-\frac{1}{2}\right) - 2 = 0$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$
$$x = 3 \quad \text{or} \quad x = -1$$

Thus vertical tangents occur at $(3, -\frac{1}{2})$ and $(-1, -\frac{1}{2})$; see Figure 11.7.

EXAMPLE 5

If $\ln xy = 6$, find dy/dx.

Solution

Using the properties of logarithms, we have

$$\ln x + \ln y = 6$$

which leads to the implicit derivative:

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 0$$

Solving gives

$$\frac{dy}{dx} = -\frac{y}{x}$$

EXAMPLE 6

Find dy/dx if $4x^2 + e^{xy} = 6y$.

Solution

We take the derivative of both sides.

$$\frac{d}{dx}(4x^2) + \frac{d}{dx}(e^{xy}) = \frac{d}{dx}(6y)$$

$$8x + e^{xy} \cdot \frac{d}{dx}(xy) = 6\frac{dy}{dx}$$

$$8x + e^{xy}\left(x\frac{dy}{dx} + y\right) = 6\frac{dy}{dx}$$

$$8x + xe^{xy}\frac{dy}{dx} + ye^{xy} = 6\frac{dy}{dx}$$

$$8x + ye^{xy} = 6\frac{dy}{dx} - xe^{xy}\frac{dy}{dx}$$

$$8x + ye^{xy} = (6 - xe^{xy})\frac{dy}{dx}$$

$$\frac{8x + ye^{xy}}{6 - xe^{xy}} = \frac{dy}{dx}$$

EXAMPLE 7

In the Application Preview, the demand for Precision, Inc., stereo headphones was given by

$$p = \frac{10{,}000}{(x+1)^2}$$

where p was the price per set in dollars and x was hundreds of headphone sets demanded. Find the rate of change of demand with respect to price when 19 (hundred) sets are demanded.

Solution

The rate of change of demand with respect to price is dx/dp. Using implicit differentiation, we get the following.

$$\frac{d}{dp}(p) = \frac{d}{dp}\left[\frac{10{,}000}{(x+1)^2}\right] = \frac{d}{dp}\left[10{,}000(x+1)^{-2}\right]$$

$$1 = 10{,}000\left[-2(x+1)^{-3}\frac{dx}{dp}\right]$$

$$1 = \frac{-20{,}000}{(x+1)^3}\frac{dx}{dp}$$

$$\frac{(x+1)^3}{-20{,}000} = \frac{dx}{dp}$$

When 19 (hundred) headphone sets are demanded we use $x = 19$, and the rate of change of demand with respect to price is

$$\left.\frac{dx}{dp}\right|_{x=19} = \frac{(19+1)^3}{-20{,}000} = \frac{8000}{-20{,}000} = -0.4$$

This result means that when 19 (hundred) headphone sets are demanded, if the price per set is increased by \$1, then the expected change in demand is a decrease of 0.4 hundred, or 40, headphone sets.

Graphing Utilities

To graph a function with a graphing utility, we need to write y as an *explicit* function of x (such as $y = \sqrt{4-x^2}$). If an equation defines y as an *implicit* function of x, we have to solve for y in terms of x before we can use the graphing utility. Sometimes we cannot solve for y, and other times, such as in

$$x^{2/3} + y^{2/3} = 8^{2/3}$$

y cannot be written as a single function of x. If this equation is solved for y and a graphing utility is used to graph that function, the resulting graph usually shows only the portion of the graph that lies in quadrants I and II, and sometimes only the part in quadrant I. The complete graph is shown in Figure 11.8. Thus, for the graph of an implicitly defined function, a graphing utility must be used carefully (and sometimes cannot be used at all).

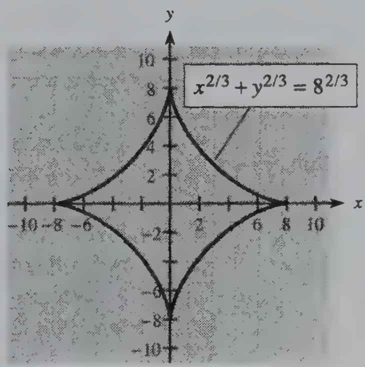


Figure 11.8

CHECKPOINT SOLUTIONS

1. (a) $\dfrac{d}{dx}(x^3) = 3x^2$ (b) $\dfrac{d}{dx}(y^4) = 4y^3\dfrac{dy}{dx}$

(c) $\dfrac{d}{dx}(x^2y^5) = x^2\left(5y^4\dfrac{dy}{dx}\right) + y^5(2x)$ (by the Product Rule)

2. For $x^3 + y^4 = x^2y^5$, we can use the answers to Question (1) to obtain

$$3x^2 + 4y^3\frac{dy}{dx} = x^2\left(5y^4\frac{dy}{dx}\right) + y^5(2x)$$

$$3x^2 + 4y^3\frac{dy}{dx} = 5x^2y^4\frac{dy}{dx} + 2xy^5$$

$$3x^2 - 2xy^5 = (5x^2y^4 - 4y^3)\frac{dy}{dx}$$

$$\frac{3x^2 - 2xy^5}{5x^2y^4 - 4y^3} = \frac{dy}{dx}$$

EXERCISE 11.3

In Problems 1–6, find dy/dx at the given point without first solving for y.

1. $x^2 - 4y - 17 = 0$ at $(1, -4)$
2. $3x^2 - 10y + 400 = 0$ at $(10, 70)$
3. $xy^2 = 8$ at $(2, 2)$
4. $e^y = x$ at $(1, 0)$
5. $x^2 + 3xy - 4 = 0$ at $(1, 1)$
6. $x^2 + 5xy + 4 = 0$ at $(1, -1)$

Find dy/dx for the functions in Problems 7–10.

7. $x^2 + 2y^2 - 4 = 0$
8. $x + y^2 - 4y + 6 = 0$
9. $x^2 + 4x + y^2 - 3y + 1 = 0$
10. $x^2 - 5x + y^3 - 3y - 3 = 0$
11. If $x^2 + y^2 = 4$, find y'.
12. If $p^2 + 4p - q = 4$, find dp/dq.
13. If $xy^2 - y^2 = 1$, find y'.
14. If $p^2 - q = 4$, find dp/dq.
15. If $p^2q = 4p - 2$, find dp/dq.
16. If $x^2 - 3y^4 = 2x^5 + 7y^3 - 5$, find dy/dx.
17. If $3x^5 - 5y^3 = 5x^2 + 3y^5$, find dy/dx.
18. If $x^2 + 3x^2y^4 = y + 8$, find dy/dx.
19. If $x^4 + 2x^3y^2 = x - y^3$, find dy/dx.
20. If $(x + y)^2 = 5x^4y^3$, find dy/dx.
21. Find dy/dx for $x^4 + 3x^3y^2 - 2y^5 = (2x + 3y)^2$.
22. Find y' for $2x + 2y = \sqrt{x^2 + y^2}$.

For Problems 23–26, find the slope of the tangent to the curve.

23. $x^2 + 4x + y^2 + 2y - 4 = 0$ at $(1, -1)$
24. $x^2 - 4x + 2y^2 - 4 = 0$ at $(2, 2)$
25. $x^2 + 2xy + 3 = 0$ at $(-1, 2)$
26. $y + x^2 = 4$ at $(0, 4)$
27. Write the equation of the line tangent to the curve $x^2 - 2y^2 + 4 = 0$ at $(2, 2)$.
28. Write the equation of the line tangent to the curve $x^2 + y^2 + 2x - 3 = 0$ at $(-1, 2)$.
29. Write the equation of the line tangent to the curve $4x^2 + 3y^2 - 4y - 3 = 0$ at $(-1, 1)$.
30. Write the equation of the line tangent to the curve $xy + y^2 = 0$ at $(3, 0)$.
31. If $\ln x = y^2$, find dy/dx.
32. If $\ln(x + y) = y^2$, find dy/dx.
33. If $y^2 \ln x = 4$, find dy/dx.
34. If $\ln xy = 2$, find dy/dx.
35. Find the slope of the tangent to the curve $x^2 + \ln y = 4$ at the point $(2, 1)$.
36. Write the equation of the line tangent to the curve $x \ln y + 2xy = 2$ at the point $(1, 1)$.
37. If $xe^y = 6$, find dy/dx.
38. If $x + e^{xy} = 10$, find dy/dx.
39. If $e^{xy} = 4$, find dy/dx.
40. If $x - xe^y = 3$, find dy/dx.
41. If $ye^x - y = 3$, find dy/dx.
42. If $x^2y = e^{x+y}$, find dy/dx.
43. Find the slope of the line tangent to the graph of $y = xe^{-x}$ at $x = 1$.
44. Find the slope of the line tangent to the curve $ye^x = 4$ at $(0, 4)$.
45. Write the equation of the line tangent to the curve $xe^y = 3$ at $(3, 0)$.
46. Write the equation of the line tangent to the curve $ye^x = 4$ at $(0, 4)$.
47. At what points does the curve defined by $x^2 + 4y^2 - 4x - 4 = 0$ have
    (a) horizontal tangents?
    (b) vertical tangents?
48. At what points does the curve defined by $x^2 + 4y^2 - 4 = 0$ have
    (a) horizontal tangents?
    (b) vertical tangents?
49. In Problem 11, the derivative y' was found to be
$$y' = \frac{-x}{y}$$
when $x^2 + y^2 = 4$. Take the implicit derivative of the equation for y' to show that
$$y'' = \frac{-y + xy'}{y^2}$$
50. Find y' implicitly for $x^3 - y^3 = 8$. Then, by taking derivatives implicitly, use it to show that
$$y'' = \frac{2x(y - xy')}{y^3}$$
51. Use the result of Problem 49 in (a) and (b) below.
    (a) Substitute $-x/y$ for y' in the expression for y'' and simplify to show that
$$y'' = -\frac{(x^2 + y^2)}{y^3}$$
    (b) Does $y'' = -4/y^3$? Why or why not?

52. Use the result of Problem 50.
(a) Substitute x^2/y^2 for y' in the expression for y'' and simplify to show that
$$y'' = \frac{2x(y^3 - x^3)}{y^5}$$
(b) Does $y'' = -16x/y^5$? Why or why not?

53. Find y'' for $\sqrt{x} + \sqrt{y} = 1$ and simplify.

54. Find y'' for $\frac{1}{x} - \frac{1}{y} = 1$.

In Problems 55 and 56, find the maximum and minimum values of y. Use a graphing utility to verify your conclusion.

55. $x^2 + y^2 - 9 = 0$

56. $4x^2 + y^2 - 8x = 0$

Applications

57. ***Advertising and sales*** Suppose that a company's sales volume y (in thousands of dollars) is related to its advertising expenditures x (in thousands of dollars) according to
$$xy - 20x + 10y = 0$$
Find the rate of change of sales volume with respect to advertising expenditures when $x = 10$ (thousand dollars).

58. ***Insect control*** Suppose that the number of mosquitoes N (in thousands) in a certain swampy area near a community is related to the number of pounds of insecticide x sprayed on the nesting areas according to
$$Nx - 10x + N = 300$$
Find the rate of change of N with respect to x when 49 pounds of insecticide is used.

59. ***Production*** Suppose that a company can produce 12,000 units when the number of hours of skilled labor y and unskilled labor x satisfy
$$384 = (x + 1)^{3/4}(y + 2)^{1/3}$$
Find the rate of change of skilled-labor hours with respect to unskilled-labor hours when $x = 255$ and $y = 214$. This can be used to approximate the change in skilled-labor hours required to maintain the same production level when unskilled-labor hours are increased by 1 hour.

60. ***Production*** Suppose that production of 10,000 units of a certain agricultural crop is related to the number of hours of labor x and the number of acres of the crop y according to
$$300x + 30{,}000y = 11xy - 0.0002x^2 - 5y$$
Find the rate of change of the number of hours with respect to the number of acres.

61. ***Demand*** If the demand function for a product is given by
$$p(q + 1)^2 = 200{,}000$$
find the rate of change of quantity with respect to price when $p = \$80$. Interpret this result.

62. ***Demand*** If the demand function for a commodity is given by
$$p^2(2q + 1) = 100{,}000$$
find the rate of change of quantity with respect to price when $p = \$50$. Interpret this result.

63. ***Radioactive decay*** The number of grams of radium, y, that will remain after t years if 100 grams existed originally can be found by using the equation
$$-0.000436t = \ln\left(\frac{y}{100}\right)$$
Use implicit differentiation to find the rate of change of y with respect to t—that is, the rate at which the radium will decay.

64. ***Disease control*** Suppose the proportion of people affected by a certain disease is described by
$$\ln\left(\frac{P}{1 - P}\right) = 0.5t$$
where t is the time in months. Find dP/dt, the rate at which P grows.

65. ***Temperature-humidity index*** The temperature-humidity index (THI) is given by
$$\text{THI} = t - 0.55(1 - h)(t - 58)$$
where t is the air temperature in degrees Fahrenheit and h is the relative humidity. If the THI remains constant, find the rate of change of humidity with respect to temperature if the temperature is 70°F (Source: "Temperature-Humidity Indices," *UMAP Journal*, Fall 1989).

11.4 Related Rates

OBJECTIVE

- *To use implicit differentiation to solve problems that involve related rates*

APPLICATION PREVIEW

According to Poiseuille's law, the flow of blood F is related to the radius r of the vessel according to

$$F = kr^4$$

where k is a constant. When the radius of a blood vessel is reduced, such as by cholesterol deposits, the flow of blood is also restricted. Drugs can be administered that increase the radius of the blood vessel and, hence, the flow of blood. The rate of change of the blood flow and the rate of change of the radius of the blood vessel are time rates of change that are related to each other, so they are called **related rates.** We can use these related rates to find the **percentage rate of change** in the blood flow that corresponds to the percentage rate of change in the radius of the blood vessel caused by the drug.

We have seen that the derivative represents the instantaneous rate of change of one variable with respect to another. When the derivative is taken with respect to time, it represents the rate at which that variable is changing with respect to time (or the velocity). For example, if distance x is measured in miles and time t in hours, then dx/dt is measured in miles per hour and indicates how fast x is changing. Similarly, if V represents the volume (in cubic feet) of water in a swimming pool and t is time (in minutes), then dV/dt is measured in cubic feet per minute (ft^3/min) and might measure the rate at which the pool is being filled with water or being emptied.

Sometimes, two (or more) quantities that depend on time are also related to each other. For example, the height of a tree h (in feet) is related to the radius r (in inches) of its trunk, and this relationship can be modeled by

$$h = kr^{2/3}$$

where k is a constant.* Of course, both h and r are also related to time, so the rates of change dh/dt and dr/dt are related to each other. Thus they are called **related rates.**

The specific relationship between dh/dt and dr/dt can be found by differentiating $h = kr^{2/3}$ implicitly with respect to time t.

EXAMPLE 1

Suppose that for a certain type of tree, the height of the tree (in feet) is related to the radius of its trunk (in inches) by

$$h = 15r^{2/3}$$

Suppose that the rate of change of r is $\frac{3}{4}$ inch per year. Find how fast the height is changing when the radius is 8 inches.

*T. McMahon, "Size and Shape in Biology," *Science* 179 (1979): 1201.

Solution

To find how the rates dh/dt and dr/dt are related, we differentiate $h = 15r^{2/3}$ implicitly with respect to time t.

$$\frac{dh}{dt} = 10r^{-1/3}\frac{dr}{dt}$$

Using $r = 8$ inches and $dr/dt = \frac{3}{4}$ inch per year gives

$$\frac{dh}{dt} = 10(8)^{-1/3}(3/4) = \frac{15}{4} = 3\tfrac{3}{4} \text{ feet per year}$$

The work in Example 1 shows how to obtain related rates, but the different units (feet per year and inches per year) may be somewhat difficult to interpret. For this reason, many applications in the life sciences deal with **percentage rates of change.** The percentage rate of change of a quantity is the rate of change of the quantity divided by the quantity.

EXAMPLE 2

As mentioned in the Application Preview, Poiseuille's law expresses the flow of blood F as a function of the radius r of the vessel according to

$$F = kr^4$$

where k is a constant. When the radius of a blood vessel is restricted, such as by cholesterol deposits, drugs can be administered that will increase the radius of the blood vessel (and hence the blood flow). Find the percentage rate of change of the flow of blood that corresponds to the percentage rate of change of the radius of a blood vessel caused by the drug.

Solution

We seek the percentage rate of change of flow, $(dF/dt)/F$, that results from a given percentage rate of change of the radius $(dr/dt)/r$. We first find the related rates of change by differentiating

$$F = kr^4$$

implicitly with respect to time.

$$\frac{dF}{dt} = k\left(4r^3\frac{dr}{dt}\right)$$

Then the percentage rate of change of flow can be found by dividing both sides of the equation by F.

$$\frac{\frac{dF}{dt}}{F} = \frac{4kr^3\frac{dr}{dt}}{F}$$

If we replace F on the right side of the equation with kr^4 and reduce, we get

$$\frac{\frac{dF}{dt}}{F} = \frac{4kr^3\frac{dr}{dt}}{kr^4} = 4\left(\frac{\frac{dr}{dt}}{r}\right)$$

Thus we see that the percentage rate of change of the flow of blood is 4 times the corresponding percentage rate of change of the radius of the blood vessel. This means that a drug that would cause a 12% increase in the radius of a blood vessel at a certain time would produce a corresponding 48% increase in blood flow through that vessel at that time.

In these examples, the equation relating the time-dependent variables has been given. For some problems, the original equation relating the variables must first be developed from the statement of the problem. These problems can be solved with the aid of the following procedure.

Solving Related-Rates Problems

Procedure	**Example**
To solve related-rates problems:	Sand falls at a rate of 5 ft^3/min on a conical pile, with the diameter always equal to the height of the pile. At what rate is the height increasing when it is 10 ft high?
1. Use geometric and/or physical conditions to write an equation that relates the time-dependent variables.	1. The conical pile has its volume given by $$V = \frac{1}{3}\pi r^2 h$$
2. Substitute into the equation values or relationships that are true *at all times.*	2. The radius $r = \frac{1}{2}h$ at all times, so $$V = \frac{1}{3}\pi\left[\frac{1}{4}h^2\right]h = \frac{\pi}{12}h^3$$
3. Differentiate both sides of the equation implicitly with respect to time. This equation is valid for all times.	3. $\dfrac{dV}{dt} = \dfrac{\pi}{12}\left[3h^2\dfrac{dh}{dt}\right] = \dfrac{\pi}{4}h^2\dfrac{dh}{dt}$
4. Substitute the values that are known at the instant specified, and solve the equation.	4. $\dfrac{dV}{dt} = 5$ at all times, so when $h = 10$, $$5 = \frac{\pi}{4}(10^2)\frac{dh}{dt}$$
5. Solve for the specified quantity at the given time.	5. $\dfrac{dh}{dt} = \dfrac{20}{100\pi} = \dfrac{1}{5\pi}$ (ft/min)

Note that you should *not* substitute numerical values for any quantity that varies with time until after the derivative is taken. If values are substituted before the derivative is taken, that quantity will have the constant value resulting from the substitution and hence will have a derivative equal to zero.

EXAMPLE 3

A hot air balloon has a velocity of 50 ft/min and is flying at a constant height of 500 ft. An observer on the ground is watching the balloon approach. How fast is the distance between the balloon and the observer changing when the balloon is 1000 ft from the observer?

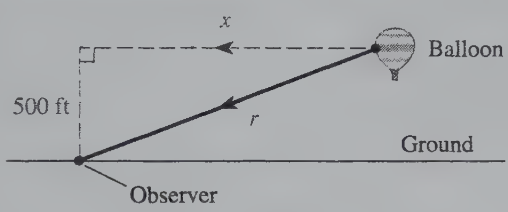


Figure 11.9

Solution

If we let r be the distance between the balloon and the observer and x be the horizontal distance from the balloon to a point directly above the observer, then we see that these quantities are related by the equation

$$x^2 + 500^2 = r^2 \qquad \text{(See Figure 11.9)}$$

Because the distance x is decreasing, we know that dx/dt must be negative. Thus we are given that $dx/dt = -50$ at all times, and we need to find dr/dt when $r = 1000$. Taking the derivative with respect to t of both sides of the equation $x^2 + 500^2 = r^2$ gives

$$2x\frac{dx}{dt} + 0 = 2r\frac{dr}{dt}$$

Using $dx/dt = -50$ and $r = 1000$, we get

$$2x(-50) = 2000\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{-100x}{2000} = \frac{-x}{20}$$

Using $r = 1000$ in $x^2 + 500^2 = r^2$ gives $x^2 = 750{,}000$. Thus $x = 500\sqrt{3}$, and

$$\frac{dr}{dt} = \frac{-500\sqrt{3}}{20} = -25\sqrt{3}\text{ ft/min}$$

CHECKPOINT

1. If V represents volume, write a mathematical symbol that represents "the rate of change of volume with respect to time."
2. (a) Differentiate $x^2 + 64 = y^2$ implicitly with respect to time.
   (b) Suppose that we know that y is increasing at 2 units per minute. Use (a) to find the rate of change of x at the instant when $x = 6$ and $y = 10$.
3. True or false: In solving a related-rates problem, we substitute all numerical values into the equation before we take derivatives.

EXAMPLE 4

Suppose that oil is spreading in a circular pattern from a leak at an offshore rig. If the rate at which the radius of the oil slick is growing is 1 ft/min, at what rate is the area of the oil slick growing when the radius is 600 ft?

Solution

The area of the circular oil slick is given by

$$A = \pi r^2$$

where r is the radius. The rate at which the area is changing is

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Using $r = 600$ ft and $dr/dt = 1$ ft/min gives

$$\frac{dA}{dt} = 2\pi(600\text{ ft})(1\text{ ft/min}) = 1200\pi\text{ ft}^2/\text{min}$$

Thus when the radius of the oil slick is 600 ft, the area is growing at the rate of 1200π ft²/min, or approximately 3770 ft²/min.

CHECKPOINT SOLUTIONS

1. dV/dt
2. (a) $2x\dfrac{dx}{dt} + 0 = 2y\dfrac{dy}{dt}$

   $x\dfrac{dx}{dt} = y\dfrac{dy}{dt}$

   (b) Use $\dfrac{dy}{dt} = 2$, $x = 6$, and $y = 10$ in (a) to obtain

   $6\dfrac{dx}{dt} = 10(2)$ or $\dfrac{dx}{dt} = \dfrac{20}{6} = \dfrac{10}{3}$
3. False. The numerical values for any quantity that is varying with time should not be substituted until after the derivative is taken.

EXERCISE 11.4

In Problems 1–4, find dy/dt using the given values.

1. $y = x^3 - 3x$, for $x = 2$, $dx/dt = 4$
2. $y = 3x^3 + 5x^2 - x$ for $x = 4$, $dx/dt = 3$
3. $xy = 4$, for $x = 8$, $dx/dt = -2$
4. $xy = x + 3$, for $x = 3$, $dx/dt = -1$

In Problems 5–8, assume that x and y are differentiable functions of t. In each case, find dx/dt given that $x = 5$, $y = 12$, and $dy/dt = 2$.

5. $x^2 + y^2 = 169$
6. $y^2 - x^2 = 119$
7. $y^2 = 2xy + 24$
8. $x^2(y - 6) = 12y + 6$
9. If $x^2 + y^2 = z^2$, find dy/dt when $x = 3$, $y = 4$, $dx/dt = 10$, and $dz/dt = 2$.
10. If $s = 2\pi r(r + h)$, find dr/dt when $r = 2$, $h = 8$, $dh/dt = 3$, and $ds/dt = 10\pi$.
11. A point is moving along the graph of the equation $y = -4x^2$. At what rate is y changing when $x = 5$ and is changing at a rate of 2 units/sec?
12. A point is moving along the graph of the equation $y = 5x^3 - 2x$. At what rate is y changing when $x = 4$ and is changing at a rate of 3 units/sec?

13. The radius of a circle is increasing at a rate of 2 ft/min. At what rate is its area changing when the radius is 3 ft? (Recall that for a circle, $A = \pi r^2$.)
14. The area of a circle is changing at a rate of 1 in²/sec. At what rate is its radius changing when the radius is 2 in.?
15. The volume of a cube is increasing at a rate of 64 in³/sec. At what rate is the length of each edge of the cube changing when the edges are 6 in. long? (Recall that for a cube, $V = x^3$.)
16. The lengths of the edges of a cube are increasing at a rate of 8 ft/min. At what rate is the surface area changing when the edges are 24 ft long? (Recall that for a cube, $S = 6x^2$.)

Applications

17. ***Profit*** Suppose that the daily profit (in dollars) from the production and sale of x units of a product is given by

$$P = 180x - \frac{x^2}{1000} - 2000$$

At what rate is the profit changing when the number of units produced and sold is 100 and is increasing at a rate of 10 units per day?

18. ***Profit*** Suppose that the monthly revenue and cost (in dollars) for x units of a product are

$$R = 400x - \frac{x^2}{20} \quad \text{and} \quad C = 5000 + 70x$$

At what rate is the profit changing if the number of units produced and sold is 100 and is increasing at a rate of 10 units per month?

19. ***Demand*** Suppose that the price p (in dollars) of a product is given by the demand function

$$p = \frac{1000 - 10x}{400 - x}$$

where x represents the quantity demanded. If the daily demand is *decreasing* at a rate of 20 units per day, at what rate is the price changing when the demand is 20 units?

20. ***Supply*** The supply function for a product is given by $p = 40 + 100\sqrt{2x + 9}$, where x is the number of units supplied and p is the price in dollars. If the price is increasing at a rate of \$1 per month, at what rate is the supply changing when $x = 20$?

21. ***Capital investment and production*** Suppose that for a particular product, the number of units x produced per month depends on the number of thousands of dollars y invested, with $x = 30y + 20y^2$. At what rate will production increase if \$10,000 is invested and if the investment capital is increasing at a rate of \$1000 per month?

22. ***Boyle's law*** Boyle's law for enclosed gases states that at a constant temperature, the pressure is related to the volume by the equation

$$P = \frac{k}{V}$$

where k is a constant. If the volume is increasing at a rate of 5 cubic inches per hour, at what rate is the pressure changing when the volume is 30 cubic inches and $k = 2$ inch-pounds?

Tumor growth For Problems 23 and 24, suppose that a tumor in a person's body has a spherical shape and that treatment is causing the radius of the tumor to decrease at a rate of 1 millimeter per month.

23. At what rate is the volume decreasing when the radius is 3 millimeters? Recall that $V = \frac{4}{3}\pi r^3$.
24. At what rate is the surface area of the tumor decreasing when the radius is 3 mm? (Recall that for a sphere, $S = 4\pi r^2$.)

25. ***Allometric relationships—fish*** For many species of fish, the allometric relationship between the weight W and the length L is approximately $W = kL^3$, where k is a constant. Find the percentage rate of change of the weight as a corresponding percentage rate of change of the length.

26. ***Blood flow*** The resistance R of a blood vessel to the flow of blood is a function of the radius r of the blood vessel and is given by

$$R = \frac{k}{r^4}$$

where k is a constant. Find the percentage rate of change of the resistance of a blood vessel in terms of the percentage rate of change in the radius of the blood vessel.

27. ***Allometric relationships—crabs*** For fiddler crabs, data gathered by Thompson* show that the allometric relationship between the weight C of the claw and the weight W of the body is given by

$$C = 0.11W^{1.54}$$

Find the percentage rate of change of the claw weight in terms of the percentage rate of change of the body weight for fiddler crabs.

28. ***Body weight and surface area*** For human beings, the surface area S of the body is related to the body's weight W according to

$$S = kW^{2/3}$$

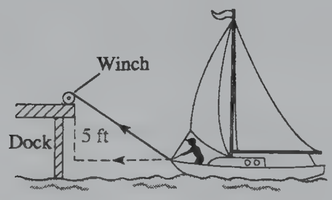
where k is a constant. Find the percentage rate of change of the body's surface area in terms of the percentage rate of change of the body's weight.

29. ***Cell growth*** A bacterial cell has a spherical shape. If the volume of the cell is increasing at a rate of 4 cubic micrometers per day, at what rate is the radius of the cell increasing when it is 2 micrometers? (Recall that for a sphere, $V = \frac{4}{3}\pi r^3$.)

30. ***Water purification*** Assume that water is being purified by flowing through a conical filter that has a height of 15 inches and a radius of 5 inches. If the depth of the water is decreasing at a rate of 1 inch per minute when the depth is 6 inches, at what rate is the (volume of) water flowing out of the filter at this instant?

31. ***Volume and radius*** Suppose that air is being pumped into a spherical balloon at a rate of 5 in^3/min. At what rate is the radius of the balloon increasing when the radius is 5 in.?

32. ***Boat docking*** Suppose that a boat is being pulled toward a dock by a winch that is 5 ft above the level of the boat deck. If the winch is pulling the cable at a rate of 3 ft/min, at what rate is the boat approaching the dock when it is 12 ft from the dock? Use the figure below.



33. ***Ladder safety*** A 30-ft ladder is leaning against a wall. If the bottom is pulled away from the wall at a rate of 1 ft/sec, at what rate is the top of the ladder sliding down the wall when the bottom is 18 ft from the wall?

34. ***Flight*** A kite is 30 ft high and is moving horizontally at a rate of 10 ft/min. If the kite string is taut, at what rate is the string being played out when 50 ft of string is out?

35. ***Flight*** A plane is flying at a constant altitude of 1 mile and a speed of 300 mph. If it is flying toward an observer on the ground, how fast is the plane approaching the observer when it is 5 miles from the observer?

36. ***Distance*** Two boats leave the same port at the same time, with boat A traveling north at 15 knots and boat B traveling east at 20 knots. How fast is the distance between them changing when boat A is 30 nautical miles from port?

37. ***Distance*** Two cars are approaching an intersection on roads that are perpendicular to each other. Car A is north of the intersection and traveling south at 40 mph. Car B is east of the intersection and traveling west at 55 mph. How fast is the distance between the cars changing when car A is 15 miles from the intersection and car B is 8 miles from the intersection?

38. ***Water depth*** Water is flowing into a barrel in the shape of a right circular cylinder at the rate of 200 in^3/min. If the radius of the barrel is 18 in., at what rate is the depth of the water changing when the water is 30 in. deep?

39. ***Water depth*** Suppose that water is being pumped into a rectangular swimming pool of uniform depth at 10 ft^3/hr. If the pool is 10 ft wide and 25 ft long, at what rate is the water rising when it is 4 ft deep?

*d'Arcy Thompson, *On Growth and Form* (Cambridge, England: Cambridge University Press, 1961).

11.5 Applications in Business and Economics

OBJECTIVES

- *To find the elasticity of demand*
- *To find the tax per unit that will maximize tax revenue*

APPLICATION PREVIEW

In this section, we consider two applications: **elasticity of demand** and **taxation in a competitive market.** Elasticity of demand measures how sensitive the demand for a product is to price changes, and it can be used to measure the effect that price changes have on total revenue. Taxation in a competitive market examines how a tax levied on goods produces shifts in market equilibrium, and it also addresses the associated problem of finding the tax per unit that, despite changes in market equilibrium, maximizes tax revenues.

Elasticity of Demand

We know from the law of demand that consumers will respond to changes in prices; if prices increase, the quantity demanded will decrease. But the degree of responsiveness of the consumers to price changes will vary widely for different products. For example, a price increase in insulin will not greatly decrease the demand for it by diabetics, but a price increase in clothes may cause consumers to buy less and wear their old clothes longer. When the response to price changes is considerable, we say the demand is *elastic*. When price changes cause relatively small changes in demand for a product, the demand is said to be *inelastic* for that product.

Economists measure the **elasticity of demand** on an interval by dividing the rate of change of demand by the rate of change of price. We may write this as

$$E_d = -\frac{\text{change in quantity demanded}}{\text{original quantity demanded}} \div \frac{\text{change in price}}{\text{original price}}$$

or

$$E_d = -\frac{\Delta q}{q} \div \frac{\Delta p}{p}$$

The demand curve usually has a negative slope, so we have introduced a negative sign into the formula to give us a positive elasticity.

We can write the equation for elasticity as

$$E_d = -\frac{p}{q} \cdot \frac{\Delta q}{\Delta p}$$

If q is a function of p, then we can write

$$\frac{\Delta q}{\Delta p} = \frac{f(p + \Delta p) - f(p)}{\Delta p}$$

and the limit as Δp approaches 0 gives the **point elasticity of demand:**

$$\eta = \lim_{\Delta p \to 0}\left(-\frac{p}{q} \cdot \frac{\Delta q}{\Delta p}\right) = -\frac{p}{q} \cdot \frac{dq}{dp}$$

Elasticity The **elasticity of demand** at the point (q_A, p_A), is

$$\eta = -\frac{p}{q} \cdot \frac{dq}{dp}\bigg|_{(q_A, p_A)}$$

EXAMPLE 1

Find the elasticity of the demand function $p + 5q = 100$ when

(a) the price is \$40, (b) the price is \$60, and (c) the price is \$50.

Solution

Solving the demand function for q gives $q = 20 - \frac{1}{5}p$. Then $dq/dp = -\frac{1}{5}$ and

$$\eta = -\frac{p}{q}\left(-\frac{1}{5}\right)$$

(a) When $p = 40$, $q = 12$ and $\eta = -\frac{p}{q}\left(-\frac{1}{5}\right)\bigg|_{(12, 40)} = -\frac{40}{12}\left(-\frac{1}{5}\right) = \frac{2}{3}$.

(b) When $p = 60$, $q = 8$ and $\eta = -\frac{p}{q}\left(-\frac{1}{5}\right)\bigg|_{(8, 60)} = -\frac{60}{8}\left(-\frac{1}{5}\right) = \frac{3}{2}$.

(c) When $p = 50$, $q = 10$ and $\eta = -\frac{p}{q}\left(-\frac{1}{5}\right)\bigg|_{(10, 50)} = -\frac{50}{10}\left(-\frac{1}{5}\right) = 1$.

Note that in Example 1 the demand equation was $p + 5q = 100$, so the demand "curve" is a straight line, with slope $m = -5$. But the elasticity was $\eta = \frac{2}{3}$ at (12, 40), $\eta = \frac{3}{2}$ at (8, 60), and $\eta = 1$ at (10, 50). This illustrates that the elasticity of demand may be different at different points on the demand curve, even though the slope of the demand "curve" is constant. (See Figure 11.10.) Economists use η to measure how responsive demand is to price at different points on the demand curve for a product.

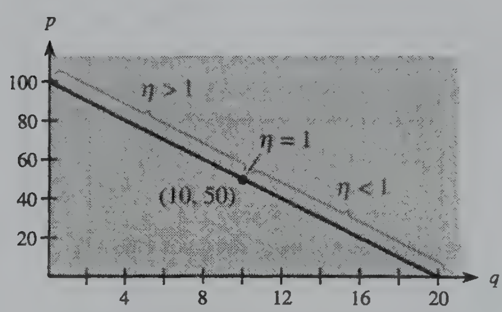


Figure 11.10

This example shows that the elasticity of demand is more than just the slope of the demand curve, which is the rate at which the demand is changing. Recall that the elasticity measures the consumers' degree of responsiveness to a price change.

Economists classify demand curves according to how responsive demand is to price changes by using elasticity.

Elasticity of Demand

- If $\eta > 1$, the demand is **elastic**, and the percent decrease in demand is greater than the corresponding percent increase in price.
- If $\eta < 1$, the demand is **inelastic**, and the percent decrease in demand will be less than the corresponding percent increase in price.
- If $\eta = 1$, the demand is **unitary elastic**, and the percent decrease in demand is approximately equal to the corresponding percent increase in price.

We can also use implicit differentiation to find dq/dp in evaluating the point elasticity of demand.

EXAMPLE 2

The demand for a certain product is given by

$$p = \frac{1000}{(q+1)^2}$$

where p is the price per unit in dollars and q is demand in units of the product. Find the elasticity of demand with respect to price when $q = 19$.

Solution

To find the elasticity, we need to find dq/dp. Using implicit differentiation, we get the following:

$$\begin{aligned} \frac{d}{dp}(p) &= \frac{d}{dp}\left[1000(q+1)^{-2}\right] \\ 1 &= 1000\left[-2(q+1)^{-3}\frac{dq}{dp}\right] \\ 1 &= \frac{-2000}{(q+1)^3}\frac{dq}{dp} \\ \frac{(q+1)^3}{-2000} &= \frac{dq}{dp} \end{aligned}$$

When $q = 19$, we have $p = 1000/(19+1)^2 = 1000/400 = 5/2$ and

$$\left.\frac{dq}{dp}\right|_{(q=19)} = \frac{(19+1)^3}{-2000} = \frac{8000}{-2000} = -4$$

The elasticity of demand when $q = 19$ is

$$\eta = \frac{-p}{q}\cdot\frac{dq}{dp} = -\frac{(5/2)}{19}\cdot(-4) = \frac{10}{19} < 1$$

Thus the demand for this product is inelastic.

Elasticity is related to revenue in a special way. We can see how by computing the derivative of the revenue function

$$R = pq$$

with respect to p.

$$\begin{aligned}\frac{dR}{dp} &= p \cdot \frac{dq}{dp} + q \cdot 1 \\ &= \frac{q}{q} \cdot p \cdot \frac{dq}{dp} + q = q \cdot \frac{p}{q} \cdot \frac{dq}{dp} + q \\ &= q(-\eta) + q \\ &= q(1 - \eta)\end{aligned}$$

From this we can summarize the relationship of elasticity and revenue.

Elasticity and Revenue

The rate of change of revenue R with respect to price p is related to elasticity in the following way.

- Elastic ($\eta > 1$) means $\frac{dR}{dp} < 0$. $\left\{\begin{array}{l}\text{Hence if price increases, revenue decreases,} \\ \text{and if price decreases, revenue increases.}\end{array}\right.$
- Inelastic ($\eta < 1$) means $\frac{dR}{dp} > 0$. $\left\{\begin{array}{l}\text{Hence if price increases, revenue increases,} \\ \text{and if price decreases, revenue decreases.}\end{array}\right.$
- Unitary elastic ($\eta = 1$) means $\frac{dR}{dp} = 0$. Hence an increase or decrease in price will not change revenue. Revenue is optimized at this point.

EXAMPLE 3

The demand for a product is given by

$$p = 10\sqrt{100 - q}, \qquad 0 \leq q \leq 100$$

(a) Find the point at which demand is of unitary elasticity, and find intervals in which the demand is inelastic and intervals in which it is elastic.
(b) Find where revenue is increasing, where it is decreasing, and where it is maximized.
(c) Use a graphing utility to show the graph of the revenue function $R = pq$, with $0 \leq q \leq 100$, and confirm the results from (b).

Solution

The elasticity is

$$\eta = -\frac{10\sqrt{100 - q}}{q} \cdot \frac{dq}{dp}$$

Finding dq/dp implicitly, we have

$$1 = 10\left[\frac{1}{2}(100-q)^{-1/2}\left(-\frac{dq}{dp}\right)\right]$$

so

$$\frac{dq}{dp} = -\frac{1}{5}\sqrt{100-q}$$

Thus

$$\eta = -\frac{10\sqrt{100-q}}{q}\left[-\frac{1}{5}\sqrt{100-q}\right] = \frac{200-2q}{q}$$

(a) Unitary elasticity occurs where $\eta = 1$.

$$1 = \frac{200-2q}{q}$$
$$q = 200 - 2q$$
$$3q = 200$$
$$q = 66\tfrac{2}{3}$$

so unitary elasticity occurs when $66\frac{2}{3}$ units are sold, at a price of \$57.74. For values of q between 0 and $66\frac{2}{3}$, $\eta > 1$ and demand is elastic. For values of q between $66\frac{2}{3}$ and 100, $\eta < 1$ and demand is inelastic.

(b) When q increases over $0 < q < 66\frac{2}{3}$, p decreases, so $\eta > 1$ means R increases. Similarly, when q increases over $66\frac{2}{3} < q < 100$, p decreases, so $\eta < 1$ means R decreases. Revenue is maximized where $\eta = 1$, at $q = 66\frac{2}{3}$, $p = 57.74$.

(c) The graph of this revenue function,

$$R = 10q\sqrt{1-q}$$

is shown in Figure 11.11 and confirms our conclusions from (b).

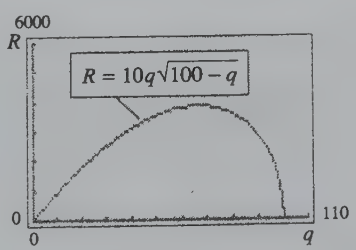


Figure 11.11

CHECKPOINT

1. Write the formula for point elasticity, η.
2. (a) If $\eta > 1$, the demand is called ________.
   (b) If $\eta < 1$, the demand is called ________.
   (c) If $\eta = 1$, the demand is called ________.
3. Find the elasticity of demand for $q = \dfrac{100}{p} - 1$ when $p = 10$ and $q = 9$.

Taxation in a Competitive Market

Many taxes imposed by governments are "hidden." That is, the tax is levied on goods produced, and the producers must pay the tax. Of course, the tax becomes a cost to the producers, and they pass that cost on to the consumer in the form of higher prices for goods.

Suppose the government imposes a tax of t dollars on each unit produced and sold by producers. If we are in pure competition in which the consumers' demand depends only on price, the *demand function* will not change. The tax will change the supply function, of course, because at each level of output q, the firm will want to charge a price higher by the amount of the tax.

The graphs of the market demand function, the original market supply function, and the market supply function after taxes are shown in Figure 11.12. Because the tax added to each item is constant, the graph of the supply function is t units above the original supply function. If $p = f(q)$ defines the original supply function, then $p = f(q) + t$ defines the supply function after taxation.

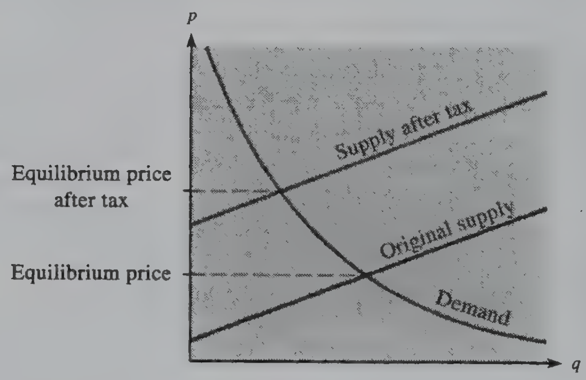


Figure 11.12

Note in this case that after the taxes are imposed, *no* items are supplied at the price that was the equilibrium price before taxation. After the taxes are imposed, the consumers simply have to pay more for the product. Because taxation does not change the demand curve, the quantity purchased at market equilibrium will be less than it was before taxation. Thus governments planning taxes should recognize that they will not collect taxes on the original equilibrium quantity. They will collect on the *new* equilibrium quantity, a quantity reduced by their taxation. Thus a large tax on each item may reduce the quantity demanded at the new market equilibrium so much that very little revenue results from the tax!

If the tax revenue is represented by $T = tq$, where t is the tax per unit and q is the equilibrium quantity of the supply and demand functions after taxation, we can use the following procedure for maximizing the total tax revenue in competitive market.

Maximizing Total Tax Revenue

Procedure	Example
To find the tax per item (under pure competition) that will maximize total tax revenue:	If the demand and supply functions are given by $p = 600 - q$ and $p = 200 + \frac{1}{3}q$, respectively, find the tax t that will maximize the total tax revenue T.
1. Write the supply function after taxation.	1. $p = 200 + \frac{1}{3}q + t$
2. Set the demand function and the new supply function equal, and solve for t.	2. $600 - q = 200 + \frac{1}{3}q + t$ $400 - \frac{4}{3}q = t$
3. Form the total tax revenue function, $T = tq$, by multiplying the equation for t by q, and then take its derivative with respect to q.	3. $T = tq = 400q - \frac{4}{3}q^2$ $T'(q) = \frac{dT}{dq} = 400 - \frac{8}{3}q$
4. Set $T' = 0$, and solve for q. This is the q that should maximize T. Use the second-derivative test to verify it.	4. $0 = 400 - \frac{8}{3}q$ $q = 150$ $T''(q) = -\frac{8}{3}$. Thus T is maximized at $q = 150$.
5. Substitute the value of q into the equation for t (in step 2). This is the value of t that will maximize T.	5. $t = 400 - \frac{4}{3}(150) = 200$ A tax of \$200 per item will maximize the total tax revenue. The total tax revenue for the period would be $\$200 \cdot (150) = \$30{,}000$.

Note that in the example just given, if a tax of \$300 were imposed, the total tax revenue the government would receive would be

$$(\$300)(75) = \$22{,}500$$

This means that consumers would spend \$100 more for each item, suppliers would sell 75 fewer items, and the government would lose \$7500 in tax revenue. Thus everyone would suffer if the tax rate were raised to \$300.

CHECKPOINT

4. For problems involving taxation in a competitive market, if supply is $p = f(q)$ and demand is $p = g(q)$, is the tax t added to $f(q)$ or to $g(q)$?

EXAMPLE 4

The demand and supply functions for a product are $p = 900 - 20q - \frac{1}{3}q^2$ and $p = 200 + 10q$, respectively. Find the tax per unit that will maximize the tax revenue T.

Solution

After taxation, the supply function is $p = 200 + 10q + t$, where t is the tax per unit. The demand function will meet the new supply function where

$$900 - 20q - \frac{1}{3}q^2 = 200 + 10q + t$$

so

$$t = 700 - 30q - \frac{1}{3}q^2$$

Then the total tax T is $T = tq = 700q - 30q^2 - \frac{1}{3}q^3$, and we maximize T as follows:

$$\begin{aligned} T'(q) &= 700 - 60q - q^2 \\ 0 &= -(q + 70)(q - 10) \\ q &= 10 \quad \text{or} \quad q = -70 \end{aligned}$$

Because $q = -70$ is meaningless, we test $q = 10$.

$$\left.\begin{array}{l} T(0) = 0 \\ T'(0) = 700 > 0 \\ T'(q) < 0 \text{ for all } q > 10 \end{array}\right\} \Rightarrow \text{absolute maximum at } q = 10$$

The maximum possible tax revenue is

$$T(10) = \$3666.67$$

The tax per unit that maximizes T is

$$t = 700 - 30(10) - \frac{1}{3}(10)^2 = \$366.67$$

An infamous example of a tax increase that resulted in decreased tax revenue and economic disaster is the "luxury tax" that went into effect in 1991. This was a 10% excise tax on the sale of more expensive jewelry, furs, airplanes, certain expensive boats, and automobiles costing over \$30,000. The Congressional Joint Tax Committee had estimated that the luxury tax would raise \$6 million from airplanes alone, but it raised only \$53,000 while it destroyed the small-airplane market (one company lost \$130 million and 480 jobs in a single year). It also capsized the boat market. The luxury tax was repealed at the end of 1993 (except for automobiles).*

**Fortune*, Sept. 6, 1993; *Motor Trend*, December 1993.

CHECKPOINT SOLUTIONS

1. $\eta = \dfrac{-p}{q} \cdot \dfrac{dq}{dp}$
2. (a) elastic (b) inelastic (c) unitary elastic
3. $\dfrac{dq}{dp} = \dfrac{-100}{p^2}$ and $\dfrac{dq}{dp} = -1$ when $p = 10, q = 9$

   $\eta = \dfrac{-10}{9}(-1) = \dfrac{10}{9}$ (elastic)
4. Tax t is added to supply: $p = f(q) + t$.

EXERCISE 11.5

Elasticity of Demand

1. (a) Find the elasticity of the demand function $p + 4q = 80$ at $(10, 40)$.
   (b) How will a price increase affect total revenue?
2. (a) Find the elasticity of the demand function $2p + 3q = 150$ at the price $p = 15$.
   (b) How will a price increase affect total revenue?
3. (a) Find the elasticity of the demand function $p^2 + 2p + q = 49$ at $p = 6$.
   (b) How will a price increase affect total revenue?
4. (a) Find the elasticity of the demand function $pq = 81$ at $p = 3$.
   (b) How will a price increase affect total revenue?
5. Suppose that the demand for a product is given by $pq + p = 5000$.
   (a) Find the elasticity when $p = \$50$ and $q = 99$.
   (b) Tell what type of elasticity this is: unitary, elastic, or inelastic.
   (c) How would revenue be affected by a price increase?
6. Suppose that the demand for a product is given by $2p^2q = 10{,}000 + 9000p^2$.
   (a) Find the elasticity when $p = \$50$ and $q = 4502$.
   (b) Tell what type of elasticity this is: unitary, elastic, or inelastic.
   (c) How would revenue be affected by a price increase?
7. Suppose that the demand for a product is given by $pq + p + 100q = 50{,}000$.
   (a) Find the elasticity when $p = \$401$.
   (b) Tell what type of elasticity this is.
   (c) How would a price increase affect revenue?
8. Suppose that the demand for a product is given by

$$(p + 1)\sqrt{q + 1} = 1000$$

   (a) Find the elasticity when $p = \$39$.
   (b) Tell what type of elasticity this is.
   (c) How would a price increase affect revenue?
9. Suppose the demand function for a product is given by

$$p = \frac{1}{2}\ln\left(\frac{5000 - q}{q + 1}\right)$$

   where p is in hundreds of dollars and q is the number of tons.
   (a) What is the elasticity of demand when the quantity demanded is 2 tons and the price is \$371?
   (b) Is the demand elastic or inelastic?
10. Suppose the weekly demand function for a product is

$$q = \frac{5000}{1 + e^{2p}} - 1$$

   where p is the price in thousands of dollars and q is the number of units demanded. What is the elasticity of demand when price is \$1000 and the quantity demanded is 595?

In Problems 11 and 12, the demand functions for specialty steel products are given. For both problems:

(a) Find the elasticity of demand as a function of quantity demanded, q.

(b) Find the point at which demand is of unitary elasticity and find intervals in which the demand is inelastic and intervals in which it is elastic.

(c) Use information about elasticity in (b) to decide where the revenue is increasing, where it is decreasing, and where it is maximized.

(d) Graph the revenue function $R = pq$, and use it to find where revenue is maximized. Is it at the same quantity as that determined in (c)?

11. $p = 120\sqrt[3]{125 - q}$ 12. $p = 30\sqrt{49 - q}$

Taxation in a Competitive Market

13. If the weekly demand function is $p = 30 - q$ and the supply function before taxation is $p = 6 + 2q$, what tax per item will maximize the total tax revenue?
14. If the demand function for a fixed period of time is given by $p = 38 - 2q$ and the supply function before taxation is $p = 8 + 3q$, what tax per item will maximize the total tax revenue?
15. If the demand and supply functions for a product are $p = 800 - 2q$ and $p = 100 + 0.5q$, respectively, find the tax per unit t that will maximize the tax revenue T.
16. If the demand and supply functions for a product are $p = 2100 - 3q$ and $p = 300 + 1.5q$, respectively, find the tax per unit t that will maximize the tax revenue T.
17. If the weekly demand function is $p = 200 - 2q^2$ and the supply function before taxation is $p = 20 + 3q$, what tax per item will maximize the total tax revenue?
18. If the monthly demand function is $p = 7230 - 5q^2$ and the supply function before taxation is $p = 30 + 30q^2$, what tax per item will maximize the total revenue?
19. Suppose the weekly demand for a product is given by $p + 2q = 840$ and the weekly supply before taxation is given by $p = 0.02q^2 + 0.55q + 7.4$. Find the tax per item that produces maximum tax revenue. Find the tax revenue.
20. If the daily demand for a product is given by the function $p + q = 1000$ and the daily supply before taxation is $p = q^2/30 + 2.5q + 920$, find the tax per item that maximizes tax revenue. Find the tax revenue.
21. If the demand and supply functions for a product are $p = 2100 - 10q - 0.5q^2$ and $p = 300 + 5q + 0.5q^2$, respectively, find the tax per unit t that will maximize the tax revenue T.
22. If the demand and supply functions for a product are $p = 5000 - 20q - 0.7q^2$ and $p = 500 + 10q + 0.3q^2$, respectively, find the tax per unit t that will maximize the tax revenue T.

KEY TERMS AND FORMULAS

Section	Key Terms	Formula
11.1	Logarithmic function	$y = \log_a x$, defined by $x = a^y$
	Natural logarithm	$\ln x = \log_e x$
	Logarithmic Properties I–V for natural logarithms	$\ln e^x = x$; $e^{\ln x} = x$; $\ln(MN) = \ln M + \ln N$; $\ln(M/N) = \ln M - \ln N$; $\ln(M^N) = N(\ln M)$
	Change-of-base formula	$\log_a x = \dfrac{\ln x}{\ln a}$
	Derivatives of logarithmic functions	$\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$; $\dfrac{d}{dx}(\ln u) = \dfrac{1}{u} \cdot \dfrac{du}{dx}$
11.2	Exponential function	$f(x) = a^x$
	e	$e = \lim\limits_{a \to 0} (1 + a)^{1/a}$
	Derivatives of exponential functions	$\dfrac{d}{dx}(e^x) = e^x$; $\dfrac{d}{dx} e^u = e^u \dfrac{du}{dx}$; $\dfrac{d}{dx} a^u = a^u \dfrac{du}{dx} \ln a$
11.3	Implicit differentiation	
11.4	Related rates	
	Percentage rates of change	
11.5	Elasticity of demand	$\eta = \dfrac{-p}{q} \cdot \dfrac{dq}{dp}$
	Elastic	$\eta > 1$
	Inelastic	$\eta < 1$
	Unitary elastic	$\eta = 1$
	Taxation in competitive market	
	Supply function after taxation	$p = f(q) + t$

REVIEW EXERCISES

Sections 11.1 and 11.2

In Problems 1–8, find the indicated derivative.

1. If $y = e^{3x^2 - x}$, find dy/dx.
2. If $y = \ln e^{x^2}$, find y'.
3. If $p = \ln\left(\dfrac{q}{q^2 - 1}\right)$, find $\dfrac{dp}{dq}$.
4. If $y = xe^{x^2}$, find dy/dx.
5. If $y = 3^{3x-4}$, find dy/dx.
6. If $y = 1 + \ln x^{10}$, find dy/dx.
7. If $y = \dfrac{\ln x}{x}$, find $\dfrac{dy}{dx}$.
8. If $y = \dfrac{1 + e^{-x}}{1 - e^{-x}}$, find $\dfrac{dy}{dx}$.
9. Write the equation of the line tangent to $y = 4e^{x^3}$ at $x = 1$.
10. Write the equation of the line tangent to $y = x \ln x$ at $x = 1$.

Section 11.3

In Problems 11–16, find the indicated derivative.

11. If $y \ln x = 5$, find dy/dx.
12. Find dy/dx for $e^{xy} = y$.
13. Find dy/dx for $y^2 = 4x - 1$.
14. Find dy/dx if $x^2 + 3y^2 + 2x - 3y + 2 = 0$.
15. Find dy/dx for $3x^2 + 2x^3y^2 - y^5 = 7$.
16. Find the second derivative y'' if $x^2 + y^2 = 1$.
17. Find the slope of the tangent to the curve $x^2 + 4x - 3y^2 + 6 = 0$ at $(3, 3)$.
18. Find the points where tangents to the graph of the equation in Problem 17 are horizontal.

Section 11.4

19. Suppose $3x^2 - 2y^3 = 10y$, where x and y are differentiable functions of t. If $dx/dt = 2$, find dy/dt when $x = 10$ and $y = 5$.
20. A right triangle with legs of lengths x and y has its area given by

$$A = \frac{1}{2}xy$$

If the rate of change of x is 2 units per minute and the rate of change of y is 5 units per minute, find the rate of change of the area when $x = 4$ and $y = 1$.

Applications

Section 11.2

21. ***Compound interest*** If the future value of \$1000 invested for n years at 12%, compounded continuously, is given by

$$S = 1000e^{0.12n}$$

find the rate at which the future value is growing after 1 year.

22. ***Compound interest***
    (a) In Problem 21, find the rate of growth of the future value after 2 years.
    (b) How much faster is the future value growing at the end of 2 years than after 1 year?
23. ***Radioactive decay*** A breeder reactor converts stable uranium-238 into the isotope plutonium-239. The decay of this isotope is given by

$$A(t) = A_0e^{-0.00002876t}$$

where $A(t)$ is the amount of isotope at time t, in years, and A_0 is the original amount. This isotope has a half-life of 24,101 years (that is, half of it will decay away in 24,101 years).
    (a) At what rate is $A(t)$ decaying at this point in time?
    (b) At what rate is $A(t)$ decaying after 1 year?
    (c) Is the rate of decay at its half-life greater or less than after 1 year?
24. ***Marginal cost*** The average cost of producing x units of a product is $\overline{C} = 600e^{x/600}$. What is the marginal cost when 600 units are produced?
25. ***Inflation*** The impact of inflation on a \$20,000 pension can be measured by the purchasing power P of \$20,000 after t years. For an inflation rate of 5% per year, compounded annually, P is given by

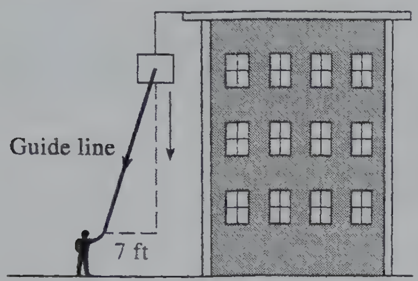
$$P = 20{,}000e^{-0.0495t}$$

At what rate is purchasing power changing when $t = 10$? (Source: *Viewpoints*, VALIC, Summer 1993)

Section 11.4

26. ***Evaporation*** A spherical droplet of water evaporates at a rate of 1 mm³/min. Find the rate of change of the radius when the droplet has a radius of 2.5 mm.

27. ***Worker safety*** A sign is being lowered over the side of a building at the rate of 2 ft/min. A worker handling a guide line is 7 ft away from a spot directly below the sign. How fast is the worker taking in the guide line at the instant the sign is 25 ft from the worker's hands? See the figure below.



28. ***Environment*** Suppose that in a study of water birds, the relationship between the square miles of wetlands A and the number of different species S of birds found in the area was determined to be

$$S = kA^{1/3}$$

where k is constant. Find the percentage rate of change of the number of species in terms of the percentage rate of change of the area.

Section 11.5

29. ***Taxes*** Can increasing the tax per unit sold actually lead to a decrease in tax revenues?

30. ***Taxes*** If the demand and supply functions for a product are

$$p = 2800 - 8q - \frac{q^2}{3} \quad \text{and} \quad p = 400 + 2q$$

respectively, find the tax per unit t that will maximize the tax revenue T.

31. ***Taxes*** If the supply and demand functions for a product are

$$p = 40 + 20q \quad \text{and} \quad p = \frac{5000}{q + 1}$$

respectively, find the tax t that maximizes the tax revenue T.

32. ***Elasticity*** A demand function is given by

$$pq = 27$$

(a) Find the elasticity of demand at (9, 3).
(b) How will a price increase affect total revenue?

33. ***Elasticity*** Suppose the demand for a product is given by

$$p^2(2q + 1) = 10{,}000$$

Find the elasticity of demand when $p = \$20$.

34. ***Elasticity*** Suppose the weekly demand function for a product is given by

$$p = 100e^{-0.1q}$$

where p is the price in dollars and q is the number of tons demanded. What is the elasticity of demand when price is \$36.79 and the quantity demanded is 10?

35. ***Revenue*** A product has the demand function

$$p = 100 - 0.5q$$

(a) Find the elasticity $\eta(q)$ as a function of q, and graph the function

$$f(q) = \eta(q) - 1$$

(b) Find the q-intercept of $f(q)$, which gives the quantity for which the product has unitary elasticity.
(c) The revenue function for this product is

$$R(q) = pq = (100 - 0.5q)q$$

Graph $R(q)$ and find the q-value for which the maximum revenue occurs.
(d) What is the relationship between elasticity and maximum revenue?

CHAPTER TEST

Find the derivatives of the following functions.

1. $y = 4 \ln(x^3 + 1)$
2. $y = \ln(x^4 + 1)^3$
3. $y = \dfrac{\ln x}{x}$
4. $y = 5e^{x^3} + x^2$
5. $S = te^{t^4}$
6. $y = \dfrac{e^{x^3+1}}{x}$
7. $f(x) = 10(3^{2x})$
8. $g(x) = 2 \log_5(4x + 7)$
9. Find y' if $3x^4 + 2y^2 + 10 = 0$.
10. Find y' if $xe^y = 10y$.
11. Let $x^2 + y^2 = 100$. If $\dfrac{dx}{dt} = 2$, find $\dfrac{dy}{dt}$ when $x = 6$ and $y = 8$.
12. The sales of a product are given by $S = 80{,}000e^{-0.4t}$, where S is the daily sales and t is the number of days after the end of an advertising campaign. Find the rate of sales decay 10 days after the end of the ad campaign.
13. Suppose the demand function for a product is given by $(p + 1)q^2 = 10{,}000$, where p is the price and q is the quantity. Find the rate of change of quantity with respect to price when $p = \$99$.
14. Suppose the weekly revenue and weekly cost for a product are given by $R(x) = 300x - 0.001x^2$ and $C(x) = 4000 + 30x$, respectively, where x is the number of units produced and sold. Find the rate at which profit is changing with respect to time when the number of units produced and sold is 50 and is increasing at a rate of 5 units per week.
15. If the demand for a product is $p^2 + 3p + q = 1500$, find the elasticity of demand at $p = 30$. If the price is raised to \$31, does revenue increase or decrease?
16. The following table gives the world population, in millions, for selected years. If the equation that models these data is

$$y = 160e^{0.00648(x - 1500)}$$

where $x = 1$ in year 1, what does this model give as the rate of growth of world population (a) in 1930? (b) in 1997?

Year	*World Population (millions)*
1	200
1650	500
1850	1000
1930	2000
1975	4000
1997	5800

Source: *World Almanac*, 1997

17. The percent of income claimed by federal, state, and local taxes for median households with two incomes is given in the table below. Using the number of years from 1950 as x and the percent as y, the logarithmic equation that models the data is

$$y = 19.08889 + 4.3271 \ln x$$

What does this model predict as the rate of increase of the percent claimed for taxes in 2000?

Year	*Percent of Income Paid in Taxes*
1955	26.7
1965	28.8
1975	34.2
1985	34.8
1997	35.6

Source: Tax Foundation

18. The U.S. national debt can be modeled by the equation

$$y = 1.5336(1.08909)^x$$

where x is the number of years from 1900 and y is in billions of dollars. Use the model to predict the rate of growth of the national debt in 2005.

19. If the demand and supply functions for a product are $p = 1100 - 5q$ and $p = 20 + 0.4q$, respectively, find the tax per unit t that will maximize the tax revenue $T = tq$.

Extended Applications Group Projects

I. Inflation

Hollingsworth Pharmaceuticals specializes in manufacturing generic medicines. Recently it developed an antibiotic with outstanding profit potential. The new antibiotic's total costs, sales, and sales growth, as well as projected inflation, are described as follows.

Total monthly costs to produce x *units (1 unit is 100 capsules):*

$$C(x) = \begin{cases} 15{,}000 + 10x & 0 \le x \le 11{,}000 \\ 15{,}000 + 10x + 0.001(x - 11{,}000)^2 & x \ge 11{,}000 \end{cases}$$

Sales: 10,000 units per month and growing at 1.25% per month
Selling price: \$17 per unit
Inflation: Approximately 0.25% per month, affecting both total costs and selling price

Company owners are pleased with the sales growth but are concerned about the projected increase in variable costs when production levels exceed 11,000 units per month. The consensus is that improvements eventually can be made that will reduce costs at higher production levels, thus altering the current cost function model. To plan properly for these changes. Hollingsworth Pharmaceuticals would like you to determine when the company's profits will begin to decrease. To help you determine this, answer the following.

1. If inflation is assumed to be compounded continuously, the selling price and total costs must be multiplied by the factor $e^{0.0025t}$. In addition, if sales growth is assumed to be compounded continuously, then sales must be multiplied by a factor of the form e^{rt}, where r is the monthly sales growth rate (expressed as a decimal) and t is time in months. Use these factors to write each of the following as a function of time t:

   (a) selling price p per unit (including inflation)
   (b) number of units x sold per month (including sales growth)
   (c) total revenue (Recall that $R = px$.)

2. Determine how many months it will be before monthly sales exceed 11,000 units.
3. If you restrict your attention to total costs when $x \ge 11{,}000$, then, after expanding and collecting like terms, $C(x)$ can be written as follows:

   $$C(x) = 136{,}000 - 12x + 0.001x^2 \quad \text{for } x \ge 11{,}000$$

   Use this form for $C(x)$ with your result from Question 1(b) and with the inflationary factor $e^{0.0025t}$ to express these total costs as a function of time.

4. Form the profit function that would be used when monthly sales exceed 11,000 units by using the total revenue function from Question 1(c) and the total cost function from Question 3. This profit function should be a function of time t.
5. Find how long it will be before the profit is maximized. You may have to solve $P'(t) = 0$ by using a graphing calculator or computer to find the t-intercept of the graph of $P'(t)$. In addition, because $P'(t)$ has large numerical coefficients, you may want to divide both sides of $P'(t) = 0$ by 1000 before solving or graphing.

II. Knowledge Workers

In January 1997, *Working Woman* made the following points about today's economy and the place of women in the economy.

- The telecommunications industry employs more people than the auto and auto parts industries combined.
- More Americans make semiconductors than make construction equipment.
- Almost twice as many Americans make surgical and medical instruments as make plumbing and heating products.
- The ratio of male to female knowledge workers (engineers, scientists, technicians, professionals, and senior managers) was 3 to 2 in 1983. The following table, which gives the number (in millions) of male and female knowledge workers from 1983 to 1997, shows how that ratio is changing.

Year	*Female Knowledge Workers (millions)*	*Male Knowledge Workers (millions)*
1983	11.0	15.4
1984	11.6	15.9
1985	12.3	16.3
1986	12.9	16.7
1987	13.6	16.8
1988	14.3	17.6
1989	15.3	18.1
1990	15.9	18.6
1991	16.1	18.4
1992	16.7	18.66
1993	17.3	18.7
1994	18.0	19.0
1995	18.5	19.8
1996	19.0	19.6
1997	19.5	19.8

Source: *Working Woman*, January 1997

To compare how the growth in the number of female knowledge workers compares with that of male knowledge workers, do the following.

1. Find a logarithmic equation (with $x = 0$ in 1980) that models the number of females, and find a logarithmic equation (with $x = 0$ in 1980) that models the number of males.
2. Find the rate of growth with respect to time of the number of female knowledge workers by taking the derivative of the equation that models the number.
3. Find the rate of growth with respect to time of the number of male knowledge workers by taking the derivative of the equation that models the number.
4. Compare the two rates of growth in the year 2000 and determine which rate is larger.
5. If these models indicate that it is possible for the number of females to equal the number of males, during what year do they indicate that this will occur?

Warm-up

Prerequisite Problem Type	For Section	Answer	Section for Review
Write as a power: (a) $\sqrt{x}$ (b) $\sqrt{x^2-9}$	12.1–12.4	(a) $x^{1/2}$ (b) $(x^2-9)^{1/2}$	0.4 Radicals
Expand $(x^2+4)^2$.	12.2	x^4+8x^2+16	0.5 Special powers
Divide $x^4-2x^3+4x^2-7x-1$ by x^2-2x.	12.3	$x^2+4+\frac{x-1}{x^2-2x}$	0.5 Division
Find the derivative of (a) $f(x)=2x^{1/2}$ (b) $u=x^3-3x$	12.1 12.2 12.3	(a) $f'(x)=x^{-1/2}$ (b) $u'=3x^2-3$	9.4 Derivatives
If $y=\frac{(x^2+4)^6}{6}$, what is y'?	12.2	$(x^2+4)^5 2x$	9.6 Derivatives
(a) If $y=\ln u$, what is y'? (b) If $y=e^u$, what is y'?	12.3	(a) $y'=\frac{1}{u}\cdot u'$ (b) $y'=e^u\cdot u'$	11.1, 11.2 Derivatives
Solve for y: $\ln y=kt+C$	12.5	$y=e^{kt+C}$	5.2 Logarithmic functions
Solve for k: $0.5=e^{5600k}$	12.5	$k\approx-0.00012378$	5.3 Exponential equations

Chapter 12

Indefinite Integrals

If the marginal cost for a product is \$36 at all levels of production, we know that the total cost function is a linear function. In particular, $C(x) = 36x + FC$, where FC is the fixed cost. But if the marginal cost changes at different levels of production, the total cost function cannot be linear. In this chapter we will use integration to find total cost functions, given information about marginal costs and fixed costs.

Accountants can use linear regression to translate information about marginal cost into a linear equation defining (approximately) the marginal cost function. By integrating this marginal cost function, it is possible to find an (approximate) function that defines the total cost.

We can also use integration to find total revenue functions from marginal revenue functions, to optimize profit from information about marginal cost and marginal revenue, and to find national consumption functions from information about marginal propensity to consume.

Integration can be used in the social and life sciences to predict growth or decay from expressions giving rates of change. For example, we can determine equations for population size from the rate of growth; we can write equations for the number of radioactive atoms remaining in a substance if we know the rate of disintegration of the substance; and we can determine the volume of blood flow from information about the rate of flow.

12.1 The Indefinite Integral

OBJECTIVE

- To find certain indefinite integrals

APPLICATION PREVIEW

In our study of the theory of the firm, we have worked with total cost, total revenue, and profit functions and have found their marginal functions. In practice, it is often easier for a company to measure marginal cost, revenue, and profit and use these data to form marginal functions from which it can find total cost, revenue, and profit functions. For example, Jarus Technologies manufactures computer memory boards, and the company's sales records show that the marginal revenue for its 64 MB memory board is given by

$$\overline{MR} = 300 - 0.2x$$

where x is the number of units sold. If we want to use this function to find the total revenue function for Jarus Technologies' 64 MB memory board, we need to find $R(x)$ from $\overline{MR} = R'(x)$. In this situation, we need to be able to reverse the process of differentiation.

We have studied procedures for and applications of finding derivatives of a given function. We now turn our attention to reversing this process of differentiation. When we know the derivative of a function, the process of finding the function itself is called **antidifferentiation.** For example, if the derivative of a function is $2x$, we know that the function could be $f(x) = x^2$ because $\frac{d}{dx}(x^2) = 2x$. But the function could also be $f(x) = x^2 + 4$ because $\frac{d}{dx}(x^2 + 4) = 2x$. It is clear that any function of the form $f(x) = x^2 + C$, where C is a constant, will have $f'(x) = 2x$ as its derivative. Thus we say that the **general antiderivative** of $f'(x) = 2x$ is $f(x) = x^2 + C$, where C is an arbitrary constant.

Antiderivative A function $F(x)$ is called an **antiderivative** of a function $f(x)$ if, for every x in the domain of f, $F'(x) = f(x)$. If C is an arbitrary constant, then $F(x) + C$ is called the **general antiderivative** of $f(x)$ because all antiderivatives of $f(x)$ have this form.

EXAMPLE 1

If $f'(x) = 3x^2$, what is $f(x)$?

Solution

The derivative of the function $f(x) = x^3$ is $f'(x) = 3x^2$. But other functions also have this derivative. They will all be of the form $f(x) = x^3 + C$, where C is a constant. Thus we say that $f(x) = x^3 + C$ is the general antiderivative of $f'(x) = 3x^2$.

EXAMPLE 2

If $f'(x) = x^3$, what is $f(x)$?

Solution

We know that the derivative of $f(x) = x^4$ is $4x^3$, so the derivative of $f(x) = \frac{1}{4}x^4$ is $f'(x) = x^3$. Thus any function of the form $f(x) = \frac{1}{4}x^4 + C$ will have the derivative $f'(x) = x^3$.

It is easily seen that

$$\text{if } f'(x) = x^4, \quad \text{then} \quad f(x) = \frac{x^5}{5} + C \text{ is the general antiderivative;}$$

$$\text{if } f'(x) = x^5, \quad \text{then} \quad f(x) = \frac{x^6}{6} + C \text{ is the general antiderivative.}$$

In general, we have the following.

Antiderivative of $f(x) = x^n$

If $f'(x) = x^n$, then $f(x) = \dfrac{x^{n+1}}{n+1} + C$, for $n \neq -1$.

We can see that this general formula applies for any $n \neq -1$ by noting that the derivative of

$$f(x) = \frac{x^{n+1}}{n+1} + C \quad \text{is} \quad f'(x) = \frac{(n+1)x^n}{n+1} + 0 = x^n$$

Later, we will discuss the case where $n = -1$.

EXAMPLE 3

What is the general antiderivative of $f'(x) = x^{-1/2}$?

Solution

Using the formula, we get

$$f(x) = \frac{x^{1/2}}{1/2} + C = 2x^{1/2} + C$$

We can check by noting that the derivative of $2x^{1/2} + C$ is $x^{-1/2}$.

Note that the general antiderivative in Example 3 is a function (actually a number of functions, one for each value of C). Several members of this family of functions are shown in Figure 12.1 on the next page. Note that at any given x-value, the tangent line to each curve would have the same slope, indicating that all family members have the same derivative.

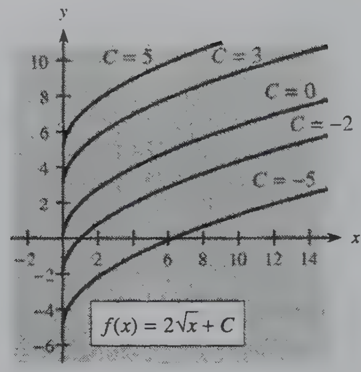


Figure 12.1

The process of finding an antiderivative is called **integration.** The function that results when integration takes place is called an **indefinite integral** or, more simply, an **integral.** We can denote the indefinite integral (that is, the general antiderivative) of a function $f(x)$ by $\int f(x)\,dx$. Thus we can write $\int x^2\,dx$ to indicate the general antiderivative of the function $f(x) = x^2$. The expression is read as "the integral of x^2 with respect to x." In this case, x^2 is called the **integrand.** The **integral sign,** $\int$, indicates the process of integration, and the dx indicates that the integral is to be taken with respect to x. Because the antiderivative of x^2 is $(x^3/3) + C$, we can write

$$\int x^2\,dx = \frac{x^3}{3} + C$$

We can now use the integral sign and rewrite the formula for integrating powers of x.

Powers of x Formula

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (\text{for } n \neq -1)$$

EXAMPLE 4

Find (a) $\int \sqrt[3]{x}\,dx$ and (b) $\int \frac{1}{x^2}\,dx$.

Solution

(a) $$\int \sqrt[3]{x}\,dx = \int x^{1/3}\,dx = \frac{x^{4/3}}{4/3} + C$$
$$= \frac{3}{4}x^{4/3} + C = \frac{3}{4}\sqrt[3]{x^4} + C$$

(b) We write the power of x in the numerator so that the integral has the form in the formula above.

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} + C = \frac{x^{-1}}{-1} + C = \frac{-1}{x} + C$$

Other formulas will be useful in evaluating integrals. The following table shows how some new integration formulas result from differentiation formulas.

Integration Formulas

Derivative	Resulting Integral
$\frac{d}{dx}(x) = 1$	$\int 1\,dx = \int dx = x + C$
$\frac{d}{dx}[c \cdot u(x)] = c \cdot \frac{d}{dx}u(x)$	$\int c\,u(x)\,dx = c\int u(x)\,dx$
$\frac{d}{dx}[u(x) \pm v(x)] = \frac{d}{dx}u(x) \pm \frac{d}{dx}v(x)$	$\int [u(x) \pm v(x)]\,dx = \int u(x)\,dx \pm \int v(x)\,dx$

The formulas above indicate that we can integrate functions term by term just as we were able to take derivatives term by term.

EXAMPLE 5

Evaluate $\int 4\,dx$.

Solution

$$\int 4\,dx = 4\int dx = 4(x + C_1) = 4x + C$$

(Because C_1 is an unknown constant, we can write $4C_1$ as the unknown constant C.)

EXAMPLE 6

Evaluate $\int 8x^5\,dx$.

Solution

$$\int 8x^5\,dx = 8\int x^5\,dx = 8\left(\frac{x^6}{6} + C_1\right) = \frac{4x^6}{3} + C$$

EXAMPLE 7

Evaluate $\int (x^3 + 4x)\,dx$.

Solution

$$\begin{aligned}\int (x^3 + 4x)\,dx &= \int x^3 dx + \int 4x\,dx \\ &= \left(\frac{x^4}{4} + C_1\right) + \left(4 \cdot \frac{x^2}{2} + C_2\right) \\ &= \frac{x^4}{4} + 2x^2 + C_1 + C_2 \\ &= \frac{x^4}{4} + 2x^2 + C\end{aligned}$$

Note that we need only one constant because the sum of C_1 and C_2 is just a new constant.

EXAMPLE 8

Evaluate $\int (x^2 - 4)^2\, dx$.

Solution

We expand $(x^2 - 4)^2$ so that the integrand is in a form that fits the basic integration formulas.

$$\int (x^2 - 4)^2\, dx = \int (x^4 - 8x^2 + 16)\, dx = \frac{x^5}{5} - \frac{8x^3}{3} + 16x + C$$

CHECKPOINT

1. True or false:
   (a) $\int (4x^3 - 2x)\, dx = \int 4x^3 dx - \int 2x\, dx$
   $= (x^4 + C) - (x^2 + C) = x^4 - x^2$
   (b) $\displaystyle\int \frac{1}{3x^2}\, dx = \frac{1}{3(x^3/3)} + C = \frac{1}{x^3} + C$
2. Evaluate $\int (2x^3 + x^{-1/2} - 4x^{-5})\, dx$.

We now return to the Application Preview problem and consider how to find total revenue from marginal revenue.

EXAMPLE 9

Sales records at Jarus Technologies show that the rate of change of the revenue (that is, the marginal revenue) for an 64 MB memory board is $\overline{MR} = 300 - 0.2x$, where x represents the quantity sold. Find the total revenue function for the product.

Solution

We know that the marginal revenue can be found by differentiating the total revenue function. That is,

$$R'(x) = 300 - 0.2x$$

Thus integrating the marginal revenue function gives the total revenue function.

$$R(x) = \int (300 - 0.2x)\, dx = 300x - 0.1x^2 + K^*$$

We can use the fact that there is no revenue when no units are sold to evaluate K. Setting $x = 0$ and $R = 0$ gives $0 = 300(0) - 0.1(0)^2 + K$, so $K = 0$. Thus the total revenue function is

$$R(x) = 300x - 0.1x^2$$

*Here we are using K rather than C to represent the constant of integration to avoid confusion between the constant C and the cost function $C = C(x)$.

Graphing Utilities

We can check that the $R(x)$ we found in Example 9 is correct by verifying that $R'(x) = 300 - 0.2x$ and $R(0) = 0$. Also, graphs can help us check the reasonableness of our result. Figure 12.2 shows the graphs of $\overline{MR} = 300 - 0.2x$ and of the $R(x)$ we found. Note that $R(x)$ passes through the origin, indicating $R(0) = 0$. Also, reading both graphs from left to right, we see that $R(x)$ increases when $\overline{MR} > 0$, attains its maximum when $\overline{MR} = 0$, and decreases when $\overline{MR} < 0$.

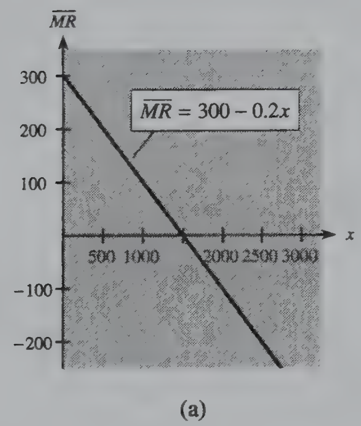


Figure 12.2

We mentioned in Chapter 9, "Derivatives," that graphing utilities have a numerical derivative feature that can be used to check graphically the derivative of a function that has been calculated with a formula. We can also use the numerical integration feature on graphing utilities to check our integration (if we assume temporarily that the constant of integration is 0). We do this by graphing the integral calculated with a formula and the numerical integral from the graphing utility on the same set of axes. If the graphs lie on top of one another, the integrals agree. Figure 12.3 illustrates this for the function $f(x) = 3x^2 - 2x + 1$. Its integral, with the constant of integration set equal to 0, is shown as $y_1 = x^3 - x^2 + x$ in Figure 12.3(a). Of course, it is often easier to use the derivative to check integration.

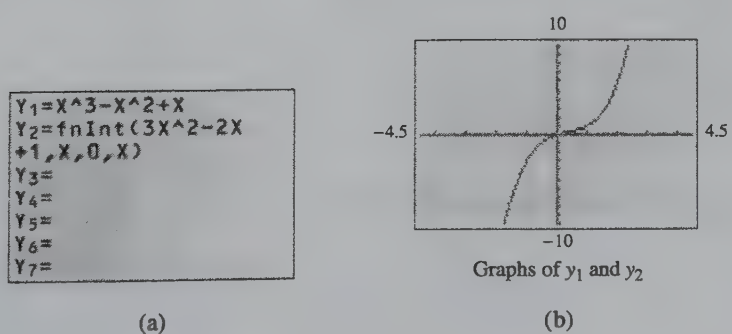


Figure 12.3

CHECKPOINT SOLUTIONS

1. (a) False, $\int (4x^3 - 2x)\,dx = \int 4x^3\,dx - \int 2x\,dx = (x^4 + C_1) - (x^2 + C_2)$
$$= x^4 - x^2 + C.$$
(b) False, $\displaystyle\int \frac{1}{3x^2}\,dx = \int \frac{1}{3}\cdot\frac{1}{x^2}\,dx = \frac{1}{3}\int x^{-2}\,dx$
$$= \frac{1}{3}\cdot\frac{x^{-1}}{-1} + C = \frac{-1}{3x} + C.$$
2. $\displaystyle\int (2x^3 + x^{-1/2} - 4x^{-5})\,dx = \frac{2x^4}{4} + \frac{x^{1/2}}{1/2} - \frac{4x^{-4}}{-4} + C$
$$= \frac{x^4}{2} + 2x^{1/2} + x^{-4} + C$$

EXERCISE 12.1

1. If $f'(x) = 4x^3$, what is $f(x)$?
2. If $f'(x) = 5x^4$, what is $f(x)$?
3. If $f'(x) = x^6$, what is $f(x)$?
4. If $g'(x) = x^4$, what is $g(x)$?

Evaluate the integrals in Problems 5–26. Check your answers by differentiating.

5. $\int x^7\,dx$
6. $\int x^5\,dx$
7. $\int 8x^5\,dx$
8. $\int 16x^9\,dx$
9. $\int (3^3 + x^{13})\,dx$
10. $\int (5^2 + x^{10})\,dx$
11. $\int (3 - x^{3/2})\,dx$
12. $\int (8 + x^{2/3})\,dx$
13. $\int (x^4 - 9x^2 + 3)\,dx$
14. $\int (3x^2 - 4x - 4)\,dx$
15. $\int (2 + 2\sqrt{x})\,dx$
16. $\int (17 + \sqrt{x^3})\,dx$
17. $\int 6\sqrt[4]{x}\,dx$
18. $\int 3\sqrt[3]{x^2}\,dx$
19. $\displaystyle\int \frac{5}{x^4}\,dx$
20. $\displaystyle\int \frac{6}{x^5}\,dx$
21. $\displaystyle\int \frac{dx}{2\sqrt[3]{x^2}}$
22. $\displaystyle\int \frac{2\,dx}{5\sqrt{x^3}}$
23. $\displaystyle\int \left(x^3 - 4 + \frac{5}{x^6}\right)dx$
24. $\displaystyle\int \left(x^3 - 7 - \frac{3}{x^4}\right)dx$
25. $\displaystyle\int \left(x^9 - \frac{1}{x^3} + \frac{2}{\sqrt[3]{x}}\right)dx$
26. $\displaystyle\int \left(3x^8 + \frac{4}{x^8} - \frac{5}{\sqrt[5]{x}}\right)dx$

In Problems 27–32, use algebra to rewrite the integrands; then integrate and simplify.

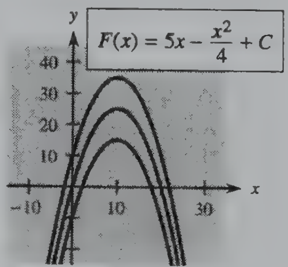
27. $\int (x + 5)^2 x\,dx$
28. $\int (2x + 1)^2 x\,dx$
29. $\int (4x^2 - 1)^2 x^3\,dx$
30. $\int (x^3 + 1)^2 x\,dx$
31. $\displaystyle\int \frac{x + 1}{x^3}\,dx$
32. $\displaystyle\int \frac{x - 3}{\sqrt{x}}\,dx$

In Problems 33 and 34, find the antiderivatives and graph the resulting family members that correspond to $C = 0$, $C = 4$, $C = -4$, $C = 8$, and $C = -8$.

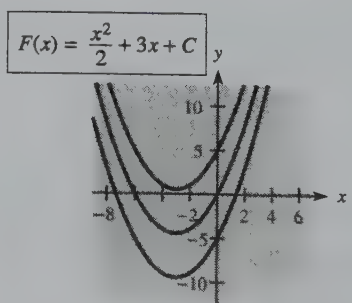
33. $\int (2x + 3)\,dx$
34. $\int (4 - x)\,dx$

In each of Problems 35–38, a family of functions is given and graphs of some members of the family are shown. Write the indefinite integral that gives the family.

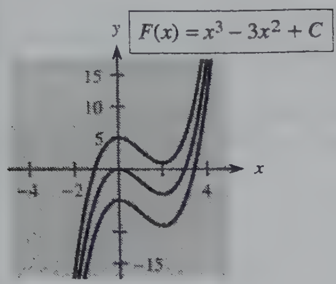
35. $F(x) = 5x - \dfrac{x^2}{4} + C$



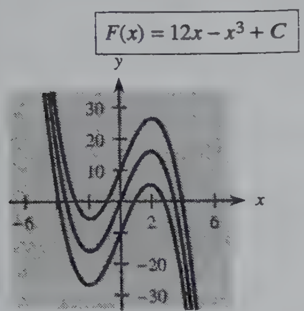
36. $F(x) = \dfrac{x^2}{2} + 3x + C$



37. $F(x) = x^3 - 3x^2 + C$



38. $F(x) = 12x - x^3 + C$



Applications

39. ***Revenue*** If the marginal revenue for a month for a commodity is $\overline{MR} = 3$, what is the total revenue function?

40. ***Revenue*** If the marginal revenue for a month for a commodity is $\overline{MR} = 5$, what is the total revenue function?

41. ***Revenue*** If the marginal revenue for a month for a commodity is $\overline{MR} = 0.4x + 3$, find the total revenue function.

42. ***Revenue*** If the marginal revenue for a month for a commodity is $\overline{MR} = 0.5x + 2$, find the total revenue function.

43. ***Revenue*** If the marginal revenue for a month is given by $\overline{MR} = 3x + 1$, what is the total revenue from the production and sale of 50 units?

44. ***Revenue*** If the marginal revenue for a month is given by $\overline{MR} = 5x + 3$, find the total revenue from the sale of 75 units.

45. ***Stimulus-response*** Suppose that when a sense organ receives a stimulus at time t, the total number of action potentials is $P(t)$. If the rate at which action potentials are produced is $t^3 + 4t^2 + 6$, and if there are 0 action potentials when $t = 0$, find the formula for $P(t)$.

46. ***Projectiles*** Suppose that a particle has been shot into the air in such a way that the rate at which its height is changing is $v = 320 - 32t$, in feet per second, and suppose that it is 1600 feet high when $t = 10$. Write the equation that describes the height of the particle at any time t.

47. ***Pollution*** A factory is dumping pollutants into a river at a rate given by $dx/dt = t^{3/4}/600$ tons per week, where t is the time in weeks since the dumping began and x is the number of tons of pollutants.
   (a) Find the equation for total tons of pollutants dumped.
   (b) How many tons were dumped during the first year?

48. ***Population growth*** The rate of growth of the population of a city is predicted to be

$$\frac{dp}{dt} = 1000t^{1.08}$$

where p is the population at time t, and t is measured in years from the present. Suppose that the current population is 100,000. What is the predicted
   (a) rate of growth 5 years from the present?
   (b) population 5 years from the present?

49. ***Average cost*** The DeWitt Company has found that the rate of change of its average cost for a product is

$$\overline{C}'(x) = \frac{1}{4} - \frac{100}{x^2}$$

in dollars, where x is the number of units. The average cost of producing 20 units is \$40.00.
   (a) Find the average cost function for the product.
   (b) Find the average cost of 100 units of the product.

50. ***Oil leakage*** An oil tanker hits a reef and begins to leak. The efforts of the workers repairing the leak cause the rate at which the oil is leaking to decrease. The oil was leaking at a rate of 31 barrels per hour at the end of the first hour after the accident, and the rate is decreasing at a rate of one barrel per hour.
   (a) What function describes the rate of loss?
   (b) How many barrels of oil will leak in the first 6 hours?
   (c) When will the oil leak be stopped? How much will have leaked altogether?

51. ***Revenue*** Assume that the rate of change of sales revenue for Scott Paper Company can be modeled by

$$\frac{dR}{dt} = -0.062t + 0.776$$

where t is the number of years past 1980, and sales revenues are in billions of dollars.

(a) If sales revenue in 1987 was \$3.9769 billion, find the function that gives sales revenue. Graph this function.
(b) The data in the table show Scott Paper Company's sales revenue in billions of dollars for selected years. Graph the data in the table.
(c) Compare the graphs in parts (a) and (b)

Year	*Sales Revenue (billions)*	*Year*	*Sales Revenue (billions)*
1983	\$2.6155	1989	4.8949
1984	2.7474	1990	5.1686
1985	2.934	1991	4.9593
1986	3.3131	1992	5.0913
1987	3.9769	1993	4.7489
1988	4.5494		

Source: Scott Paper Company, *1993 Annual Report*

52. ***Revenue*** Assume that the rate of change of AT&T's total revenues can be modeled by

$$\frac{dR}{dt} = 0.506t - 4.03$$

where t is the number of years past 1980, and total revenues are in billions of dollars.

(a) If AT&T's total revenue in 1985 was \$63.13 billion, find the function for total revenue. Graph this function.
(b) The data in the table give AT&T's total revenue for selected years. Graph the data in the table.
(c) Compare the graphs in parts (a) and (b).

Year	*Total Revenue (billions)*	*Year*	*Total Revenue (billions)*
1985	\$63.13	1990	62.191
1986	69.906	1991	63.089
1987	60.53	1992	64.904
1989	61.1	1993	67.156

Source: AT&T *Annual Report, 1993*

53. ***Personal savings*** Suppose the rate of personal savings in the United States is given by

$$\frac{dPS}{dt} = 2.1t^2 - 65.4t + 491.6$$

where t is the number of years past 1970, and personal savings PS is in billions of dollars.

(a) If personal savings in 1980 was \$153.8 billion, find the function that models personal savings. Graph this function.
(b) The data in the table show personal savings in the United States for selected years. Graph the data in the table.
(c) Compare the graphs in parts (a) and (b).

Year	*Personal Savings (billions)*	*Year*	*Personal Savings (billions)*
1980	\$153.8	1988	155.7
1985	189.3	1989	152.1
1986	187.5	1990	175.6
1987	142.0	1991	199.6

Source: *Survey of Current Business*, March 1993

54. ***Consumer debt*** Assume that the rate of change of consumer debt as a percentage of disposable income can be modeled by

$$\frac{dD}{dt} = 0.0112t^3 - 0.243t^2 + 1.36t - 1.3$$

where t is the number of years past 1980.

(a) If consumer debt in 1980 was 18.2% of disposable income, find the function that models consumer debt as a percentage of disposable income. Graph this function.
(b) The data in the table show consumer debt as a percentage of disposable income for selected years. Graph the data in the table.
(c) Compare the graphs in parts (a) and (b).

Year	*Consumer Debt as a Percentage of Disposable Income*	*Year*	*Consumer Debt as a Percentage of Disposable Income*
1980	18.2	1990	20.0
1982	16.9	1991	18.9
1983	17.7	1994	19.7
1985	20.8	1995	21.3
1986	21.4	1996	23.7
1988	21.0		

Source: Federal Reserve System

12.2 The Power Rule

OBJECTIVE

- To evaluate integrals of the form $\int u^n \cdot u' \, dx = \int u^n \, du$ if $n \neq -1$

APPLICATION PREVIEW

In the previous section, we saw that total revenue could be found by integrating marginal revenue. That is,

$$R(x) = \int \overline{MR} \, dx$$

For example, if the marginal revenue for a product is given by

$$\overline{MR} = \frac{600}{\sqrt{3x+1}} + 2$$

then

$$R(x) = \int \left[\frac{600}{\sqrt{3x+1}} + 2 \right] dx$$

To evaluate this integral, however, we need a more general formula than the Powers of x Formula.

In this section, we will extend the Powers of x Formula to a rule for powers of a function of x.

Our goal in this section is to extend the Powers of x Formula,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

to powers of a function of x. In order to do this, we must understand the symbol dx.

Recall from Section 9.3, "The Derivative," that the derivative of $y = f(x)$ with respect to x can be denoted by dy/dx. As we will see, there are advantages to using dy and dx as separate quantities whose ratio dy/dx equals $f'(x)$.

Differentials If $y = f(x)$ is a differentiable function with derivative $dy/dx = f'(x)$, then the **differential of x** is dx, and the **differential of y** is dy, where

$$dy = f'(x) \, dx$$

Although differentials are useful in certain approximation problems, we are interested in the differential notation at this time.

EXAMPLE 1

Find the differential dy if $y = x^3 - 4x^2 + 5$.

Solution

$$dy = f'(x)\,dx = (3x^2 - 8x)\,dx$$

If the dependent variable in a function is u, then $du = u'(x)\,dx$.

EXAMPLE 2

If $u = x^2 + 4$, find du.

Solution

Because u is a function of x (that is, $u = u(x)$),

$$du = u'(x)\,dx = 2x\,dx$$

In terms of our goal of extending the Powers of x Formula, we would suspect that if x is replaced by a function of x, then dx should be replaced by the differential of that function. Let's see whether this is true.

Recall that if $y = [u(x)]^n$, the derivative of y is

$$\frac{dy}{dx} = n[u(x)]^{n-1} \cdot u'(x)$$

Using this formula for derivatives, we can see that

$$\int n[u(x)]^{n-1} \cdot u'(x)\,dx = [u(x)]^n + C$$

It is easy to see that this formula is equivalent to the following formula, which is called the **Power Rule for Integration.**

Power Rule for Integration

$$\int [u(x)]^n \cdot u'(x)\,dx = \frac{[u(x)]^{n+1}}{n+1} + C, \qquad n \neq -1$$

Using the fact that

$$du = u'(x)\,dx \quad \text{or} \quad du = u'\,dx$$

we can write the Power Rule in the following alternative form.

Power Rule (Alternative Form)

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1$$

Note that this formula has the same form as the formula

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

with the function u *substituted for x and du substituted for dx.*

EXAMPLE 3

Evaluate $\int (x^2 + 4)^5 \cdot 2x\,dx$.

Solution

To use the Power Rule, we must be sure that we have the function $u(x)$, its derivative $u'(x)$, and n.

$$u = x^2 + 4, \qquad n = 5$$
$$u' = 2x$$

All required parts are present, so the integral is of the form

$$\int (x^2+4)^5 2x\,dx = \int u^5 \cdot u'\,dx = \int u^5 du$$
$$= \frac{u^6}{6} + C = \frac{(x^2+4)^6}{6} + C$$

We can check the integration by noting that the derivative of

$$\frac{(x^2+4)^6}{6} + C \quad \text{is} \quad (x^2+4)^5 \cdot 2x$$

EXAMPLE 4

Evaluate $\int \sqrt{2x+3} \cdot 2\,dx$.

Solution

If we let $u = 2x + 3$, then $u' = 2$, and so we have

$$\int \sqrt{2x+3} \cdot 2\,dx = \int \sqrt{u}\,u'\,dx = \int \sqrt{u}\,du$$
$$= \int u^{1/2} du = \frac{u^{3/2}}{3/2} + C$$

Because $u = 2x + 3$, we have

$$\int \sqrt{2x+3} \cdot 2\,dx = \frac{2}{3}(2x+3)^{3/2} + C$$

CHECK: The derivative of $\frac{2}{3}(2x + 3)^{3/2} + C$ is $(2x + 3)^{1/2} \cdot 2$.

Some members of the family of functions given by

$$\int \sqrt{2x+3} \cdot 2\,dx = \frac{2}{3}(2x+3)^{3/2} + C$$

are shown in Figure 12.4 on the next page. Note from the graphs that the domain of each function is $x \geq -3/2$. This is because $2x + 3$ must be nonnegative so that $(2x + 3)^{3/2} = (\sqrt{2x+3})^3$ is a real number.

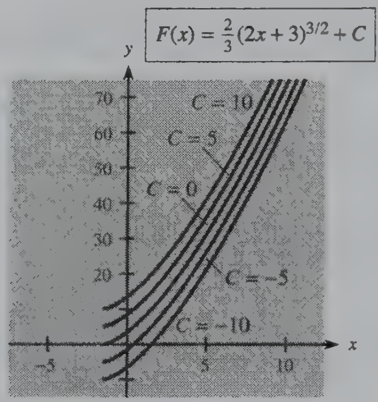


Figure 12.4

EXAMPLE 5

Evaluate $\int (x^2 + 4)^4 \cdot x\, dx$.

Solution

If we let $u = x^2 + 4$, then $u' = 2x$. Thus we do not have an integral of the form $\int u^n \cdot u'\, dx$, as we had in Example 3 and Example 4; the factor 2 is not in the integrand. To get the integrand in the correct form, we can multiply by 2 and divide it out as follows:

$$\int (x^2+4)^4 \cdot x\, dx = \int (x^2+4)^4 \cdot \frac{1}{2}(2x)\, dx$$

Because $\frac{1}{2}$ is a constant factor, we can factor it outside the integral sign, getting

$$\frac{1}{2}\int (x^2+4)^4 \cdot 2x\, dx$$

Now the integral is in the form $\frac{1}{2}\int u^4 \cdot u'\, dx$. Thus

$$\begin{aligned}\int (x^2+4)^4 \cdot x\, dx &= \frac{1}{2}\int (x^2+4)^4 \cdot 2x\, dx\\ &= \frac{1}{2}\,\frac{(x^2+4)^5}{5} + C\\ &= \frac{1}{10}(x^2+4)^5 + C\end{aligned}$$

EXAMPLE 6

Evaluate $\int \sqrt{x^3 - 4} \cdot 5x^2\, dx$.

Solution

If we let $u = x^3 - 4$, then $u' = 3x^2$. Thus we need the factor 3, rather than 5, in the integrand. If we multiply by the constant factor 3 (and divide it out), we have

$$\begin{aligned}\int \sqrt{x^3-4} \cdot 5x^2\, dx &= \int \sqrt{x^3-4} \cdot \frac{5}{3}(3x^2)\, dx\\ &= \frac{5}{3}\int (x^3-4)^{1/2} \cdot 3x^2\, dx\end{aligned}$$

This integral is of the form $\frac{5}{3} \int u^{1/2} \cdot u' \, dx$, resulting in

$$\frac{5}{3} \cdot \frac{u^{3/2}}{3/2} + C = \frac{5}{3} \cdot \frac{(x^3 - 4)^{3/2}}{3/2} + C = \frac{10}{9}(x^3 - 4)^{3/2} + C$$

Note that we can factor a constant outside the integral sign to obtain the integrand in the form we seek, but if the integral requires the introduction of a variable to obtain the form $u^n \cdot u' \, dx$, we *cannot* use this form and must try something else.

EXAMPLE 7

Evaluate $\int (x^2 + 4)^2 \, dx$.

Solution

If we let $u = x^2 + 4$, then $u' = 2x$. Because we would have to introduce a variable to get u' in the integral, we cannot solve this problem by using the Power Rule. We must find another method. We can evaluate this integral by squaring and then integrating term by term.

$$\begin{aligned} \int (x^2 + 4)^2 \, dx &= \int (x^4 + 8x^2 + 16) \, dx \\ &= \frac{x^5}{5} + \frac{8x^3}{3} + 16x + C \end{aligned}$$

Note that if we tried to introduce the factor $2x$ into the integral of Example 7, we would get

$$\begin{aligned} \int (x^2 + 4)^2 \, dx &= \int (x^2 + 4)^2 \cdot \frac{1}{2x}(2x) \, dx \\ &= \frac{1}{2} \int (x^2 + 4)^2 \cdot \frac{1}{x}(2x) \, dx \end{aligned}$$

But we cannot factor the $1/x$ outside the integral, so we do not have the proper form. Again, we can only introduce *a constant factor* to get an integral in the proper form.

EXAMPLE 8

Evaluate $\int (2x^2 - 4x)^2 (x - 1) \, dx$.

Solution

If we want to treat this as an integral of the form $\int u^n u' \, dx$, we will have to let $u = 2x^2 - 4x$. Then u' will be $4x - 4$. Multiplying and dividing by 4 will give us this form, as follows.

$$\int (2x^2 - 4x)^2 (x - 1)\, dx = \int (2x^2 - 4x)^2 \cdot \frac{1}{4} \cdot 4(x - 1)\, dx$$
$$= \frac{1}{4} \int (2x^2 - 4x)^2 (4x - 4)\, dx$$
$$= \frac{1}{4} \int u^2 u'\, dx = \frac{1}{4} \cdot \frac{u^3}{3} + C$$
$$= \frac{1}{4} \frac{(2x^2 - 4x)^3}{3} + C$$
$$= \frac{1}{12}(2x^2 - 4x)^3 + C$$

EXAMPLE 9

Evaluate $\int \frac{x^2 - 1}{(x^3 - 3x)^3}\, dx.$

Solution

This integral can be treated as $\int u^{-3}u'\, dx$ if we let $u = x^3 - 3x$. Then we can multiply and divide by 3 to get $u' = 3(x^2 - 1)$.

$$\int \frac{x^2 - 1}{(x^3 - 3x)^3}\, dx = \int (x^3 - 3x)^{-3} \cdot \frac{1}{3} \cdot 3(x^2 - 1)\, dx$$
$$= \frac{1}{3} \int (x^3 - 3x)^{-3} (3x^2 - 3)\, dx$$
$$= \frac{1}{3}\left[\frac{(x^3 - 3x)^{-2}}{-2}\right] + C$$
$$= \frac{-1}{6(x^3 - 3x)^2} + C$$

CHECKPOINT

1. Which of the following can be evaluated with the Power Rule?
   (a) $\int (4x^2 + 1)^{10} (8x\, dx)$ (b) $\int (4x^2 + 1)^{10} (x\, dx)$
   (c) $\int (4x^2 + 1)^{10} (8\, dx)$ (d) $\int (4x^2 + 1)^{10}\, dx$
2. Which of the following is equal to $\int (2x^3 + 5)^{-2} (6x^2\, dx)$?
   (a) $\frac{[(2x^4)/4 + 5x]^{-1}}{-1} \cdot \frac{6x^3}{3} + C$ (b) $\frac{(2x^3 + 5)^{-1}}{-1} \cdot \frac{6x^3}{3} + C$
   (c) $\frac{(2x^3 + 5)^{-1}}{-1} + C$
3. True or false: Constants can be factored outside the integral sign.
4. Evaluate:
   (a) $\int (x^3 + 9)^5 (3x^2\, dx)$ (b) $\int (x^3 + 9)^{15} (x^2\, dx)$ (c) $\int (x^3 + 9)^2 (x\, dx)$

We now return to the problem introduced in the Application Preview.

EXAMPLE 10

Suppose that the marginal revenue for a product is given by

$$\overline{MR} = \frac{600}{\sqrt{3x+1}} + 2$$

Find the total revenue function.

Solution

$$\begin{aligned} R(x) &= \int \overline{MR}\,dx = \int \left[\frac{600}{(3x+1)^{1/2}} + 2\right] dx \\ &= \int 600(3x+1)^{-1/2}\,dx + \int 2\,dx \\ &= 600\left(\frac{1}{3}\right)\int (3x+1)^{-1/2}(3\,dx) + 2\int dx \\ &= 200\,\frac{(3x+1)^{1/2}}{1/2} + 2x + K \\ &= 400\sqrt{3x+1} + 2x + K \end{aligned}$$

We know that $R(0) = 0$, so we have

$$0 = 400\sqrt{1} + 0 + K \quad \text{or} \quad K = -400$$

Thus the total revenue function is

$$R(x) = 400\sqrt{3x+1} + 2x - 400$$

Note in Example 10 that even though $R(0) = 0$, the constant of integration K was *not* 0. This is because $x = 0$ does not necessarily mean that $u(x)$ will also be 0.

CHECKPOINT SOLUTIONS

1. Expressions (a) and (b) can be evaluated with the Power Rule. For (a), we let $u = 4x^2 + 1$ so that the integral becomes

$$\int u^{10}\,u'\,dx = \int u^{10}\,du$$

For (b), we let $u = 4x^2 + 1$ again, and the integral becomes

$$\frac{1}{8}\int u^{10}\,u'\,dx = \frac{1}{8}\int u^{10}\,du$$

Expressions (c) and (d) do not fit the format of the Power Rule, because neither integral has an x with the dx, outside the power (so they need to be multiplied out before integrating).

2. $u = 2x^3 + 5$, so $u' = 6x^2$

$$\begin{aligned} \int (2x^3+5)^{-2}(6x^2)\,dx &= \int u^{-2}u'\,dx = \int u^{-2}\,du \\ &= -u^{-1} + C = -(2x^3+5)^{-1} + C \end{aligned}$$

Thus (c) is the correct choice.

3. True

4. (a) $\int (x^3+9)^5(3x^2)\,dx = \int u^5 u'\,dx = \dfrac{u^6}{6} + C = \dfrac{(x^3+9)^6}{6} + C$

(b) $\int (x^3+9)^{15}(x^2\,dx) = \dfrac{1}{3}\int (x^3+9)^{15}(3x^2\,dx)$

$$= \frac{1}{3}\cdot\frac{(x^3+9)^{16}}{16} + C = \frac{(x^3+9)^{16}}{48} + C$$

(c) The Power Rule does not fit, so we expand the integrand.

$$\int (x^3+9)^2(x\,dx) = \int (x^6+18x^3+81)x\,dx$$
$$= \int (x^7+18x^4+81x)\,dx$$
$$= \frac{x^8}{8} + \frac{18x^5}{5} + \frac{81x^2}{2} + C$$

EXERCISE 12.2

Evaluate the integrals in Problems 1–32. Check your results by differentiation.

1. $\int (x^2+3)^3 2x\,dx$
2. $\int (3x^3+1)^4 9x^2\,dx$
3. $\int (15x^2+10)^4(30x)\,dx$
4. $\int (8x^4+5)^3(32x^3)\,dx$
5. $\int (3x-x^3)^2(3-3x^2)\,dx$
6. $\int (4x^2-3x)^4(8x-3)\,dx$
7. $\int (x^2+5)^3 x\,dx$
8. $\int (3x^2-4)^6 x\,dx$
9. $\int 7(4x-1)^6\,dx$
10. $\int 3(5-x)^{-3}\,dx$
11. $\int (x^2+1)^{-3}\,x\,dx$
12. $\int (3x^4+7)^{-4}(5x^3\,dx)$
13. $\int (x-1)(x^2-2x+5)^4\,dx$
14. $\int (2x^3-x)(x^4-x^2)^6\,dx$
15. $\int 2(x^3-1)(x^4-4x+3)^{-5}\,dx$
16. $\int 3(x^5-2x)(x^6-6x^2+7)^{-2}\,dx$
17. $\int 7x^3\sqrt{x^4+6}\,dx$
18. $\int 3x\sqrt{5-x^2}\,dx$
19. $\int (x^3+1)^2(3x\,dx)$
20. $\int (x^2-5)^2(2x^2\,dx)$
21. $\int (3x^2-1)^2(8x^2\,dx)$
22. $\int (2x^4+3)^2(8x\,dx)$
23. $\int \sqrt{x^3-3x}\,(x^2-1)\,dx$
24. $\int \sqrt[3]{x^2+2x}\,(x+1)\,dx$
25. $\displaystyle\int \frac{x^2\,dx}{(x^3-1)^2}$
26. $\displaystyle\int \frac{x\,dx}{(x^2-1)^3}$
27. $\displaystyle\int \frac{3x^4\,dx}{(2x^5-5)^4}$
28. $\displaystyle\int \frac{5x^3\,dx}{(x^4-8)^3}$
29. $\displaystyle\int \frac{x^3-1}{(x^4-4x)^3}\,dx$
30. $\displaystyle\int \frac{3x^5-2x^3}{(x^6-x^4)^5}\,dx$
31. $\displaystyle\int \frac{x^2-4x}{\sqrt{x^3-6x^2+2}}\,dx$
32. $\displaystyle\int \frac{x^2+1}{\sqrt{x^3+3x+10}}\,dx$

In Problems 33 and 34, (a) evaluate each integral and (b) graph the members of the solution family for $C=-5$, $C=0$, and $C=5$.

33. $\int x(x^2-1)^3\,dx$
34. $\int (3x-11)^{1/3}\,dx$

Each of Problems 35 and 36 has the form $\int f(x)\,dx$.

(a) Evaluate each integral to obtain a family of functions.

(b) Find and graph the family member that passes through the point (0, 2). Call that function $F(x)$.

(c) Find any x-values where $f(x)$ is not defined but $F(x)$ is.

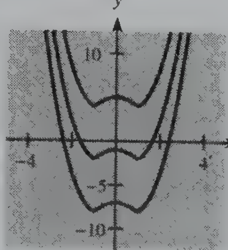
(d) At the x-values found in (c), what kind of tangent line does $F(x)$ have?

35. $\displaystyle\int \frac{3dx}{(2x-1)^{3/5}}$
36. $\displaystyle\int \frac{x^2\,dx}{(x^3-1)^{1/3}}$

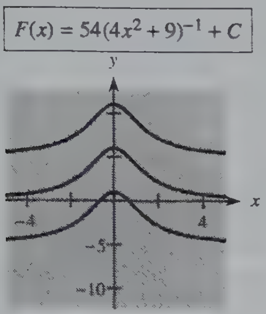
In each of Problems 37 and 38, a family of functions is given, together with the graphs of some functions in the family. Write the indefinite integral that gives the family.

37. $F(x) = (x^2-1)^{4/3} + C$

$F(x) = (x^2-1)^{4/3} + C$



38. $F(x) = 54(4x^2 + 9)^{-1} + C$



Applications

39. ***Revenue*** Suppose that the marginal revenue for a product is given by

$$\overline{MR} = \frac{-30}{(2x+1)^2} + 30$$

Find the total revenue.

40. ***Revenue*** The marginal revenue for a new calculator is given by

$$\overline{MR} = 60{,}000 - \frac{40{,}000}{(10+x)^2}$$

where x represents hundreds of calculators. Find the total revenue function for these calculators.

41. ***Physical productivity*** The total physical output of a number of machines or workers is called *physical productivity* and is a function of the number of machines or workers. If $P = f(x)$ is the productivity, dP/dx is the marginal physical productivity. If the marginal physical productivity for bricklayers is $dP/dx = 90(x + 1)^2$, where P is the number of bricks laid per day, find the physical productivity of 4 bricklayers. *Note:* $P = 0$ when $x = 0$.

42. ***Production*** The rate of production of a new line of products is given by

$$\frac{dx}{dt} = 200\left[1 + \frac{400}{(t+40)^2}\right]$$

where x is the number of items and t is the number of weeks the product has been in production.
(a) Assuming that $x = 0$ when $t = 0$, find the total number of items produced as a function of time t.
(b) How many items were produced in the fifth week?

43. ***Typing speed*** The rate of change in typing speed of the average student is $ds/dx = 5(x + 1)^{-1/2}$, where x is the number of typing lessons the student has had.
(a) Find the typing speed as a function of the number of lessons if the average student can type 10 words per minute with no lessons ($x = 0$).
(b) How many words per minute can the average student type after 24 lessons?

44. ***Productivity*** Because a new employee must learn an assigned task, production will increase with time. Suppose that for the average new employee, the rate of performance is given by

$$\frac{dN}{dt} = \frac{1}{2\sqrt{t+1}}$$

where N is the number of units completed t hours after beginning a new task. If 2 units are completed after 3 hours, how many units are completed after 8 hours?

45. ***Film attendance*** An excellent film with a very small advertising budget must depend largely on word-of-mouth advertising. In this case, the rate at which weekly attendance might grow can be given by

$$\frac{dA}{dt} = \frac{-100}{(t+10)^2} + \frac{2000}{(t+10)^3}$$

where t is the time in weeks since release and A is attendance in millions.
(a) Find the function that describes weekly attendance at this film.
(b) Find the attendance at this film in the tenth week.

46. ***Product quality and advertising*** An inferior product with a large advertising budget does well when it is introduced, but sales decline as people discontinue use of the product. Suppose that the rate of weekly sales is given by

$$S'(t) = \frac{400}{(t+1)^3} - \frac{200}{(t+1)^2}$$

where S is sales in millions and t is time in weeks.
(a) Find the function that describes the weekly sales.
(b) Find the sales for the first week and the ninth week.

47. ***Demographics*** Because of the decline of the steel industry, a western Pennsylvania town predicts that its public school population will decrease at the rate

$$\frac{dN}{dx} = \frac{-300}{\sqrt{x+9}}$$

where x is the number of years and N is the total school population. If the present population ($x = 0$) is 8000, what population size is planned for in 7 years?

48. ***Franchise growth*** A new fast-food firm predicts that the number of franchises for its products will grow at the rate

$$\frac{dn}{dt} = 9\sqrt{t+1}$$

where t is the number of years, $0 \le t \le 10$. If there is one franchise ($n = 1$) at present ($t = 0$), how many franchises are predicted for 8 years from now?

49. ***Poverty line*** Suppose the rate of change of the number of people (in millions) in the United States who lived below the poverty level can be modeled by

$$\frac{dp}{dt} = -0.004635(t-60)^2 + 0.2412(t-60) - 2.4165$$

where t is the number of years past 1900.

Year	Persons Below the Poverty Level (millions)	Year	Persons Below the Poverty Level (millions)
1960	39.9	1990	33.6
1965	33.2	1991	35.7
1970	25.4	1992	38
1975	25.9	1993	39.3
1980	29.3	1994	38.1
1986	32.4	1995	36.4
1989	31.5	1996	36.5

Source: Bureau of the Census, U.S. Dept. of Commerce

(a) If 33.6 million people lived below the poverty level in 1990, find the function that models the number of people, in millions, in the United States who lived below the poverty level. Graph this function.

(b) The data in the table show the number of people, in millions, in the United States who lived below the poverty level for selected years. Graph the data in the table with $t = 0$ in 1900.

(c) Compare the graphs in (a) and (b).

50. ***U.S. public debt*** Suppose the rate of change of the interest paid on the public debt of the United States as a percentage of federal expenditures can be modeled by

$$\frac{dD}{dt} = -0.06172(0.1t+3)^3 + 1.2823(0.1t+3)^2 - 8.43(0.1t+3) + 17.58$$

where t is the number of years past 1930.

Year	Interest Paid as a Percentage of Federal Expenditures	Year	Interest Paid as a Percentage of Federal Expenditures
1930	0.0	1970	9.9
1940	10.5	1975	9.8
1950	13.4	1980	12.7
1955	9.4	1985	18.9
1960	10.0	1990	21.1
1965	9.6	1995	22.0

Source: Bureau of Public Debt, U.S. Dept. of the Treasury

(a) If 9.6% of federal expenditures was devoted to interest payments on the public debt in 1965, find the function $D(t)$ that models the interest paid on the public debt of the United States as a percentage of federal expenditures. Graph this function.

(b) The data in the table give the interest paid on the public debt of the United States as a percentage of federal expenditures for selected years. Graph the data in the table with $t = 0$ in 1930.

(c) Compare the graphs in (a) and (b).

51. ***Union membership*** Suppose the rate of change of the percentage of U.S. workers who belonged to unions can be modeled by

$$\frac{dU}{dt} = 0.1415(0.1t + 2)^2 - 2.1317(0.1t + 2) + 7.3900$$

where t is the number of years past 1920.

(a) If 18.0% of U.S. workers belonged to unions in 1985, find the function $U(t)$ that models the percentage of U.S. workers who belonged to unions. Graph this function.

(b) The data in the table show the percentage of U.S. workers who belonged to unions for selected years. Graph the data ($t = 0$ in 1920).

(c) Compare the graphs in (a) and (b).

Year	*Union Membership as a Percentage of U.S. Labor Force*
1930	11.6
1935	13.2
1940	26.9
1945	35.5
1950	31.5
1955	33.2
1960	31.4
1965	28.4
1970	27.3
1975	25.5
1980	21.9
1985	18.0
1990	16.1
1993	15.8
1994	15.5
1995	14.9
1996	14.5

Source: Bureau of Labor Statistics, U.S. Dept. of Labor

12.3 Integrals Involving Logarithmic and Exponential Functions

OBJECTIVES

- *To evaluate integrals of the form* $\int \frac{u'}{u}\,dx$ *or, equivalently,* $\int \frac{1}{u}\,du$
- *To evaluate integrals of the form* $\int e^u u'\,dx$ *or, equivalently,* $\int e^u\,du$

APPLICATION PREVIEW

The rate of growth of the market value of a home has typically exceeded the inflation rate. Suppose, for example, that the real estate market has an average annual inflation rate of 8%. Then the rate of change of the value of a house that cost $100,000 can be modeled by

$$\frac{dV}{dt} = 7.7e^{0.077t}$$

where V is the value of the home in hundreds of thousands of dollars and t is the time in years since the home was purchased. To find the market value of such a home 10 years after it was purchased, we would first have to integrate dV/dt. That is, we must be able to integrate an exponential.

In this section, we consider integration formulas that result in natural logarithms and formulas for integrating exponentials.

Recall that the Power Rule for integrals applies only if $n \neq -1$. That is,

$$\int u^n u'\,dx = \frac{u^{n+1}}{n+1} + C \qquad \text{if } n \neq -1$$

The following formula applies when $n = -1$.

Logarithmic Formula If u is a function of x, then

$$\int u^{-1}u'\,dx = \int \frac{u'}{u}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$$

This formula is a direct result of the fact that

$$\frac{d}{dx}(\ln|u|) = \frac{1}{u}\cdot u'$$

We use the absolute value of u because the logarithm is defined only when the quantity is positive. We can see this result by considering the following.

For $u > 0$: $\quad \dfrac{d}{dx}(\ln|u|) = \dfrac{d}{dx}(\ln u) = \dfrac{1}{u}\cdot u'$

For $u < 0$: $\quad \dfrac{d}{dx}(\ln|u|) = \dfrac{d}{dx}[\ln(-u)] = \dfrac{1}{(-u)}\cdot(-u') = \dfrac{1}{u}\cdot u'$

In addition to this verification, we can graphically illustrate the need for the absolute value sign. Figure 12.5(a) shows that $f(x) = 1/x$ is defined for $x \neq 0$, and from Figures 12.5(b) and 12.5(c), we see that $F(x) = \int 1/x\,dx = \ln|x|$ is also defined for $x \neq 0$. But $y = \ln x$ is defined only for $x > 0$.

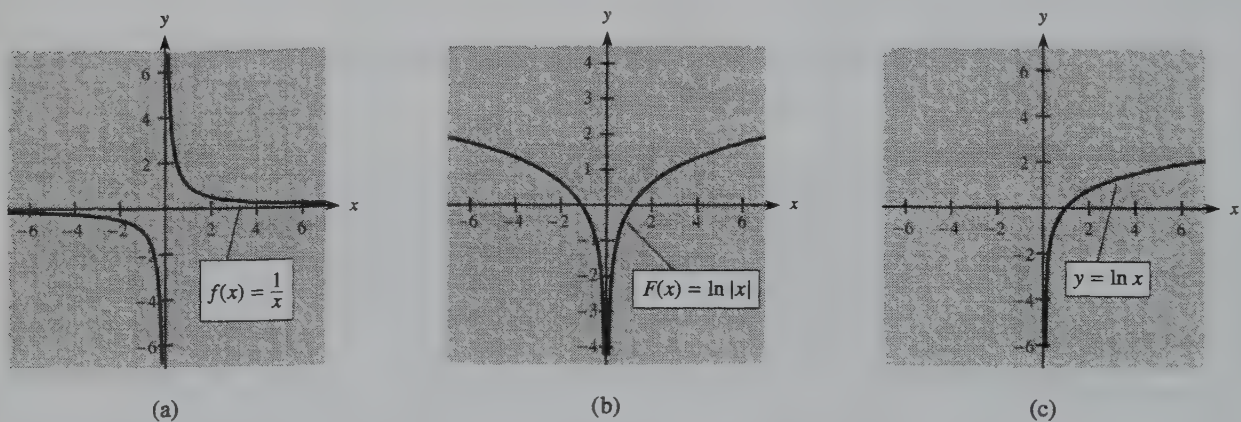


Figure 12.5

EXAMPLE 1

Evaluate $\displaystyle\int \frac{4}{4x+8}\,dx$.

Solution

This integral is of the form

$$\int \frac{u'}{u}\,dx = \ln|u| + C$$

with $u = 4x + 8$ and $u' = 4$. Thus

$$\int \frac{4}{4x+8}\,dx = \ln|4x+8| + C$$

Figure 12.6 shows several members of the family

$$F(x) = \int \frac{4\,dx}{4x+8} = \ln|4x+8| + C$$

We can choose different values for C and use a graphing utility to graph families of curves such as those in Figure 12.6.

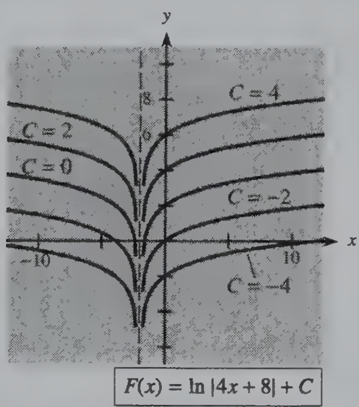


Figure 12.6

EXAMPLE 2

Evaluate $\displaystyle\int \frac{x-3}{x^2-6x+1}\,dx.$

Solution

This integral is of the form $\int (u'/u)\,dx$, *almost.* If we let $u = x^2 - 6x + 1$, then $u' = 2x - 6$. If we multiply (and divide) the numerator by 2, we get

$$\begin{aligned}\int \frac{x-3}{x^2-6x+1}\,dx &= \frac{1}{2}\int \frac{2(x-3)}{x^2-6x+1}\,dx \\ &= \frac{1}{2}\int \frac{2x-6}{x^2-6x+1}\,dx \\ &= \frac{1}{2}\int \frac{u'}{u}\,dx = \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln|x^2-6x+1| + C\end{aligned}$$

If an integral contains a fraction in which the degree of the numerator is equal to or greater than that of the denominator, we should divide the denominator into the numerator as a first step.

EXAMPLE 3

Evaluate $\displaystyle\int \frac{x^4 - 2x^3 + 4x^2 - 7x - 1}{x^2 - 2x}\,dx.$

Solution

Because the numerator is of higher degree than the denominator, we begin by dividing $x^2 - 2x$ into the numerator.

$$\begin{array}{r|l} & x^2 \qquad\qquad + 4 \\ \hline x^2 - 2x & x^4 - 2x^3 + 4x^2 - 7x - 1 \\ & x^4 - 2x^3 \\ \hline & \qquad\qquad 4x^2 - 7x - 1 \\ & \qquad\qquad 4x^2 - 8x \\ \hline & \qquad\qquad\qquad\quad x - 1 \end{array}$$

Thus

$$\begin{aligned}\int \frac{x^4 - 2x^3 + 4x^2 - 7x - 1}{x^2 - 2x}\,dx &= \int \left(x^2 + 4 + \frac{x-1}{x^2 - 2x}\right) dx \\ &= \int (x^2 + 4)\,dx + \frac{1}{2}\int \frac{2(x-1)\,dx}{x^2 - 2x} \\ &= \frac{x^3}{3} + 4x + \frac{1}{2}\ln|x^2 - 2x| + C\end{aligned}$$

CHECKPOINT

1. True or false:

(a) $\displaystyle\int \frac{3x^2\,dx}{x^3 + 4} = \ln|x^3 + 4| + C$ (b) $\displaystyle\int \frac{2x\,dx}{\sqrt{x^2+1}} = \ln\left|\sqrt{x^2+1}\right| + C$

(c) $\displaystyle\int \frac{2}{x}\,dx = 2\ln|x| + C$

(d) $\displaystyle\int \frac{x}{x+1}\,dx = x\int \frac{1}{x+1}\,dx = x\ln|x+1| + C$

(e) To evaluate $\displaystyle\int \frac{4x}{4x+1}\,dx$, our first step is to divide $4x + 1$ into $4x$.

2. (a) Divide $4x + 1$ into $4x$. (b) Evaluate $\displaystyle\int \frac{4x}{4x+1}\,dx$.

We know that

$$\frac{d}{dx}(e^u) = e^u \cdot u'$$

The corresponding integral is given by the following.

Exponential Formula If u is a function of x,

$$\int e^u \cdot u'\,dx = \int e^u\,du = e^u + C$$

EXAMPLE 4

Evaluate $\int 5e^x\,dx$.

Solution

$\int 5e^x\,dx = 5\int e^x\,dx = 5e^x + C$

EXAMPLE 5

Evaluate $\int 2xe^{x^2}\,dx$.

Solution

Letting $u = x^2$ implies that $u' = 2x$, and the integral is of the form $\int e^u \cdot u'\,dx$. Thus

$$\int 2xe^{x^2}\,dx = \int e^{x^2}(2x)\,dx = \int e^u \cdot u'\,dx = e^u + C = e^{x^2} + C$$

EXAMPLE 6

Evaluate $\displaystyle\int \frac{x^2dx}{e^{x^3}}$.

Solution

In order to use $\int e^u \cdot u'\,dx$, we write the exponential in the numerator. Thus

$$\int \frac{x^2\,dx}{e^{x^3}} = \int e^{-x^3}\,(x^2\,dx)$$

This is *almost* of the form $\int e^u \cdot u'\,dx$. Letting $u = -x^3$ gives $u' = -3x^2$. Thus

$$\int e^{-x^3}\,(x^2\,dx) = -\frac{1}{3}\int e^{-x^3}\,(-3x^2\,dx) = -\frac{1}{3}e^{-x^3} + C = \frac{-1}{3e^{x^3}} + C$$

CHECKPOINT

3. True or false:

(a) $\int e^{x^2}\,(2x\,dx) = e^{x^2} \cdot x^2 + C$ (b) $\int e^{-3x}\,dx = -\frac{1}{3}e^{-3x} + C$

(c) $\displaystyle\int \frac{dx}{e^{3x}} = \frac{1}{3}\left(\frac{1}{e^{3x}}\right) + C$ (d) $\displaystyle\int e^{3x+1}\,(3\,dx) = \frac{e^{3x+2}}{3x+2} + C$

We now return to the real estate inflation rate problem in the Application Preview.

EXAMPLE 7

Suppose the rate of change of the value of a house that cost \$100,000 can be modeled by

$$\frac{dV}{dt} = 7.7e^{0.077t}$$

where V is the market value of the home in hundreds of thousands of dollars and t is the time in years since the home was purchased.

(a) Find the function that expresses the value V in terms of t.
(b) Find the predicted value after 10 years.

Solution

(a) $V = \int \frac{dV}{dt}\,dt = \int 7.7e^{0.077t}\,dt$

$$V = 7.7 \int e^{0.077t}\left(\frac{1}{0.077}\right)(0.077\,dt)$$

$$V = 7.7\left(\frac{1}{0.077}\right) \int e^{0.077t}(0.077\,dt)$$

$$V = 100e^{0.077t} + C$$

Using $V = 100$ (thousand) when $t = 0$, we have

$$100 = 100 + C$$
$$0 = C$$

Thus we have the value as a function of time given by

$$V = 100e^{0.077t}$$

(b) The value after 10 years is found by using $t = 10$.

$$V = 100e^{0.077(10)} = 100e^{0.77} \approx 215.98$$

Thus, after 10 years, the predicted value of the home is \$215,980.

EXAMPLE 8

Figure 12.7 shows the graphs of $g(x) = 5e^{-x^2}$ and $h(x) = -10xe^{-x^2}$. One of these functions is $f(x)$ and the other is $\int f(x)\,dx$ with $C = 0$.

(a) Decide which of $g(x)$ and $h(x)$ is $f(x)$ and which is $\int f(x)\,dx$.
(b) How can the graph of $f(x)$ be used to locate and classify the extrema of $\int f(x)\,dx$?
(c) What feature of the graph of $f(x)$ occurs at the same x-values as the inflection points of the graph of $\int f(x)\,dx$?

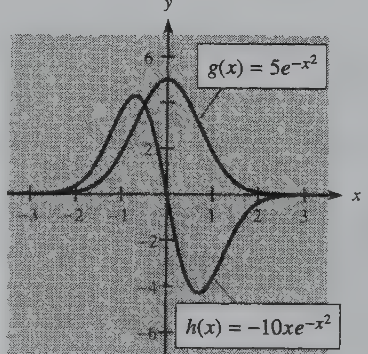


Figure 12.7

Solution

(a) The graph of $h(x)$ looks like the graph of $g'(x)$ because $h(x) > 0$ where $g(x)$ is increasing, $h(x) < 0$ where $g(x)$ is decreasing, and $h(x) = 0$ where $g(x)$ has its maximum. However, if $h(x) = g'(x)$, then, equivalently,

$$\int h(x)\,dx = \int g'(x)\,dx = g(x) + C$$

so $h(x) = f(x)$ and $g(x) = \int f(x)\,dx$. We can verify this by noting that

$$\int -10xe^{-x^2}\,dx = 5\int e^{-x^2}(-2x\,dx) = 5e^{-x^2} + C$$

(b) We know that $f(x)$ is the derivative of $\int f(x)\,dx$, so, as we saw in (a), the x-intercepts of $f(x)$ locate the critical values and extrema of $\int f(x)\,dx$.

(c) The first derivative of $\int f(x)\,dx$ is $f(x)$, and its second derivative is $f'(x)$. Hence the inflection points of $\int f(x)\,dx$ occur where $f'(x) = 0$. But $f(x)$ has its extrema where $f'(x) = 0$. Thus the extrema of $f(x)$ occur at the same x-values as the inflection points of $\int f(x)\,dx$.

CHECKPOINT SOLUTIONS

1. (a) True

(b) False; $\displaystyle\int \frac{2x\,dx}{\sqrt{x^2+1}} = \int (x^2+1)^{-1/2}(2x\,dx)$

$$= \frac{(x^2+1)^{1/2}}{1/2} + C = 2(x^2+1)^{1/2} + C$$

(c) True

(d) False. We cannot factor the variable x outside the integral sign.

(e) True

2. (a) $$\begin{array}{r|l} & \ \ 1 \\ \hline 4x+1 & 4x \\ & \underline{4x+1} \\ & \ \ -1 \end{array}$$

so $\dfrac{4x}{4x+1} = 1 - \dfrac{1}{4x+1}$.

(b) $\displaystyle\int \frac{4x\,dx}{4x+1} = \int \left(1 - \frac{1}{4x+1}\right)dx = x - \frac{1}{4}\ln|4x+1| + C$

3. (a) False. The correct solution is $e^{x^2} + C$ (see Example 5).

(b) True

(c) False; $\displaystyle\int \frac{dx}{e^{3x}} = \int e^{-3x}\,dx = -\frac{1}{3}e^{-3x} + C = \frac{-1}{3e^{3x}} + C$

(d) False; $\displaystyle\int e^{3x+1}(3\,dx) = e^{3x+1} + C$

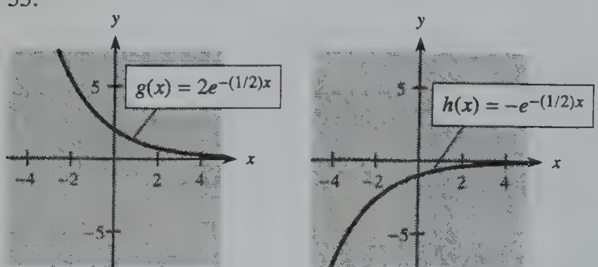
EXERCISE 12.3

Evaluate the integrals in Problems 1–32.

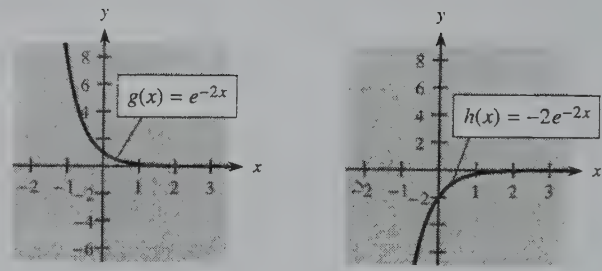
1. $\int \frac{3x^2}{x^3+4}\,dx$
2. $\int \frac{8x^7}{x^8-1}\,dx$
3. $\int \frac{dz}{4z+1}$
4. $\int \frac{y}{y^2+1}\,dy$
5. $\int \frac{x^3}{x^4+1}\,dx$
6. $\int \frac{x^2}{x^3-9}\,dx$
7. $\int \frac{4x}{x^2-4}\,dx$
8. $\int \frac{5x^2}{x^3-1}\,dx$
9. $\int \frac{3x^2-2}{x^3-2x}\,dx$
10. $\int \frac{4x^3+2x}{x^4+x^2}\,dx$
11. $\int \frac{z^2+1}{z^3+3z+17}\,dz$
12. $\int \frac{(x+2)dx}{x^2+4x-9}$
13. $\int \frac{x^3-x^2+1}{x-1}\,dx$
14. $\int \frac{2x^3+x^2+2x+3}{2x+1}\,dx$
15. $\int \frac{x^2+x+3}{x^2+3}\,dx$
16. $\int \frac{x^4-2x^2+x}{x^2-2}\,dx$
17. $\int 3e^{3x}\,dx$
18. $\int 4e^{4x}\,dx$
19. $\int e^{-x}\,dx$
20. $\int e^{2x}\,dx$
21. $\int 1000e^{0.1x}\,dx$
22. $\int 1600e^{0.4x}\,dx$
23. $\int 840e^{-0.7x}\,dx$
24. $\int 250e^{-0.5x}\,dx$
25. $\int x^3e^{3x^4}\,dx$
26. $\int xe^{2x^2}\,dx$
27. $\int \frac{3}{e^{2x}}\,dx$
28. $\int \frac{4}{e^{1-2x}}\,dx$
29. $\int \frac{x^5}{e^{2-3x^6}}\,dx$
30. $\int \frac{x^3}{e^{4x^4}}\,dx$
31. $\int \left(e^{4x}-\frac{3}{e^{x/2}}\right)dx$
32. $\int \left(xe^{3x^2}-\frac{5}{e^{x/3}}\right)dx$

In Problems 33 and 34, graphs of two functions labeled $g(x)$ and $h(x)$ are given. Decide which is the graph of $f(x)$ and which is one member of the family $\int f(x)\,dx$. Check your conclusions by evaluating the integral.

33.

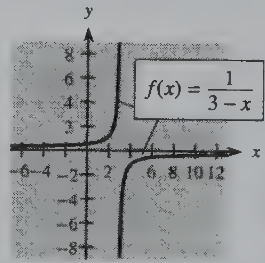


34.

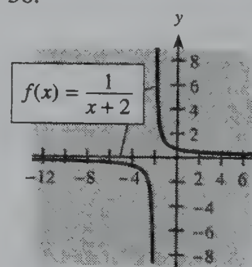


In Problems 35 and 36, a function $f(x)$ and its graph are given. Find the family $F(x) = \int f(x)\,dx$ and graph the member that satisfies $F(0) = 0$.

35.

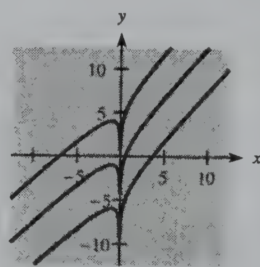


36.



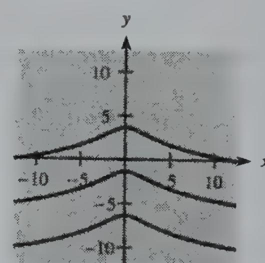
In Problems 37–40, a family of functions is given and graphs of some members are shown. Find the function $f(x)$ such that the family is given by $\int f(x)\,dx$.

37. $F(x) = x + \ln|x| + C$



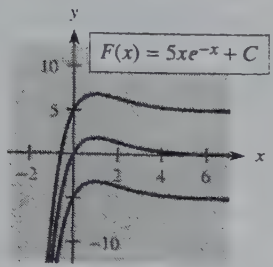
$F(x) = x + \ln|x| + C$

38. $F(x) = -\ln(x^2+4) + C$

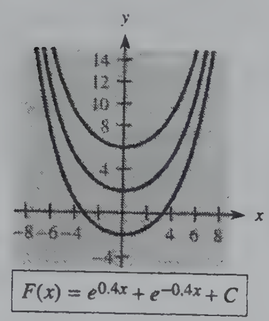


$F(x) = -\ln(x^2+4) + C$

39. $F(x) = 5xe^{-x} + C$



40. $F(x) = e^{0.4x} + e^{-0.4x} + C$



Applications

41. ***Revenue*** Suppose that the marginal revenue from the sale of a product is $\overline{MR} = R'(x) = 6e^{0.01x}$. What is the revenue on the sale of 100 units of the product?

42. ***Concentration of a drug*** Suppose that the rate at which the concentration of a drug in the blood changes with respect to time t is given by

$$C'(t) = \frac{c}{b-a}(be^{-bt} - ae^{-at}), \qquad t \ge 0$$

where a, b, and c are constants depending on the drug administered, with $b > a$. Assuming that $C(t) = 0$ when $t = 0$, find the formula for the concentration of the drug in the blood at any time t.

43. ***Radioactive decay*** The rate of disintegration of a radioactive substance can be described by

$$\frac{dn}{dt} = n_0(-K)e^{-Kt}$$

where n_0 is the number of radioactive atoms present when time t is 0, and K is a positive constant that depends on the substance involved. Using the fact that the constant of integration is 0, integrate dn/dt to find the number of atoms n that are still radioactive after time t.

44. ***World population*** Because the world contains only about 10 billion acres of arable land, world population is limited. Suppose that the world population is limited to 40 billion people and that the rate of population growth is proportional to how close the world is to this upper limit. Then the rate of growth would be given by $dP/dt = K(40 - P)$, where K is a positive constant. This means that

$$t = \frac{1}{K}\int \frac{1}{40-P}\,dP$$

(a) Evaluate this integral to find an expression relating P and t.

(b) Use the properties relating logarithms and exponential functions to write P as a function of t.

45. ***Memorization*** The rate of vocabulary memorization of the average student in a foreign language is given by

$$\frac{dv}{dt} = \frac{40}{t+1}$$

where t is the number of continuous hours of study, $0 < t \le 4$. How many words would the average student memorize in 3 hours?

46. ***Population growth*** The rate of growth of world population can be modeled by

$$\frac{dn}{dt} = N_0(1+r)^t \ln(1+r), \qquad r < 1$$

where t is the time in years from the present and N_0 and r are constants. What function describes world population if the present population is N_0? Use the formula $\int (a^u \ln a)u'\,dx = a^u + C$.

47. ***Compound interest*** If \$$P$ is invested for n years at 10%, compounded continuously, the rate at which the future value is growing is

$$\frac{dS}{dn} = 0.1Pe^{0.1n}$$

(a) What function describes the future value at the end of n years?

(b) In how many years will the future value double?

48. ***Temperature changes*** When an object is moved from one environment to another, its temperature T changes at a rate given by

$$\frac{dT}{dt} = kCe^{kt}$$

where t is the time in the new environment (in hours), C is the temperature difference (old − new) between the two environments, and k is a constant. If the temperature of the body (and the old environment) is 70°F, and $C = -10$°F, what function describes the temperature T of the object t hours after it is moved?

49. ***Blood pressure in the aorta*** The rate at which blood pressure decreases in the aorta of a normal adult after a heartbeat is

$$\frac{dp}{dt} = -46.645e^{-0.491t}$$

where t is time in seconds.

(a) What function describes the blood pressure in the aorta if $p = 95$ when $t = 0$?

(b) What is the blood pressure 0.1 second after a heartbeat?

50. ***Sales and advertising*** A store finds that its sales decline after the end of an advertising campaign, with its daily sales for the period declining at the rate $S'(t) = -147.78e^{-0.2t}$, $0 \le t \le 100$, where t is the number of days since the end of the campaign. Suppose that $S = 7389$ when $t = 0$.

(a) Find the function that describes the number of daily sales t days after the end of the campaign.

(b) Find the total number of sales 10 days after the end of the advertising campaign.

51. ***Life expectancy*** Suppose the rate of change of the expected life span l at birth of people born in the United States can be modeled by

$$\frac{dl}{dt} = \frac{14.1372}{t + 20}$$

where t is the number of years past 1920.

(a) If the expected life span was 72.6 years for people born in 1975, find the function that models the life span. Graph this function.

(b) The data in the table give the expected life span for people born in various years. Graph the data.

(c) Compare the graphs in (a) and (b).

Year	*Life Span (years)*	Year	*Life Span (years)*
1920	54.1	1988	74.9
1930	59.7	1989	75.2
1940	62.9	1990	75.4
1950	68.2	1991	75.5
1960	69.7	1992	75.8
1970	70.8	1993	75.5
1975	72.6	1994	75.7
1980	73.7	1996	76.1
1987	75.0		

Source: *World Almanac*, 1991

52. ***Violent crime*** Suppose the rate of change of the number of violent crimes per 100,000 people in the United States can be modeled by

$$\frac{dc}{dt} = \frac{321}{t + 5}$$

when t is the number of years past 1985.

(a) If there were 758 violent crimes per 100,000 people in 1991, find the function $c(t)$ that models the number of violent crimes. Graph this function.

(b) The data in the table give the number of violent crimes per 100,000 people for selected years. Graph the data in the table.

(c) Compare the graphs in (a) and (b).

Year	*Violent Crimes (per 100,000)*
1987	610
1988	637
1989	663
1990	732
1991	758
1992	765

Source: FBI Crime Report, 1993

53. ***Revenue*** Suppose the rate of change of revenue from wireless technology can be modeled by

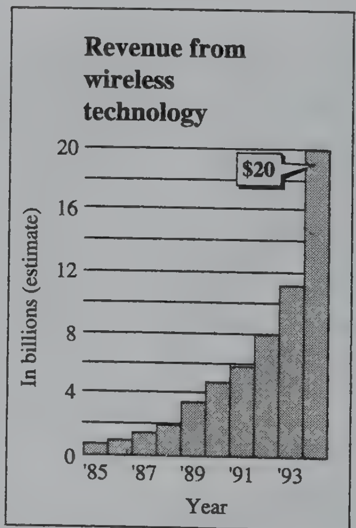
$$\frac{dR}{dt} = 0.1994e^{0.3486t}$$

where t is the number of years past 1984 and R is revenue in billions of dollars.

(a) Find a model for revenue, $R(t)$, if revenue from wireless technology was \$11 billion in 1993.

(b) Graph the function you found in (a) and compare it to the graph in the figure.

(c) What does your model predict for revenue from wireless technology in 1995?



Source: *USA Today*, March 1, 1994

54. ***Health care costs*** Suppose the rate of change of national health care costs in the United States can be modeled by

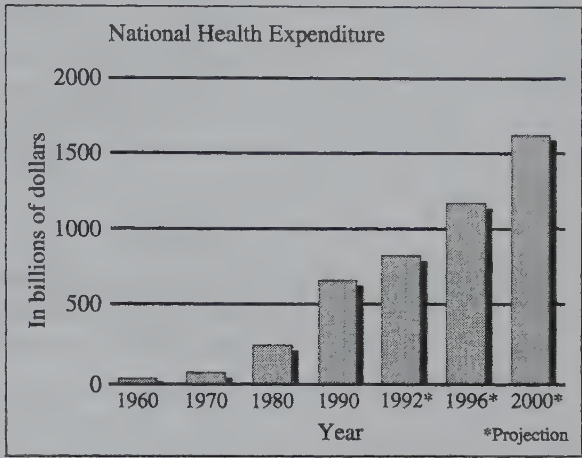
$$\frac{dHC}{dt} = 4.0428e^{0.08984t}$$

where t is the number of years past 1960 and HC is the national health care expenditure in billions of dollars.

(a) Use the fact that 1991 health care costs totaled \$752 billion to find a function $HC(t)$ that models health care costs.

(b) Graph the function you found in (a) and compare it to the graph in the figure.

(c) What does your model project for health care costs for the year 2000?



Source: Congressional Budget Office, printed in *Newsweek*, October 4, 1993

12.4 Applications of the Indefinite Integral in Business and Economics

OBJECTIVES

- *To use integration to find total cost functions from information involving marginal cost*
- *To optimize profit, given information regarding marginal cost and marginal revenue*
- *To use integration to find national consumption functions from information about marginal propensity to consume and marginal propensity to save*

APPLICATION PREVIEW

We turn our attention to applications involving cost, profit, and consumption. A thorough understanding of these concepts is imperative for success in any study of economics and business.

In this section, we will use integration to derive total cost and profit functions from the marginal cost and marginal revenue functions. One of the reasons for the marginal approach in economics is that firms can observe marginal changes in real life. If they know the marginal cost and the total cost when a given quantity is sold, they can develop their total cost function.

In this section, we also use integration to obtain the national consumption function from functions that describe the marginal propensity to consume or the marginal propensity to save.

Total Cost and Profit

We know that the marginal cost for a commodity is $\overline{MC} = C'(x)$, where $C(x)$ is the total cost function. Thus if we have the marginal cost function, we can integrate to find the total cost. That is, $C(x) = \int \overline{MC}\, dx$.

If, for example, the marginal cost is $\overline{MC} = 4x + 3$, the total cost is given by

$$\begin{aligned} C(x) &= \int \overline{MC}\, dx \\ &= \int (4x+3)\, dx \\ &= 2x^2 + 3x + K \end{aligned}$$

where K represents the constant of integration. Now, we know that the total revenue is 0 if no items are produced, but the total cost may not be 0 if nothing is produced. The fixed costs accrue whether goods are produced or not. Thus the value for the constant of integration depends on the fixed costs FC of production.

Thus we cannot determine the total cost function from the marginal cost unless additional information is available to help us determine the fixed costs.

EXAMPLE 1

If the marginal cost function for a month for a certain product is $\overline{MC} = 3x + 50$, and if the fixed costs related to the product amount to \$100 per month, find the total cost function for the month.

Solution

The total cost function is

$$\begin{aligned} C(x) &= \int (3x + 50)\, dx \\ &= \frac{3x^2}{2} + 50x + K \end{aligned}$$

But the constant of integration K is found by using the fact that $C(0) = FC = 100$. Thus

$$3(0)^2 + 50(0) + K = 100, \quad \text{so } K = 100$$

and the total cost for the month is given by

$$C(x) = \frac{3x^2}{2} + 50x + 100$$

EXAMPLE 2

Suppose monthly records show that the rate of change of the cost (that is, the marginal cost) for a product is $\overline{MC} = 3(2x + 25)^{1/2}$ and that the fixed costs for the month are \$11,125. What would be the total cost of producing 300 items per month?

Solution

We can integrate the marginal cost to find the total cost function.

$$\begin{aligned} C(x) &= \int \overline{MC}\, dx = \int 3(2x + 25)^{1/2}\, dx \\ &= 3 \cdot \left(\frac{1}{2}\right) \int (2x + 25)^{1/2}(2\, dx) \\ &= \left(\frac{3}{2}\right) \frac{(2x + 25)^{3/2}}{3/2} + K \\ &= (2x + 25)^{3/2} + K \end{aligned}$$

We can find K by using the fact that fixed costs are \$11,125.

$$\begin{aligned} C(0) = 11{,}125 &= (25)^{3/2} + K \\ 11{,}125 &= 125 + K, \quad \text{or} \quad K = 11{,}000 \end{aligned}$$

Thus the total cost function is

$$C(x) = (2x + 25)^{3/2} + 11{,}000$$

and the cost of producing 300 items per month is

$$\begin{aligned} C(300) &= (625)^{3/2} + 11{,}000 \\ &= 26{,}625 \quad \text{(dollars)} \end{aligned}$$

It can be shown that the profit is usually maximized when $\overline{MR} = \overline{MC}$. To see that this does not always give us a maximum *positive* profit, consider the following facts concerning the manufacture of widgets over the period of a month:

1. The marginal revenue is $\overline{MR} = 400 - 30x$.
2. The marginal cost is $\overline{MC} = 20x + 50$.
3. When 5 widgets are produced and sold, the total cost is \$1750. The profit *should* be maximized when $\overline{MR} = \overline{MC}$, or when $400 - 30x = 20x + 50$. Solving for x gives $x = 7$. To see whether our profit is maximized when 7 units are produced and sold, let us examine the profit function.

The profit function is given by $P(x) = R(x) - C(x)$, where

$$R(x) = \int \overline{MR}\, dx \qquad \text{and} \qquad C(x) = \int \overline{MC}\, dx$$

Integrating, we get

$$R(x) = \int (400 - 30x)\, dx = 400x - 15x^2 + K$$

but $K = 0$ for this total revenue function, so

$$R(x) = 400x - 15x^2$$

The total cost function is

$$C(x) = \int (20x + 50)\, dx = 10x^2 + 50x + K$$

The value of fixed cost can be determined by using the fact that 5 widgets cost \$1750. This tells us that $C(5) = 1750 = 250 + 250 + K$, so $K = 1250$.

Thus the total cost is $C(x) = 10x^2 + 50x + 1250$. Now, the profit is

$$P(x) = R(x) - C(x)$$

or

$$P(x) = (400x - 15x^2) - (10x^2 + 50x + 1250)$$

Simplifying gives

$$P(x) = 350x - 25x^2 - 1250$$

We have found that $\overline{MR} = \overline{MC}$ if $x = 7$, and the graph of $P(x)$ is a parabola that opens downward, so profit is maximized at $x = 7$. But if $x = 7$, profit is

$$P(7) = 2450 - 1225 - 1250 = -25$$

That is, the production and sale of 7 items result in a loss of \$25.

The preceding discussion indicates that, although setting $\overline{MR} = \overline{MC}$ may optimize profit, it does not indicate the level of profit or loss, as forming the profit function does.

If the widget firm is in a competitive market, and its optimal level of production results in a loss, it has two options. It can continue to produce at the optimal level in the short run until it can lower or eliminate its fixed costs, even though it is losing money; or it can take a larger loss (its fixed cost) by stopping production. Producing 7 units causes a loss of \$25 per month, and ceasing production results in a loss of \$1250 (the fixed cost) per month. If this firm and many others like it cease production, the supply will be reduced, causing an eventual increase in price. The firm can resume production when the price increase indicates that it can make a profit.

EXAMPLE 3

Given that $\overline{MR} = 200 - 4x$, $\overline{MC} = 50 + 2x$, and the total cost of producing 10 Wagbats is \$700, at what level should the Wagbat firm hold production in order to maximize the profits?

Solution

Setting $\overline{MR} = \overline{MC}$, we can solve for the production level that maximizes profit.

$$\begin{aligned} 200 - 4x &= 50 + 2x \\ 150 &= 6x \\ 25 &= x \end{aligned}$$

The level of production that should optimize profit is 25 units. To see whether 25 units maximizes profits or minimizes the losses (in the short run), we must find the total revenue and total cost functions.

$$\begin{aligned} R(x) &= \int (200 - 4x)\,dx = 200x - 2x^2 + K \\ &= 200x - 2x^2, \quad \text{because } K = 0 \\ C(x) &= \int (50 + 2x)\,dx = 50x + x^2 + K \end{aligned}$$

We find K by noting that $C(x) = 700$ when $x = 10$.

$$700 = 50(10) + (10)^2 + K$$

so $K = 100$.

Thus the cost is given by $C = C(x) = 50x + x^2 + 100$. At $x = 25$, $R = R(25) = 200(25) - 2(25)^2 = \$3750$ and $C = C(25) = 50(25) + (25)^2 + 100 = \$1975$.

We see that the total revenue is greater than the total cost, so production should be held at 25 units, which results in a maximum profit.

Graphing Utilities

If it is difficult to solve $\overline{MC} = \overline{MR}$ analytically, we could use a graphing utility to solve this equation by finding the point of intersection of the graphs of $\overline{MC}$ and $\overline{MR}$. We may also be able to integrate $\overline{MC}$ and $\overline{MR}$ to find the functions $C(x)$ and $R(x)$ and then use a graphing utility to graph them. From the graphs of $C(x)$ and $R(x)$ we can learn about these functions—and hence about profit.

EXAMPLE 4

Suppose that $\overline{MC} = 1.01(x + 190)^{0.01}$ and $\overline{MR} = (1/\sqrt{2x + 1}) + 2$, where x is the number of thousands of units and both revenue and cost are in thousands of dollars. Suppose further that fixed costs are \$100,236 and that production is limited to at most 180 thousand units.

(a) Determine $C(x)$ and $R(x)$ and graph them to determine whether a profit can be made.
(b) Estimate the level of production that yields maximum profit, and find the maximum profit.

Solution

(a) $C(x) = \int \overline{MC}\, dx = \int 1.01\,(x + 190)^{0.01}\, dx$

$$= 1.01\,\frac{(x+190)^{1.01}}{1.01} + K$$

When we say that fixed costs equal \$100,236, we mean $C(0) = 100.236$.

$$100.236 = C(0) = (190)^{1.01} + K$$
$$100.236 = 200.236 + K$$
$$-100 = K$$

Thus $C(x) = (x + 190)^{1.01} - 100$.

$$R(x) = \int \overline{MR}\, dx = \int [(2x+1)^{-1/2} + 2]\, dx$$
$$= \frac{1}{2}\int (2x+1)^{-1/2}\,(2\, dx) + \int 2\, dx$$
$$= \frac{\frac{1}{2}(2x+1)^{1/2}}{1/2} + 2x + K$$

$R(0) = 0$ means

$$0 = R(0) = (1)^{1/2} + 0 + K, \quad \text{or} \quad K = -1$$

Thus $R(x) = (2x + 1)^{1/2} + 2x - 1$.

The graphs of $C(x)$ and $R(x)$ are shown in Figure 12.8 on the next page. (The x-range is chosen to include the production range from 0 to 180 (thousand) units. The y-range is chosen to extend beyond fixed costs of about 100 thousand dollars.)

From the figure we see that a profit can be made as long as the number of units sold exceeds about 95 (thousand). We could locate this value more precisely by using INTERSECT or TRACE and ZOOM.

(b) From the graph we also see that $R(x) - C(x) = P(x)$ is at its maximum at the right edge of the graph. Because production is limited to at most 180 thousand units, profit will be maximized when $x = 180$ and the maximum profit is

$$\begin{aligned}P(180) &= R(180) - C(180)\\ &= [(361)^{1/2} + 360 - 1] - [(370)^{1.01} - 100]\\ &\approx 85.46 \quad \text{(thousand dollars)}\end{aligned}$$

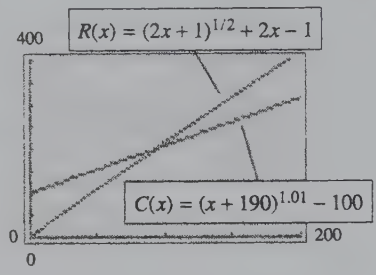


Figure 12.8

CHECKPOINT

1. True or false:
   (a) If $C(x) = \int \overline{MC}\, dx$, then the constant of integration equals the fixed costs.
   (b) If $R(x) = \int \overline{MR}\, dx$, then the constant of integration equals 0.
2. Find $C(x)$ if $\overline{MC} = \dfrac{100}{\sqrt{x+1}}$ and fixed costs are \$8000.

National Consumption and Savings

The consumption function is one of the basic ingredients in a larger discussion of how an economy can have persistent high unemployment or persistent high inflation. This study is often called **Keynesian analysis,** after its founder John Maynard Keynes.

If C represents national consumption (in billions of dollars), then a **national consumption function** has the form $C = f(y)$, where y is disposable national income (also in billions of dollars). The **marginal propensity to consume** is the derivative of the national consumption function with respect to y, or $dC/dy = f'(y)$. For example, suppose that

$$C = f(y) = 0.8y + 6$$

is a national consumption function; then the marginal propensity to consume is $f'(y) = 0.8$.

If we know the marginal propensity to consume, we can integrate with respect to y to find national consumption:

$$C = \int f'(y)\, dy = f(y) + K$$

We can find the unique national consumption function if we have additional information to help us determine the value of K, the constant of integration.

EXAMPLE 5

If consumption is \$6 billion when disposable income is 0, and if the marginal propensity to consume is $dC/dy = 0.3 + 0.4/\sqrt{y}$ (in billions of dollars), find the national consumption function.

Solution

If

$$\frac{dC}{dy} = 0.3 + \frac{0.4}{\sqrt{y}}$$

then

$$C = \int \left(0.3 + \frac{0.4}{\sqrt{y}}\right) dy = 0.3y + 0.8y^{1/2} + K$$

Now, if $C = 6$ when $y = 0$, then $6 = 0.3(0) + 0.8\sqrt{0} + K$. Thus the constant of integration is $K = 6$, and the consumption function is

$$C = 0.3y + 0.8\sqrt{y} + 6 \qquad \text{(billions of dollars)}$$

If S represents national savings, we can assume that the disposable national income is given by $y = C + S$, or $S = y - C$. Then the **marginal propensity to save** is $dS/dy = 1 - dC/dy$.

EXAMPLE 6

If the consumption is \$9 billion when income is 0, and if the marginal propensity to save is 0.25, find the consumption function.

Solution

If $dS/dy = 0.25$, then $0.25 = 1 - dC/dy$, or $dC/dy = 0.75$. Thus

$$C = \int 0.75\,dy = 0.75y + K$$

If $C = 9$ when $y = 0$, then $9 = 0.75(0) + K$, or $K = 9$. Then the consumption function is $C = 0.75y + 9$ (billions of dollars).

CHECKPOINT

3. If the marginal propensity to save is

$$\frac{dS}{dy} = 0.7 - \frac{0.4}{\sqrt{y}}$$

find the marginal propensity to consume.

4. Find the national consumption function if the marginal propensity to consume is

$$\frac{dC}{dy} = \frac{1}{\sqrt{y+4}} + 0.2$$

and national consumption is \$6.8 billion when disposable income is 0.

CHECKPOINT SOLUTIONS

1. (a) False. $C(0)$ equals the fixed costs. It may or may not be the constant of integration. [In Problem 2, $C(0) = 8000$, but $K = 7800$.]
   (b) False. We use $R(0) = 0$ to determine the constant of integration, but it may be nonzero. See Example 4.

2. $C(x) = 100\int (x+1)^{-1/2}\,dx = 100\left[\dfrac{(x+1)^{1/2}}{1/2}\right] + K$

   $C(x) = 200\sqrt{x+1} + K$

   When $x = 0$, $C(x) = 8000$, so

$$8000 = 200\sqrt{1} + K$$
$$7800 = K$$
$$C(x) = 200\sqrt{x+1} + 7800$$

3. $\dfrac{dC}{dy} = 1 - \dfrac{dS}{dy} = 1 - \left(0.7 - \dfrac{0.4}{\sqrt{y}}\right) = 0.3 + \dfrac{0.4}{\sqrt{y}}$

4. $C(y) = \displaystyle\int \left(\frac{1}{\sqrt{y+4}} + 0.2\right) dy$

   $= \int [(y+4)^{-1/2} + 0.2]\,dy$

   $= \dfrac{(y+4)^{1/2}}{1/2} + 0.2y + K$

   $C(y) = 2\sqrt{y+4} + 0.2y + K$

   Using $C(0) = 6.8$ gives $6.8 = 2\sqrt{4} + 0 + K$, or $K = 2.8$.

   Thus $C(y) = 2\sqrt{y+4} + 0.2y + 2.8$.

EXERCISE 12.4

Total Cost and Profit

1. If the monthly marginal cost for a product is $\overline{MC} = 2x + 100$, with fixed costs amounting to \$200, find the total cost function for the month.
2. If the monthly marginal cost for a product is $\overline{MC} = x + 30$, and the related fixed costs are \$50, find the total cost function for the month.
3. If the marginal cost for a product is $\overline{MC} = 4x + 2$, and the production of 10 units results in a total cost of \$300, find the total cost function.
4. If the marginal cost for a product is $\overline{MC} = 3x + 50$, and the total cost of producing 20 units is \$2000, what will be the total cost function?
5. If the marginal cost for a product is $\overline{MC} = 4x + 40$, and the total cost of producing 25 units is \$3000, what will be the cost of producing 30 units?
6. If the marginal cost for producing a product is $\overline{MC} = 5x + 10$, with a fixed cost of \$800, what will be the cost of producing 20 units?
7. A firm knows that its marginal cost for a product is $\overline{MC} = 3x + 20$, that its marginal revenue is $\overline{MR} = 44 - 5x$, and that the cost of production and sale of 80 units is \$11,400.
   (a) Find the optimal level of production.
   (b) Find the profit function.
   (c) Find the profit or loss at the optimal level.
8. A certain firm's marginal cost for a product is $\overline{MC} = 6x + 60$, its marginal revenue is $\overline{MR} = 180 - 2x$, and its total cost of production of 10 items is \$1000.
   (a) Find the optimal level of production.
   (b) Find the profit function.
   (c) Find the profit or loss at the optimal level of production.
   (d) Should production be continued for the short run?
   (e) Should production be continued for the long run?
9. Suppose that the marginal revenue for a product is $\overline{MR} = 900$ and the marginal cost is $\overline{MC} = 30\sqrt{x+4}$, with a fixed cost of \$1000.

(a) Find the profit or loss from the production and sale of 5 units.
(b) How many units will result in a maximum profit?

10. Suppose that the marginal cost for a product is $\overline{MC} = 60\sqrt{x+1}$ and its fixed cost is \$340.00. If the marginal revenue for it is $\overline{MR} = 80x$, find the profit or loss from production and sale of:
(a) 3 units.
(b) 8 units.

11. The average cost of a product changes at the rate

$$\overline{C}'(x) = -6x^{-2} + 1/6$$

and the average cost of 6 units is \$10.00.
(a) Find the average cost function.
(b) Find the average cost of 12 units.

12. The average cost of a product changes at the rate

$$\overline{C}'(x) = \frac{-10}{x^2} + \frac{1}{10}$$

and the average cost of 10 units is \$20.00.
(a) Find the average cost function.
(b) Find the average cost of 20 units.

13. Suppose for a certain product that marginal cost is given by $\overline{MC} = 1.05(x + 180)^{0.05}$ and marginal revenue is given by $\overline{MR} = (1/\sqrt{0.5x+4}) + 2.8$, where x is in thousands of units and both revenue and cost are in thousands of dollars. Fixed costs are \$200,000 and production is limited to at most 200 thousand units.
(a) Find $C(x)$ and $R(x)$.
(b) Graph $C(x)$ and $R(x)$ to determine whether a profit can be made.
(c) Determine the level of production that yields maximum profit, and find the maximum profit (or minimum loss).

14. Suppose for a certain product that the marginal cost is given by $\overline{MC} = 1.02(x + 200)^{0.02}$ and marginal revenue is given by $\overline{MR} = (2/\sqrt{4x+1}) + 1.75$, where x is in thousands of units and revenue and cost are in thousands of dollars. Suppose further that fixed costs are \$150,000 and production is limited to at most 200 thousand units.
(a) Find $C(x)$ and $R(x)$.
(b) Graph $C(x)$ and $R(x)$ to determine whether a profit can be made.
(c) Determine what level of production yields maximum profit, and find the maximum profit (or minimum loss).

National Consumption and Savings

15. If consumption is \$5 billion when disposable income is 0, and if the marginal propensity to consume is

$$\frac{dC}{dy} = 0.4 + \frac{0.3}{\sqrt{y}} \qquad \text{(in billions of dollars)}$$

find the national consumption function.

16. If consumption is \$7 billion when disposable income is 0, and if the marginal propensity to consume is 0.80, find the national consumption function (in billions of dollars).

17. If consumption is \$8 billion when income is 0, and if the marginal propensity to consume is

$$\frac{dC}{dy} = 0.3 + \frac{0.2}{\sqrt{y}} \qquad \text{(in billions of dollars)}$$

find the national consumption function.

18. If national consumption is \$9 billion when income is 0, and if the marginal propensity to consume is 0.30, what is consumption when disposable income is \$20 billion?

19. If consumption is \$6 billion when disposable income is 0, and if the marginal propensity to consume is

$$\frac{dC}{dy} = \frac{1}{\sqrt{y+1}} + 0.4 \qquad \text{(in billions of dollars)}$$

find the national consumption function.

20. If consumption is \$5.8 billion when disposable income is 0, and if the marginal propensity to consume is

$$\frac{dC}{dy} = \frac{1}{\sqrt{2y+9}} + 0.8 \qquad \text{(in billions of dollars)}$$

find the national consumption function.

21. Suppose that the marginal propensity to consume is

$$\frac{dC}{dy} = 0.7 - e^{-2y} \qquad \text{(in billions of dollars)}$$

and that consumption is \$5.65 billion when disposable income is 0. Find the national consumption function.

22. Suppose that the marginal propensity to consume is

$$\frac{dC}{dy} = 0.04 + \frac{\ln(y+1)}{y+1} \qquad \text{(in billions of dollars)}$$

and that consumption is \$6.04 billion when disposable income is 0. Find the national consumption function.

23. Suppose that the marginal propensity to save is

$$\frac{dS}{dy} = 0.15 \qquad \text{(in billions of dollars)}$$

and that consumption is \$5.15 billion when disposable income is 0. Find the national consumption function.

24. Suppose that the marginal propensity to save is

$$\frac{dS}{dy} = 0.22 \qquad \text{(in billions of dollars)}$$

and that consumption is \$8.6 billion when disposable income is 0. Find the national consumption function.

25. Suppose that the marginal propensity to save is

$$\frac{dS}{dy} = 0.2 - \frac{1}{\sqrt{3y+7}} \qquad \text{(in billions of dollars)}$$

and that consumption is \$6 billion when disposable income is 0. Find the national consumption function.

26. If consumption is \$3 billion when disposable income is 0, and if the marginal propensity to save is

$$\frac{dS}{dy} = 0.2 + e^{-1.5y} \qquad \text{(in billions of dollars)}$$

find the national consumption function.

12.5 Differential Equations

OBJECTIVES

- *To show that a function is the solution to a differential equation*
- *To use integration to find the general solution of a differential equation*
- *To find particular solutions of differential equations using given conditions*
- *To solve separable differential equations*
- *To solve applied problems involving separable differential equations*

APPLICATION PREVIEW

Carbon-14 dating, used to determine the age of fossils, is based on three facts. First, the half-life of carbon-14 is 5600 years. Second, the amount of carbon-14 in any living organism is essentially constant. Third, when an organism dies, the rate of change of carbon-14 in the organism is proportional to the amount present. If y represents the amount of carbon-14 present in the organism, then we can express the rate of change of carbon-14 by the **differential equation**

$$\frac{dy}{dt} = ky$$

where k is a constant and t is time in years. In this section, we study methods that allow us to find a function y that satisfies this differential equation, and then we use that function to date a fossil.

Recall that we introduced the derivative as an instantaneous rate of change and denoted the instantaneous rate of change of y with respect to time as dy/dt. For many growth or decay processes, such as carbon-14 decay, the rate of change of the amount of a substance with respect to time is proportional to the amount present. As we noted above, this can be represented by the equation

$$\frac{dy}{dt} = ky \qquad (k = \text{constant})$$

An equation of this type, where y is an unknown function of x or t, is called a **differential equation** because it contains derivatives (or differentials). In this section, we restrict ourselves to differential equations where the highest derivative present in the equation is the first derivative. These differential equations are called **first-order differential equations.** Examples are

$$f'(x) = \frac{1}{x+1}, \qquad \frac{dy}{dt} = 2t, \quad \text{and} \quad x\,dy = (y+1)\,dx$$

Solution of Differential Equations

The solution to a differential equation is a function [say $y = f(x)$] that, when used in the differential equation, results in an identity.

EXAMPLE 1

Show that $y = 4e^{-5t}$ is a solution of $dy/dt + 5y = 0$.

Solution

We must show that substituting $y = 4e^{-5t}$ into the equation $dy/dt + 5y = 0$ results in an identity:

$$\begin{aligned}\frac{d}{dt}(4e^{-5t}) + 5(4e^{-5t}) &= 0\\ -20e^{-5t} + 20e^{-5t} &= 0\\ 0 &= 0\end{aligned}$$

Thus $y = 4e^{-5t}$ is a solution.

Now that we know what it means for a function to be a solution to a differential equation, let us consider how to find solutions.

The most elementary differential equations are of the form

$$\frac{dy}{dx} = f(x)$$

where $f(x)$ is a continuous function. These equations are elementary to solve because the solutions are found by integration:

$$y = \int f(x)\,dx$$

EXAMPLE 2

Find the solution of

$$f'(x) = \frac{1}{x+1}$$

Solution

The solution is

$$f(x) = \int f'(x)\,dx = \int \frac{1}{x+1}\,dx = \ln|x+1| + C$$

The solution in Example 2, $f(x) = \ln|x + 1| + C$, is called the **general solution** because every solution to the equation has this form, and different values of C give different **particular solutions.** Figure 12.9 on the next page shows the graphs of several members of the family of solutions to this differential equation. (We cannot, of course, show all of them.)

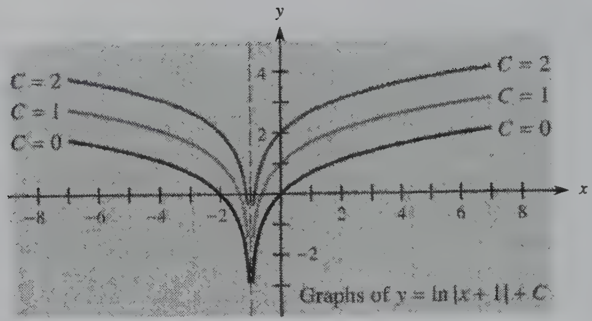


Figure 12.9

We can find a particular solution to a differential equation when we know that the solution must satisfy additional conditions, such as **initial conditions** or **boundary conditions.** For instance, to find the particular solution to

$$f'(x) = \frac{1}{x+1} \qquad \text{with the condition that} \qquad f(-2) = 2$$

we use $f(-2) = 2$ in the general solution, $f(x) = \ln|x + 1| + C$.

$$2 = f(-2) = \ln|-2 + 1| + C$$
$$2 = \ln|-1| + C \quad \text{so} \quad C = 2$$

Thus the particular solution is

$$f(x) = \ln|x + 1| + 2$$

and is shown in Figure 12.9 with $C = 2$.

We frequently denote the value of the solution function $y = f(t)$ at the initial time $t = 0$ as $y(0)$ instead of $f(0)$.

CHECKPOINT

1. Given $f'(x) = 2x - [1/(x + 1)], f(0) = 4$,
   (a) find the general solution to the differential equation.
   (b) find the particular solution that satisfies $f(0) = 4$.

Just as we can find the differential of both sides of an equation, we can find the solution to a differential equation of the form

$$G(y)\,dy = f(x)\,dx$$

by integrating both sides.

EXAMPLE 3

Solve $3y^2\,dy = 2x\,dx$, if $y(1) = 2$.

Solution

We find the general solution by integrating both sides.

$$\int 3y^2\,dy = \int 2x\,dx$$

$$y^3 + C_1 = x^2 + C_2$$

$$y^3 = x^2 + C, \qquad \text{where } C = C_2 - C_1$$

By using $y(1) = 2$, we can find C.

$$2^3 = 1^2 + C$$

$$7 = C$$

Thus the particular solution is given implicitly by

$$y^3 = x^2 + 7$$

Separable Differential Equations

It is frequently necessary to change the form of a differential equation before it can be solved by integrating both sides.

For example, the equation

$$\frac{dy}{dx} = y^2$$

cannot be solved by simply integrating both sides of the equation with respect to x because we cannot evaluate $\int y^2\,dx$.

However, we can multiply both sides of $dy/dx = y^2$ by dx/y^2 to obtain an equation that has all terms containing y on one side of the equation and all terms containing x on the other side. That is, we obtain

$$\frac{dy}{y^2} = dx$$

Separable Differential Equations

When a differential equation can be equivalently expressed in the form

$$g(y)\,dy = f(x)\,dx$$

we say that the equation is **separable.**

The solution of a separable differential equation is obtained by integrating both sides of the equation after the variables have been separated.

EXAMPLE 4

Solve the differential equation

$$(x^2y + x^2)\,dy = x^3\,dx$$

Solution

To write the equation in separable form, we first factor x^2 from the left side and divide both sides by it.

$$x^2(y+1)\,dy = x^3\,dx$$
$$(y+1)\,dy = \frac{x^3}{x^2}\,dx$$

The equation is now separated, so we integrate both sides.

$$\int (y+1)\,dy = \int x\,dx$$
$$\frac{y^2}{2} + y + C_1 = \frac{x^2}{2} + C_2$$

This equation, as well as the equation

$$y^2 + 2y - x^2 = C, \quad \text{where } C = 2(C_2 - C_1)$$

gives the solution implicitly.

Note that we need not write both C_1 and C_2 when we integrate, because it is always possible to combine the two constants into one.

EXAMPLE 5

Solve the differential equation

$$\frac{dy}{dt} = ky \qquad (k = \text{constant})$$

Solution

To solve the equation, we write it in separated form as

$$\frac{dy}{y} = k\,dt$$

and integrate both sides as follows:

$$\int \frac{dy}{y} = \int k\,dt$$
$$\ln|y| = kt + C_1$$

Assuming that $y > 0$ and writing this equation in exponential form gives

$$y = e^{kt + C_1}$$
$$y = e^{kt} \cdot e^{C_1} = Ce^{kt}, \quad \text{where } C = e^{C_1}$$

This solution,

$$y = Ce^{kt}$$

is the general solution of the differential equation $dy/dt = ky$ because all solutions have this form, with different values of C giving different particular solutions. The case of $y < 0$ is covered by values of $C < 0$.

CHECKPOINT

2. True or false:
   (a) The general solution to $dy = (x/y)\,dx$ can be found from

$$\int y\,dy = \int x\,dx$$

   (b) The first step in solving $dy/dx = -2xy^2$ is to separate it.
   (c) The equation $dy/dx = -2xy^2$ separates as $y^2\,dy = -2x\,dx$.
3. Suppose that $(xy + x)(dy/dx) = x^2y + y$.
   (a) Separate this equation. (b) Find the general solution.

In many applied problems that can be modeled with differential equations, we know of conditions that allow us to obtain a particular solution.

Applications of Differential Equations

We now consider two applications that can be modeled by differential equations. These are radioactive decay (as introduced in the Application Preview) and one-container mixture problems (as a model for drugs in an organ).

EXAMPLE 6

In the Application Preview, we introduced carbon-14 dating and the facts that the process is based on. We said that when an organism dies, the rate of change of the amount of carbon-14 present is proportional to the amount present and is represented by the differential equation

$$\frac{dy}{dt} = ky$$

where y is the amount present, k is a constant, and t is time in years. If we denote the initial amount of carbon-14 in an organism as y_0, then $y = y_0$ represents the amount present at time $t = 0$ (when the organism died). Suppose that anthropologists discover a fossil that contains 1% of the initial amount of carbon-14. Find the age of the fossil. (Recall that the half-life of carbon-14 is 5600 years.)

Solution

We must find a particular solution to

$$\frac{dy}{dt} = ky$$

subject to the fact that when $t = 0$, $y = y_0$, and we must determine the value of k on the basis of the half-life of carbon-14 ($t = 5600$ years, $y = \frac{1}{2}y_0$ units.) From Example 5, we know that the general solution to the differential equation $dy/dt = ky$ is $y = Ce^{kt}$. Using $y = y_0$ when $t = 0$, we obtain $y_0 = C$, so the equation becomes $y = y_0e^{kt}$. Using $t = 5600$ and $y = \frac{1}{2}y_0$ in this equation gives

$$\frac{1}{2}y_0 = y_0e^{5600k} \quad \text{or} \quad 0.5 = e^{5600k}$$

Rewriting this equation in logarithmic form and then solving for k, we get

$$\ln(0.5) = 5600k$$
$$-0.69315 = 5600k$$
$$-0.00012378 = k$$

Thus the equation we seek is

$$y = y_0e^{-0.00012378t}$$

Using the fact that $y = 0.01y_0$ when the fossil was discovered, we can find its age t by solving

$$0.01y_0 = y_0e^{-0.00012378t} \quad \text{or} \quad 0.01 = e^{-0.00012378t}$$

Rewriting this in logarithmic form and then solving gives

$$\ln(0.01) = -0.00012378t$$
$$-4.6051702 = -0.00012378t$$
$$37{,}204 = t$$

Thus the fossil is approximately 37,200 years old.

Another application of differential equations comes from a group of applications called *one-container mixture problems.* In problems of this type, there is a substance whose amount in a container is changing with time, and the goal is to determine the amount of the substance at any time t. The differential equations that model these problems are of the following form:

$$\begin{bmatrix}\text{Rate of change}\\ \text{of the amount}\\ \text{of the substance}\end{bmatrix} = \begin{bmatrix}\text{Rate at which}\\ \text{the substance}\\ \text{enters the container}\end{bmatrix} - \begin{bmatrix}\text{Rate at which}\\ \text{the substance}\\ \text{leaves the container}\end{bmatrix}$$

We consider this application as it applies to the amount of a drug in an organ.

EXAMPLE 7

A liquid carries a drug into an organ of volume 300 cc at a rate of 5 cc/s, and the liquid leaves the organ at the same rate. If the concentration of the drug in the entering liquid is 0.1 g/cc, and if x represents the amount of drug in the organ at any time t, then using the fact that the rate of change of the amount of the drug in the organ, dx/dt, equals the rate at which the drug enters minus the rate at which it leaves, we have

$$\frac{dx}{dt} = \left(\frac{5\text{ cc}}{\text{s}}\right)\left(\frac{0.1\text{ g}}{\text{cc}}\right) - \left(\frac{5\text{ cc}}{\text{s}}\right)\left(\frac{x\text{ g}}{300\text{ cc}}\right)$$

or

$$\frac{dx}{dt} = 0.5 - \frac{x}{60} = \frac{30}{60} - \frac{x}{60} = \frac{30-x}{60}, \quad \text{in g/s}$$

Find the amount of the drug in the organ as a function of time t.

Solution

Multiplying both sides of the equation $\frac{dx}{dt} = \frac{30-x}{60}$ by $\frac{dt}{(30-x)}$ gives

$$\frac{dx}{30-x} = \frac{1}{60}\,dt$$

The equation is now separated, so we can integrate both sides.

$$\int \frac{dx}{30-x} = \int \frac{1}{60}\,dt$$

$$-\ln(30-x) = \frac{1}{60}t + C_1 \qquad (30-x>0)$$

$$\ln(30-x) = -\frac{1}{60}t - C_1$$

Rewriting this in exponential form gives

$$30 - x = e^{-t/60 - C_1} = e^{-t/60} \cdot e^{-C_1}$$

Letting $C = e^{-C_1}$ yields

$$30 - x = Ce^{-t/60}$$

so

$$x = 30 - Ce^{-t/60}$$

and we have the desired function.

CHECKPOINT SOLUTIONS

1. (a) $f(x) = \int \left[2x - \frac{1}{x+1}\right] dx = x^2 - \ln|x+1| + C$

   (b) If $f(0) = 4$, then $4 = 0^2 - \ln|1| + C$, so $C = 4$.
   Thus $f(x) = x^2 - \ln|x+1| + 4$ is the particular solution.

2. (a) True, and the solution is

$$\int y\,dy = \int x\,dx \qquad \frac{y^2}{2} = \frac{x^2}{2} + C$$

   (b) True

   (c) False. It separates as $dy/y^2 = -2x\,dx$, and the solution is

$$\int \frac{-dy}{y^2} = \int 2x\,dx \qquad y^{-1} = x^2 + C$$

$$\frac{1}{y} = x^2 + C$$

$$y = \frac{1}{x^2 + C}$$

3. (a) $x(y+1)\frac{dy}{dx} = (x^2+1)y$

$$\frac{y+1}{y}\,dy = \frac{x^2+1}{x}\,dx$$

$$\left(1 + \frac{1}{y}\right) dy = \left(x + \frac{1}{x}\right) dx$$

   (b) $\int \left(1 + \frac{1}{y}\right) dy = \int \left(x + \frac{1}{x}\right) dx$

$$y + \ln|y| = \frac{x^2}{2} + \ln|x| + C$$

   This is the general solution.

EXERCISE 12.5

In Problems 1–4, show that the given function is a solution of the differential equation.

1. $y = x^2$; $\quad 4y - 2xy' = 0$
2. $y = x^3$; $\quad 3y - xy' = 0$
3. $y = 3x^2 + 1$; $\quad 2y\,dx - x\,dy = 2\,dx$
4. $y = 4x^3 + 2$; $\quad 3y\,dx - x\,dy = 6\,dx$

In Problems 5–10, use integration to find the general solution to each differential equation.

5. $dy = xe^{x^2} + 1\,dx$
6. $dy = x^2e^{x^3 - 1}\,dx$
7. $2y\,dy = 4x\,dx$
8. $4y\,dy = 4x^3\,dx$
9. $3y^2\,dy = (2x - 1)\,dx$
10. $4y^3\,dy = (3x^2 + 2x)\,dx$

In Problems 11–14, find the particular solutions.

11. $y' = e^{x-3}$, $\quad y(0) = 2$
12. $y' = e^{2x+1}$, $\quad y(0) = e$
13. $dy = \left(\dfrac{1}{x} - x\right) dx$, $\quad y(1) = 0$
14. $dy = \left(x^2 - \dfrac{1}{x+1}\right) dx$, $\quad y(0) = \dfrac{1}{3}$

In Problems 15–28, find the general solution to the given differential equation.

15. $\dfrac{dy}{dx} = \dfrac{x^2}{y}$
16. $y^3\,dx = \dfrac{dy}{x^3}$
17. $dx = x^3y\,dy$
18. $dy = x^2y^3\,dx$
19. $dx = (x^2y^2 + x^2)\,dy$
20. $dy = (x^2y^3 + xy^3)\,dx$
21. $y^2\,dx = x\,dy$
22. $y\,dx = x\,dy$
23. $\dfrac{dy}{dx} = \dfrac{x}{y}$
24. $\dfrac{dy}{dx} = \dfrac{x^2 + x}{y + 1}$
25. $(x + 1)\dfrac{dy}{dx} = y$
26. $x^2y\dfrac{dy}{dx} = y^2 + 1$
27. $e^{2x}y\,dy = (y + 1)\,dx$
28. $e^{4x}(y + 1)\,dx + e^{2x}y\,dy = 0$

In Problems 29–36, find the particular solution to each differential equation.

29. $\dfrac{dy}{dx} = \dfrac{x^2}{y^3}$, when $x = 1, y = 1$
30. $\dfrac{dy}{dx} = \dfrac{x+1}{xy}$, when $x = 1, y = 3$
31. $2y^2\,dx = 3x^2\,dy$, when $x = 2, y = -1$
32. $(x + 1)\,dy = y^2\,dx$, when $x = 0, y = 2$
33. $x^2e^{2y}\,dy = (x^3 + 1)\,dx$, when $x = 1, y = 0$
34. $y' = \dfrac{1}{xy}$, when $x = 1, y = 3$
35. $2xy\dfrac{dy}{dx} = y^2 + 1$, when $x = 1, y = 2$
36. $xe^y\,dx = (x + 1)dy$, when $x = 0, y = 0$

Applications

37. ***Allometric growth*** If x and y are measurements of certain parts of an organism, then the rate of change of y with respect to x is proportional to the ratio of y to x. That is, these measurements satisfy

$$\frac{dy}{dx} = k\frac{y}{x}$$

which is referred to as an allometric law of growth. Solve this differential equation.

38. ***Bimolecular chemical reactions*** A bimolecular chemical reaction is one in which two chemicals react to give another substance. Suppose that one molecule of each of the two chemicals react to form two molecules of a new substance. If x represents the number of molecules of the new substance at time t, then the rate of change of x is proportional to the product of the numbers of molecules of the original chemicals available to be converted. That is, if each of the chemicals initially contained A molecules, then

$$\frac{dx}{dt} = k(A - x)^2$$

If 40% of the initial amount A is converted after 1 hour, how long will it be before 90% is converted?

Compound interest In Problems 39 and 40, use the following information.

When interest is compounded continuously, the rate of change of the amount x of the investment is proportional to the amount present. In this case, the proportionality constant is the annual interest rate r (as a decimal); that is,

$$\frac{dx}{dt} = rx$$

39. (a) If \$10,000 is invested at 6%, compounded continuously, find an equation for the future value of the investment as a function of time t in years.
    (b) What is the future value of the investment after 1 year? After 5 years?
    (c) How long will it take for investment to double?
40. (a) If \$2000 is invested at 8%, compounded continuously, find an equation for the future value of the investment as a function of time t, in years.
    (b) How long will it take for investment to double?
    (c) What will be the future value of this investment after 35 years?

41. ***Bacterial growth*** Suppose that the growth of a certain population of bacteria satisfies

$$\frac{dy}{dt} = ky$$

where y is the number of organisms and t is the number of hours. If initially there are 10,000 organisms and the number triples after 2 hours, how long will it be before there is 100 times the original population?

42. ***Bacterial growth*** Suppose that, for a certain population of bacteria, growth occurs according to

$$\frac{dy}{dt} = ky \quad (t \text{ in hours})$$

If the doubling rate depends on temperature, find how long it takes for the number of bacteria to reach 50 times the original number at each given temperature in (a) and (b).
(a) At 90°F, the number doubles after 30 minutes $\left(\frac{1}{2}\text{ hour}\right)$.
(b) At 40°F, the number doubles after 3 hours.

43. ***Half-life*** A breeder reactor converts uranium-238 into an isotope of plutonium-239 at a rate proportional to the amount present at any time. After 10 years, 0.03% of the radioactivity has dissipated (that is, 0.9997 of the initial amount remains). Suppose that initially there is 100 pounds of this substance. Find the half-life.

44. ***Radioactive decay*** A certain radioactive substance has a half-life of 50 hours. Find how long it will take for 90% of the radioactivity to be dissipated if the amount of material x satisfies

$$\frac{dx}{dt} = kx \quad (t \text{ in hours})$$

45. ***Drug in an organ*** Suppose that a liquid carries a drug into a 100-cc organ at a rate of 5 cc/s and leaves the organ at the same rate. Suppose that the concentration of the drug entering is 0.06 g/cc. If initially there is no drug in the organ, find the amount of drug in the organ as a function of time t.

46. ***Drug in an organ*** Suppose that a liquid carries a drug into a 250-cc organ at a rate of 10 cc/s and leaves the organ at the same rate. Suppose that the concentration of the drug entering is 0.15 g/cc. Find the amount of drug in the organ as a function of time t if initially there is none in the organ.

47. ***Drug in an organ*** Suppose that a liquid carries a drug with concentration 0.1 g/cc into a 200-cc organ at a rate of 5 cc/s and leaves the organ at the same rate. If initially there is 10 g of the drug in the organ, find the amount of drug in the organ as a function of time t.

48. ***Drug in an organ*** Suppose that a liquid carries a drug with concentration 0.05 g/cc into a 150-cc organ at a rate of 6 cc/s and leaves at the same rate. If initially there is 1.5 g of drug in the organ, find the amount of drug in the organ as a function of time t.

49. ***Sales and pricing*** Suppose that in a certain company, the relationship between the price per unit p of its product and the weekly sales volume y, in thousands of dollars, is given by

$$\frac{dy}{dp} = -\frac{2}{5}\left(\frac{y}{p+8}\right)$$

Solve this differential equation if $y = 8$ when $p =$ \$24.

50. ***Sales and pricing*** Suppose that a chain of auto service stations, Quick-Oil, Inc., has found that the relationship between its price p for a oil change and its monthly sales volume y, in thousands of dollars, is

$$\frac{dy}{dp} = -\frac{1}{2}\left(\frac{y}{p+5}\right)$$

Solve this differential equation if $y = 18$ when $p =$ \$20.

51. ***Tumor volume*** Let V denote the volume of a tumor, and suppose that the growth rate of the tumor satisfies

$$\frac{dV}{dt} = 0.2Ve^{-0.1t}$$

If the initial volume of the tumor is 1.86 units, find an equation for V as a function of t.

52. ***Gompertz curves*** The differential equation

$$\frac{dx}{dt} = x(a - b\ln x)$$

where x represents the number of objects at time t, and a and b are constants, is the model for Gompertz curves. Recall from Section 5.3, "Solution of Exponential Equations," that Gompertz curves can be used to study growth or decline of populations, organizations, and revenue from sales of a product, as well as forecast equipment maintenance costs. Solve the differential equation to obtain the Gompertz curve formula

$$x = e^{a/b}e^{-ce^{-bt}}$$

53. ***Cell growth*** If V is the volume of a spherical cell, then in certain cell growth and for some fetal growth models, the rate of change of V is given by

$$\frac{dV}{dt} = kV^{2/3}$$

where k is a constant depending on the organism. If $V = 0$ when $t = 0$, find V as a function of t.

54. ***Atmospheric pressure*** The rate of change of atmospheric pressure P with respect to the altitude above sea level h is proportional to the pressure. That is,

$$\frac{dP}{dh} = kP$$

Suppose that the pressure at sea level is denoted by P_0, and at 18,000 ft the pressure is half what it is at sea level. Find the pressure, as a percentage of P_0, at 25,000 ft.

55. ***Newton's law of cooling*** Newton's law of cooling states that the rate of change of temperature $u = u(t)$ of an object is proportional to the temperature difference between the object and its surroundings, where T is the constant temperature of the surroundings. That is,

$$\frac{du}{dt} = k(u - T)$$

Suppose an object at 0°C is placed in a room where the temperature is 20°C. If the temperature of the object is 8°C after 1 hour, how long will it take for the object to reach 18°C?

56. ***Newton's law of cooling*** Newton's law of cooling can be used to estimate time of death. (Actually the estimate may be quite rough, because cooling does not begin until metabolic processes have ceased.) Suppose a corpse is discovered at noon in a 70°F room, and at that time the body temperature is 96.1°F. If at 1:00 P.M. the body temperature is 94.6°F, use Newton's law of cooling to estimate the time of death.

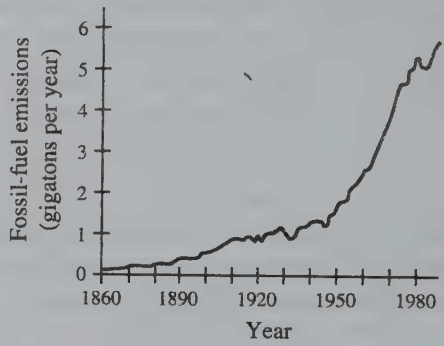
57. ***Fossil-fuel emissions*** The amount of carbon in the atmosphere has been estimated to have increased at a rate of about 4.3% per year from 1860 until 1973, except during the Great Depression and the world wars.* This increase is due to carbon emissions from fossil-fuel burning and can be modeled by the differential equation

$$\frac{dE}{dt} = 0.043E$$

where t is the number of years since 1860 and E is fossil-fuel emissions in gigatons per year.

(a) Solve this differential equation, and find a particular solution that satisfies $E(0) = 0.1$.

(b) Graph your solution and compare it with the graph below, which shows actual emissions since 1860.



Source: *American Scientist*, Vol. 79, (July–August 1990), p. 313

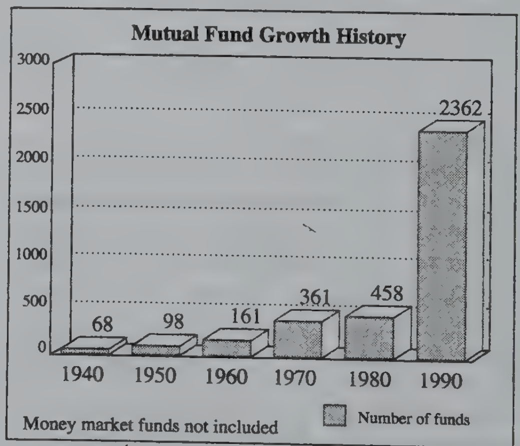
58. ***Mutual fund growth*** The number of mutual funds established since 1940 can be modeled by

$$\frac{dy}{dt} = 0.662y, \quad y(4) = 68$$

where t is the number of decades since 1900 and y is the number of mutual funds.

(a) Find the particular solution to this differential equation.

(b) Check your model against the data in the figure for 1960 and 1990.



Source: Investment Company Institute, published in *Investment Digest of Valic Co.*, Vol. 5, No. 2 (Summer), 1992.

**American Scientist*, Vol. 79 (July–August 1990), p. 313.

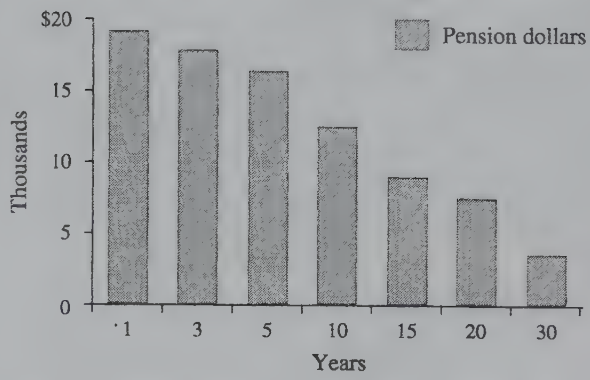
59. ***Impact of inflation*** The impact of a 5% inflation rate on a \$20,000-per-year pension is shown in the accompanying figure. If P represents the dollars of purchasing power of a \$20,000 pension, then the effect of a 5% inflation rate can be modeled by the differential equation

$$\frac{dP}{dt} = -0.05P, \quad P(0) = 20{,}000$$

where t is in years.

(a) Find the particular solution to this differential equation.

(b) Check your model against the graph in the figure. In particular, find the purchasing power after 30 years. Does your calculation match the graph?



Source: *Viewpoints*, Financial news from Valic Co., Summer 1993

KEY TERMS AND FORMULAS

Section	Key Terms	Formula
12.1	General antiderivative of $f'(x)$	$f(x) + C$
	Integral	$\int f(x)\,dx$
	Powers of x Formula	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
	Integration Formulas	$\int dx = x + C$
		$\int cu(x)\,dx = c\int u(x)\,dx$; c = a constant
		$\int [u(x) \pm v(x)]\,dx = \int u(x)\,dx \pm \int v(x)\,dx$
12.2	Power Rule	$\int [u(x)]^n u'(x)\,dx = \frac{[u(x)]^{n+1}}{n+1} + C \quad (n \neq -1)$
12.3	Logarithmic Formula	$\int \frac{u'}{u}\,dx = \int \frac{1}{u}\,du = \ln\|u\| + C$
	Exponential Formula	$\int e^u u'\,dx = \int e^u\,du = e^u + C$
12.4	Total cost	$C(x) = \int \overline{MC}\,dx$
	Total revenue	$R(x) = \int \overline{MR}\,dx$
	Profit	$P(x) = R(x) - C(x)$
	Marginal propensity to consume	$\frac{dC}{dy}$
	Marginal propensity to save	$\frac{dS}{dy} = 1 - \frac{dC}{dy}$

Section	Key Terms	Formula
	National consumption	$C = \int f'(y)\,dy = \int \frac{dC}{dy}\,dy$
12.5	Differential equations Solutions General Particular	
	First order	$\frac{dy}{dx} = f(x) \Rightarrow y = \int f(x)\,dx$
	Separable	$g(y)\,dy = f(x)\,dx$ $\Rightarrow \int g(y)\,dy = \int f(x)\,dx$
	Radioactive decay	$\frac{dy}{dt} = ky$
	Drugs in an organ	Rate = (rate in) − (rate out)

REVIEW EXERCISES

Sections 12.1–12.3

Evaluate the integrals in Problems 1–26.

1. $\int x^6 dx$
2. $\int x^{1/2}\,dx$
3. $\int (x^3 - 3x^2 + 4x + 5)\,dx$
4. $\int (x^2 - 1)^2\,dx$
5. $\int (x^2 - 1)^2 x\,dx$
6. $\int (x^3 - 3x^2)(x^2 - 2x)\,dx$
7. $\int (x^3 + 4)^2 x\,dx$
8. $\int (x^3 + 4)^6 x^2\,dx$
9. $\int \frac{x^2}{x^3 + 1}\,dx$
10. $\int \frac{x^2}{(x^3 + 1)^2}\,dx$
11. $\int \frac{x^2\,dx}{\sqrt[3]{x^3 - 4}}$
12. $\int \frac{x^2\,dx}{x^3 - 4}$
13. $\int \frac{x^3 + 1}{x^2}\,dx$
14. $\int \frac{x^3 - 3x + 1}{x - 1}\,dx$
15. $\int y^2 e^{y^3}\,dy$
16. $\int (x - 1)^2\,dx$
17. $\int \frac{3x^2}{2x^3 - 7}\,dx$
18. $\int \frac{5\,dx}{e^{4x}}$
19. $\int (x^3 - e^{3x})\,dx$
20. $\int xe^{1 + x^2}\,dx$
21. $\int \frac{6x^7}{(5x^8 + 7)^3}\,dx$
22. $\int \frac{7x^3}{\sqrt{1 - x^4}}\,dx$
23. $\int \left(\frac{e^{2x}}{2} + \frac{2}{e^{2x}}\right)dx$
24. $\int \left[x - \frac{1}{(x + 1)^2}\right]dx$
25. (a) $\int (x^2 - 1)^4 x\,dx$ (b) $\int (x^2 - 1)^{10} x\,dx$
    (c) $\int (x^2 - 1)^7 3x\,dx$ (d) $\int (x^2 - 1)^{-2/3} x\,dx$
26. (a) $\int \frac{2x\,dx}{x^2 - 1}$ (b) $\int \frac{2x\,dx}{(x^2 - 1)^2}$
    (c) $\int \frac{3x\,dx}{\sqrt{x^2 - 1}}$ (d) $\int \frac{3x\,dx}{x^2 - 1}$

Section 12.5

In Problems 27–32, find the general solution to each differential equation.

27. $\frac{dy}{dt} = 4.6e^{-0.05t}$
28. $dy = (64 + 76x - 36x^2)\,dx$
29. $\frac{dy}{dx} = \frac{4x}{y - 3}$
30. $t\,dy = \frac{dt}{y + 1}$
31. $\frac{dy}{dx} = \frac{x}{e^y}$
32. $\frac{dy}{dt} = \frac{4y}{t}$

In Problems 33 and 34, find the particular solution to each differential equation.

33. $y' = \frac{x^2}{y + 1}$ $y(0) = 4$
34. $y' = \frac{2x}{1 + 2y}$ $y(2) = 0$

Applications

Section 12.1

35. ***Revenue*** If the marginal revenue for a month for a product is $\overline{MR} = 6x + 12$, find the total revenue from the sale of 4 units of the product.

36. ***Productivity*** Suppose that the rate of change of production of the average worker at a factory is given by

$$\frac{dp}{dt} = 27 + 24t - 3t^2, \qquad 0 \le t \le 8$$

where p is the number of units the worker produces in t hours. How many units will the average worker produce in an 8-hour shift? (Assume that $p = 0$ when $t = 0$.)

Section 12.2

37. ***Oxygen levels in water*** The rate of change of the oxygen level in a body of water after an oil spill is given by

$$P'(t) = 400\left[\frac{5}{(t+5)^2} - \frac{50}{(t+5)^3}\right]$$

where t is the number of months after the spill. What function gives the oxygen level P at any time t if $P = 400$ when $t = 0$?

38. ***Bacterial growth*** A population of bacteria grows at the rate

$$r = \frac{100{,}000}{(t+100)^2}$$

where t is time. If the population is 1000 when $t = 1$, write the equation that gives the size of the population at any time t.

Section 12.3

39. ***Market share*** The rate of change of the market share (as a percentage) a firm expects for a new product is

$$\frac{dy}{dt} = 2.4e^{-0.04t}$$

where t is the number of months after the product is introduced.
   (a) Write the equation that gives the expected market share y at any time t. (Note that $y = 0$ when $t = 0$.)
   (b) What market share does the firm expect after one year?

40. ***Revenue*** If the marginal revenue for a product is $\overline{MR} = \dfrac{800}{x+1}$, find the total revenue function.

Section 12.4

41. ***Cost*** The marginal cost for a product is $\overline{MC} = 6x + 4$ and the cost of producing 100 items is \$31,400.
   (a) Find the fixed costs.
   (b) Find the total cost function.

42. ***Profit*** Suppose a product has a daily marginal revenue $\overline{MR} = 46$ and a daily marginal cost $\overline{MC} = 30 + \frac{1}{5}x$. If the daily fixed cost is \$200.00, how many units will give maximum profit and what is the maximum profit?

43. ***National consumption*** If consumption is \$8.5 billion when disposable income is 0, and if the marginal propensity to consume is

$$\frac{dC}{dy} = \frac{1}{\sqrt{2y+16}} + 0.6 \quad \text{(in billions of dollars)}$$

find the national consumption function.

44. ***National consumption*** Suppose that the marginal propensity to save is

$$\frac{dS}{dy} = 0.2 - 0.1e^{-2y} \quad \text{(in billions of dollars)}$$

and consumption is \$7.8 billion when disposable income is 0. Find the national consumption function.

Section 12.5

45. ***Allometric growth*** For many species of fish, the length L and weight W of a fish are related by

$$\frac{dW}{dL} = \frac{3W}{L}$$

The general solution to this differential equation expresses the allometric relationship between the length and weight of a fish. Find the general solution.

46. ***Fossil dating*** Radioactive beryllium is sometimes used to date fossils found in deep-sea sediment. The amount of radioactive material x satisfies

$$\frac{dx}{dt} = kx$$

Suppose that 10 units of beryllium are present in a living organism and that the half-life of beryllium is 4.6 million years. Find the age of a fossil if 20% of the original radioactivity is present when the fossil is discovered.

47. ***Drug in an organ*** Suppose that a liquid carries a drug into a 120-cc organ at a rate of 4 cc/s and leaves the organ at the same rate. If initially there is no drug in the organ and if the concentration of drug in the liquid is 3 g/cc, find the amount of drug in the organ as a function of time.

48. ***Chemical mixture*** A 300-gal tank initially contains a solution with 100 lb of a chemical. A mixture containing 2 lb/gal of the chemical enters the tank at 3 gal/min, and the well-stirred mixture leaves at the same rate. Find an equation that gives the amount of the chemical in the tank as a function of time. How long will it be before there are 500 lb of chemical in the tank?

CHAPTER TEST

Evaluate the integrals in Problems 1–8.

1. $\int (6x^2 + 8x - 7)\,dx$
2. $\int \left(4 + \sqrt{x} - \frac{1}{x^2}\right)dx$
3. $\int 5x^2 (4x^3 - 7)^9\,dx$
4. $\int (3x^2 - 6x + 1)^9(2x - 2)\,dx$
5. $\int \frac{s^3}{2s^4 - 5}\,ds$
6. $\int 100e^{-0.01x}\,dx$
7. $\int 5y^3e^{2y^4-1}\,dy$
8. $\int \left(e^x + \frac{5}{x} - 1\right)dx$
9. Evaluate $\int \frac{x^2}{x+1}\,dx$. Use long division.
10. If $\int f(x)dx = 2x^3 - x + 5e^x + C$, find $f(x)$.

In Problems 11 and 12, find the particular solution to each differential equation.

11. $y' = 4x^3 + 3x^2$, if $y(0) = 4$
12. $\frac{dy}{dx} = e^{4x}$, if $y(0) = 2$
13. Find the general solution of the separable differential equation $\frac{dy}{dx} = x^3y^2$.
14. Suppose the rate of growth of the population of a city is predicted to be
$$\frac{dp}{dt} = 2000t^{1.04}$$
where p is the population and t is the number of years past 2000. If the population in the year 2000 is 50,000, what is the predicted population in the year 2010?
15. Suppose that the marginal cost for a product is $\overline{MC} = 4x + 50$, the marginal revenue is $\overline{MR} = 500$, and the cost of the production and sale of 10 units is \$1000. What is the profit function for this product?
16. Suppose the marginal propensity to save is given by
$$\frac{dS}{dy} = 0.22 - \frac{0.25}{\sqrt{0.5y + 1}} \quad \text{(in billions of dollars)}$$
and national consumption is \$6.6 billion when disposable income is \$0. Find the national consumption function.
17. A certain radioactive material has a half-life of 100 days. If the amount of material present, x, satisfies $\frac{dx}{dt} = kx$, where t is in days, how long will it take for 90% of the radioactivity to dissipate?

Extended Applications Group Projects

I. Employee Production Rate

The manager of a plant has been instructed to hire and train additional employees to manufacture a new product. She must hire a sufficient number of new employees so that within 30 days they will be producing 2500 units of the product each day.

Because a new employee must learn an assigned task, production will increase with training. Suppose that research on similar projects indicates that production increases with training according to the learning curve, so that for the average employee, the rate of production per day is given by

$$\frac{dN}{dt} = be^{-at}$$

where N is the number of units produced per day after t days of training. Because of experience with a very similar project, the manager expects the rate for this project to be

$$\frac{dN}{dt} = 2.5e^{-0.05t}$$

The manager tested her training program with 5 employees and learned that the average employee could produce 11 units per day after 5 days of training. On the basis of this information, she must decide how many employees to hire and begin to train so that a month from now they will be producing 2500 units of the product per day. She estimates that it will take her 10 days to hire the employees, and thus she will have 15 days remaining to train them. She also expects a 10% attrition rate during this period.

How many employees would you advise the plant manager to hire? Check your advice by answering the following questions.

1. Use the expected rate of production and the results of the manager's test to find the function relating N and t—that is, $N = N(t)$.
2. Find the number of units the average employee can produce after 15 days of training. How many such employees would be needed to maintain a production rate of 2500 units per day?
3. Explain how you would revise this last result to account for the expected 10% attrition rate. How many new employees should the manager hire?

II. Supply and Demand

If p is the price of a given commodity at time t, then we can think of price as a function of time. Similarly, the number of units demanded by consumers q_d at any time, and the number of units supplied by producers q_s at any time, may also be considered as functions of time as well as functions of price.

Both the quantity demanded and the quantity supplied depend not only on the price at the time, but also on the direction and rate of change that consumers and producers ascribe to prices. For example, even when prices are high, if consumers feel that prices are rising, the demand may risè. Similarly, if prices are low but producers feel they may go lower, the supply may rise.

If we assume that prices are determined in the marketplace by supply and demand, then the equilibrium price is the one we seek.

Suppose the supply and demand functions for a certain commodity in a competitive market are given, in hundreds of units, by

$$q_s = 30 + p + 5\frac{dp}{dt}$$

$$q_d = 51 - 2p + 4\frac{dp}{dt}$$

where dp/dt denotes the rate of change of the price with respect to time. If, at $t = 0$, the market equilibrium price is 12, we can express the market equilibrium price as a function of time.

Our goals are

A. To express the market equilibrium price as a function of time.
B. To determine whether there is price stability in the marketplace for this item (that is, to determine whether the equilibrium price approaches a constant over time).

To achieve these goals, do the following.

1. Set the expressions for q_s and q_d equal to each other.
2. Solve this equation for $\frac{dp}{dt}$.
3. Write this equation in the form $f(p)\,dp = g(t)\,dt$.
4. Integrate both sides of this separated differential equation.
5. Solve the resulting equation for p in terms of t.
6. Use the fact that $p = 12$ when $t = 0$ to find C, the constant of integration, and write the market equilibrium price p as a function of time t.
7. Find the $\lim_{t\to\infty} p$, which gives the price we can expect this product to approach. If this limit is finite, then for this item there is price stability in the marketplace. If $\lim_{t\to\infty} p = \infty$, then price will continue to increase until economic conditions change.

Warm-up

Prerequisite Problem Type	*For Section*	*Answer*	*Section for Review*
Simplify: $\frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - \frac{2n(n+1)}{2} + n\right]$	**13.1**	$\frac{2n^2 - 3n + 1}{6n^2}$	0.7 Fractions
(a) If $F(x) = \frac{x^4}{4} + 4x + C$, what is $F(4) - F(2)$? (b) If $F(x) = -\frac{1}{9} \ln \frac{9 + \sqrt{81 - 9x^2}}{3x}$, what is $F(3) - F(2)$?	**13.2** **13.5**	(a) 68 (b) $\frac{1}{9} \ln \left(\frac{3 + \sqrt{5}}{2}\right)$	1.2 Functional notation
Find the limit: (a) $\lim_{n\to+\infty} \frac{n^2 + n}{2n^2}$ (b) $\lim_{n\to+\infty} \frac{2n^2 - 3n + 1}{6n^2}$ (c) $\lim_{b\to\infty} \left(1 - \frac{1}{b}\right)$ (d) $\lim_{b\to\infty} \left(\frac{-100{,}000}{e^{0.10b}} + 100{,}000\right)$	**13.1** **13.7**	(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) 1 (d) 100,000	9.2 Limits at infinity
Find the derivative of $y = \ln x$.	**13.6**	$\frac{1}{x}$	11.1 Derivatives of logarithmic functions
Integrate: (a) $\int (x^3 + 4)\, dx$ (b) $\int x\sqrt{x^2 - 9}\, dx$ (c) $\int e^{2x}\, dx$	**13.2–** **13.7**	(a) $\frac{x^4}{4} + 4x + C$ (b) $\frac{1}{3}(x^2 - 9)^{3/2} + C$ (c) $\frac{1}{2}e^{2x} + C$	12.1, 12.2, 12.3 Integration

Chapter 13

Definite Integrals; Techniques of Integration

We saw some applications of the indefinite integral in Chapter 12. In this chapter we define the definite integral and discuss a theorem that is useful for evaluating it. We will also see how it can be used to find the areas under certain curves. The definite integral is used to solve many interesting types of problems from economics, finance, and probability.

Definite integrals can be used to approximate the ***total value,*** *the* ***present value,*** *and the* ***future value*** *of a* ***continuous income stream.*** *Improper integrals can be used to find the* ***capital values*** *of continuous income streams.*

Some consumers in a competitive market are willing to pay more than the market equilibrium price, and some producers are willing to sell for less than this price. The savings are called ***consumer's surplus*** *and* ***producer's surplus,*** *respectively, and areas under the demand and supply curves are used to calculate them.*

Evaluation of improper integrals is one of three new techniques of integration introduced in this chapter. The other two are use of integral tables and use of integration by parts.

13.1 Area Under a Curve

OBJECTIVES

- To use the sum of areas of rectangles to approximate the area under a curve
- To use Σ notation to denote sums
- To find the exact area under a curve

APPLICATION PREVIEW

One way to find the accumulated production (such as the production of ore from a mine) over a period of time is to graph the rate of production as a function of time and find the area under the resulting curve over a specified time interval. For example, if a coal mine produces at a rate of 30 tons per day, the production over 10 days $(30 \cdot 10 = 300)$ could be represented by the area under the line $y = 30$ between $x = 0$ and $x = 10$ (see Figure 13.1).

Using area to determine the accumulated production is very useful when the rate-of-production function varies at different points in time. For example, if the rate of production is represented by

$$y = 100e^{-0.1x}$$

where x represents the number of days, then the area under the curve (and above the x-axis) from $x = 0$ to $x = 10$ represents the total production over the 10-day period (see Figure 13.2a). In order to determine the accumulated production and to solve other types of problems, we need a method for finding areas under curves. That is the goal of this section.

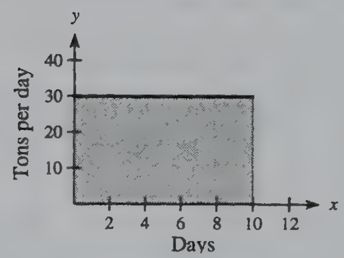


Figure 13.1

To estimate the accumulated production for the example in the Application Preview, we approximate the area under the graph of the production rate function. We can find a rough approximation of the area under this curve by fitting two rectangles to the curve as shown in Figure 13.2(b). The area of the first rectangle is $5 \cdot 100 = 500$ square units, and the area of the second rectangle is $(10 - 5)[100e^{-0.1(5)}] \approx 5(60.65) = 303.25$ square units, so this rough approximation is 803.25 square units. This approximation is clearly larger than the exact area under the curve. Why?

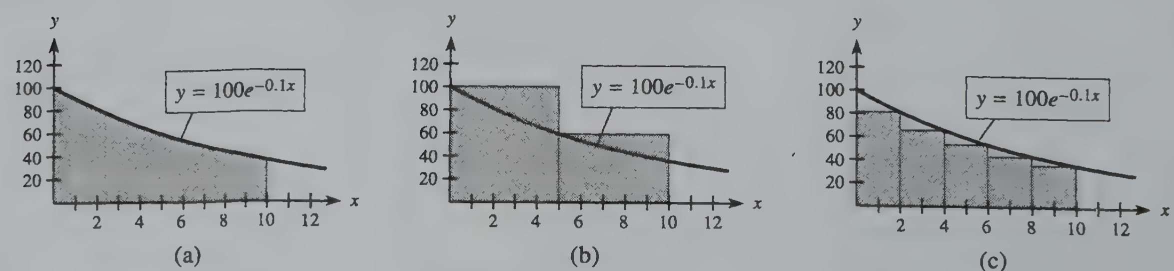


Figure 13.2

EXAMPLE 1

Fit five rectangles with equal bases inside the area under the curve $y = 100e^{-0.1x}$, and use them to approximate the area under the curve from $x = 0$ to $x = 10$ (see Figure 13.2c).

Solution

Each of the five rectangles has base 2, and the height of each rectangle is the value of the function at the right-hand endpoint of the interval forming its base. Thus the areas of the rectangles are as follows:

Rectangle	*Base*	*Height*	*Area = Base × Height*
1	2	$100e^{-0.1(2)} \approx 81.87$	$2(81.87) = 163.74$
2	2	$100e^{-0.1(4)} \approx 67.03$	$2(67.03) = 134.06$
3	2	$100e^{-0.1(6)} \approx 54.88$	$2(54.88) = 109.76$
4	2	$100e^{-0.1(8)} \approx 44.93$	$2(44.93) = 89.86$
5	2	$100e^{-0.1(10)} \approx 36.79$	$2(36.79) = 73.58$

The area under the curve is approximately equal to

$$163.74 + 134.06 + 109.76 + 89.86 + 73.58 = 571$$

The area is actually 632.12, to two decimal places, so this approximation is much better than the one we obtained with just two rectangles. In general, if we use bases of equal width, the approximation of the area under a curve improves when more rectangles are used.

Suppose that we wish to find the area between the curve $y = x$ and the x-axis from $x = 0$ to $x = 1$ (see Figure 13.3). One way to approximate this area is to use the areas of rectangles whose bases are on the x-axis and whose heights are the vertical distances from points on their bases to the curve. We can divide the interval $[0, 1]$ into n equal subintervals and use them as the bases of n rectangles whose heights are determined by the curve (see Figure 13.4). The width of each of these rectangles is $1/n$. Using the functional value at the right-hand endpoint of each subinterval as the height of the rectangle, we get n rectangles as shown in Figure 13.4. Because part of each rectangle lies above the curve, the sum of the areas of the rectangles will overestimate the area.

Then, with $y = f(x) = x$ and subinterval width $1/n$, the areas of the rectangles are as follows:

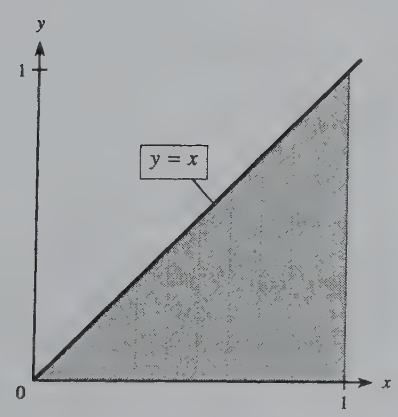


Figure 13.3

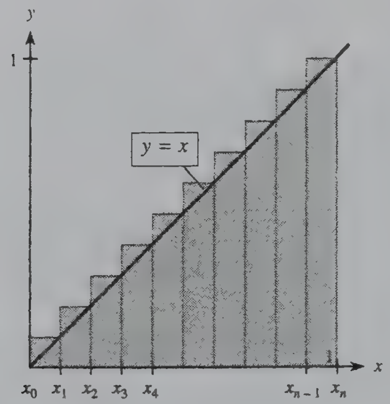


Figure 13.4

Rectangle	*Base*	*Endpoint*	*Height*	*Area = Base × Height*
1	$\frac{1}{n}$	$x_1 = \frac{1}{n}$	$f(x_1) = \frac{1}{n}$	$\frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}$
2	$\frac{1}{n}$	$x_2 = \frac{2}{n}$	$f(x_2) = \frac{2}{n}$	$\frac{1}{n} \cdot \frac{2}{n} = \frac{2}{n^2}$
3	$\frac{1}{n}$	$x_3 = \frac{3}{n}$	$f(x_3) = \frac{3}{n}$	$\frac{1}{n} \cdot \frac{3}{n} = \frac{3}{n^2}$
⋮				
i	$\frac{1}{n}$	$x_i = \frac{i}{n}$	$f(x_i) = \frac{i}{n}$	$\frac{1}{n} \cdot \frac{i}{n} = \frac{i}{n^2}$
⋮				
n	$\frac{1}{n}$	$x_n = \frac{n}{n}$	$f(x_n) = \frac{n}{n}$	$\frac{1}{n} \cdot \frac{n}{n} = \frac{n}{n^2}$

Note that i/n^2 gives the area of the ith rectangle for *any* value of i. Thus for any value of n, this area can be approximated by the sum

$$A \approx \frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \cdots + \frac{i}{n^2} + \cdots + \frac{n}{n^2}$$

In particular, we have the following approximations of this area for specific values of n (the number of rectangles).

$$n = 5: \quad A \approx \frac{1}{25} + \frac{2}{25} + \frac{3}{25} + \frac{4}{25} + \frac{5}{25} = \frac{15}{25} = 0.60$$

$$n = 10: \quad A \approx \frac{1}{100} + \frac{2}{100} + \frac{3}{100} + \cdots + \frac{10}{100} = \frac{55}{100} = 0.55$$

$$n = 100: \quad A \approx \frac{1}{10{,}000} + \frac{2}{10{,}000} + \frac{3}{10{,}000} + \cdots + \frac{100}{10{,}000}$$

$$= \frac{5050}{10{,}000} = 0.505$$

We can find this sum for any n more easily if we observe that the common denominator is n^2 and that the numerator is the sum of the first n terms of an arithmetic sequence with first term 1 and last term n. As you may recall from

Section 6.1, "Simple Interest; Sequences," the first n terms of this arithmetic sequence add to $(n + 1)/2$. Thus the area is approximated by

$$A \approx \frac{1 + 2 + 3 + \cdots + n}{n^2} = \frac{n(n + 1)/2}{n^2} = \frac{n + 1}{2n}$$

Using this formula, we see the following.

$$n = 5: \quad A \approx \frac{5 + 1}{2(5)} = \frac{6}{10} = 0.60$$

$$n = 10: \quad A \approx \frac{10 + 1}{2(10)} = \frac{11}{20} = 0.55$$

$$n = 100: \quad A \approx \frac{100 + 1}{2(100)} = \frac{101}{200} = 0.505$$

Note that as n gets larger, the number of rectangles increases, the area of each rectangle decreases, and the approximation becomes more accurate. If we let n increase without bound, the approximation approaches the exact area.

$$A = \lim_{n \to +\infty} \frac{n + 1}{2n} = \lim_{n \to +\infty} \frac{1 + 1/n}{2} = \frac{1}{2}$$

We can see that this area is correct, for we are computing the area of a triangle with base 1 and height 1. The formula for the area of a triangle gives

$$A = \frac{1}{2}bh = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

A special notation exists that uses the Greek letter Σ (sigma) to express the sum of numbers or expressions. (We used sigma notation informally in Chapter 8, "Further Topics in Probability; Statistics.") We may indicate the sum of the n numbers $a_1, a_2, a_3, a_4, \ldots, a_n$ by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

This may be read as "The sum of a_i as i goes from 1 to n." The subscript i in a_i is replaced first by 1, then by 2, then by 3, . . . , until it reaches the value above the sigma. The i is called the **index of summation,** and it starts with the lower limit, 1, and ends with the upper limit, n. For example, if $x_1 = 2, x_2 = 3, x_3 = -1$, and $x_4 = -2$, then

$$\sum_{i=1}^{4} x_i = x_1 + x_2 + x_3 + x_4 = 2 + 3 + (-1) + (-2) = 2$$

The area of the triangle under $y = x$ that we discussed above was approximated by

$$A \approx \frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \cdots + \frac{i}{n^2} + \cdots + \frac{n}{n^2}$$

Using **sigma notation,** we can write this sum as

$$A \approx \sum_{i=1}^{n} \left(\frac{i}{n^2}\right)$$

Sigma notation allows us to represent the sums of the areas of the rectangles in an abbreviated fashion. Some formulas that simplify computations involving sums follow.

Sum Formulas

I. $\sum_{i=1}^{n} 1 = n$

II. $\sum_{i=1}^{n} cx_i = c \sum_{i=1}^{n} x_i \quad (c = \text{constant})$

III. $\sum_{i=1}^{n} (x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$

IV. $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$

V. $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$

We have found that the area of the triangle discussed above was approximated by

$$A \approx \sum_{i=1}^{n} \frac{i}{n^2}$$

We can use these formulas to simplify this sum as follows.

$$\begin{aligned}\sum_{i=1}^{n} \frac{i}{n^2} &= \frac{1}{n^2} \sum_{i=1}^{n} i && \text{(Formula II)}\\ &= \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right] && \text{(Formula IV)}\\ &= \frac{n+1}{2n}\end{aligned}$$

Note that this is the same formula we obtained previously using other methods.

The following example shows that we can find the area by evaluating the function at the left-hand endpoints of the subintervals.

EXAMPLE 2

Use rectangles to find the area under $y = x^2$ (and above the x-axis) from $x = 0$ to $x = 1$.

Solution

We again divide the interval [0, 1] into n equal subintervals of length $1/n$. If we evaluate the function at the left-hand endpoints of these subintervals to determine the heights of the rectangles, the sum of the areas of the rectangles will underestimate the area (see Figure 13.5). Thus we have the following:

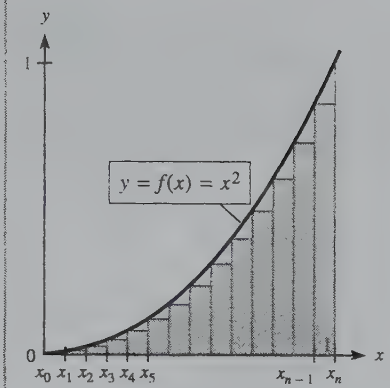


Figure 13.5

Rectangle	*Base*	*Endpoint*	*Height*	*Area = Base × Height*
1	$\frac{1}{n}$	$x_0 = 0$	$f(x_0) = 0$	$\frac{1}{n} \cdot 0 = 0$
2	$\frac{1}{n}$	$x_1 = \frac{1}{n}$	$f(x_1) = \frac{1}{n^2}$	$\frac{1}{n} \cdot \frac{1}{n^2} = \frac{1}{n^3}$
3	$\frac{1}{n}$	$x_2 = \frac{2}{n}$	$f(x_2) = \frac{4}{n^2}$	$\frac{1}{n} \cdot \frac{4}{n^2} = \frac{4}{n^3}$
4	$\frac{1}{n}$	$x_3 = \frac{3}{n}$	$f(x_3) = \frac{9}{n^2}$	$\frac{1}{n} \cdot \frac{9}{n^2} = \frac{9}{n^3}$
$\vdots$				
i	$\frac{1}{n}$	$x_{i-1} = \frac{i-1}{n}$	$\frac{(i-1)^2}{n}$	$\frac{(i-1)^2}{n^3}$
$\vdots$				
n	$\frac{1}{n}$	$x_{n-1} = \frac{n-1}{n}$	$\frac{(n-1)^2}{n^2}$	$\frac{(n-1)^2}{n^3}$

Note that $(i - 1)^2/n^3 = (i^2 - 2i + 1)/n^3$ gives the area of the ith rectangle for *any* value of i. The sum of these areas may be written as

$$S = \sum_{i=1}^{n} \frac{i^2 - 2i + 1}{n^3} = \frac{1}{n^3}\left(\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \right) \qquad \text{(Formulas II and III)}$$

$$= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - \frac{2n(n+1)}{2} + n \right] \qquad \text{(Formulas V, IV, and I)}$$

$$= \frac{2n^3 + 3n^2 + n}{6n^3} - \frac{n^2 + n}{n^3} + \frac{n}{n^3} = \frac{2n^2 - 3n + 1}{6n^2}$$

If n is large, there will be a large number of smaller rectangles, and the approximation of the area under the curve will be good; the larger n, the better the approximation. For example, if $n = 10$, the area approximation is

$$S(10) = \frac{200 - 30 + 1}{600} = 0.285$$

whereas if $n = 100$,

$$S(100) = \frac{20{,}000 - 300 + 1}{60{,}000} = 0.328$$

If we let n increase without bound, we find the exact area.

$$A = \lim_{n\to\infty}\left(\frac{2n^2 - 3n + 1}{6n^2}\right) = \lim_{n\to\infty}\left(\frac{2 - \frac{3}{n} + \frac{1}{n^2}}{6}\right) = \frac{1}{3}$$

Note that the approximations with $n = 10$ and $n = 100$ were less than $\frac{1}{3}$. This is because all the rectangles were *under* the curve (see Figure 13.5 on the previous page).

Thus we see that we can determine the area under a curve $y = f(x)$ from $x = a$ to $x = b$ by dividing the interval $[a, b]$ into n equal subintervals of width $(b - a)/n$ and evaluating

$$A = \lim_{n\to\infty} S_R = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right) \qquad \text{(using right-hand endpoints)}$$

or

$$A = \lim_{n\to\infty} S_L = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\left(\frac{b-a}{n}\right) \qquad \text{(using left-hand endpoints)}$$

CHECKPOINT

1. For the interval [0, 2], determine whether the following are true or false.
   (a) For 4 subintervals, each subinterval has width $\frac{1}{2}$.
   (b) For 200 subintervals, each subinterval has width $\frac{1}{100}$.
   (c) For n subintervals, each subinterval has width $\frac{2}{n}$.
   (d) For n subintervals, $x_0 = 0, x_1 = \frac{2}{n}, x_2 = 2\left(\frac{2}{n}\right), \dots, x_i = i\left(\frac{2}{n}\right), \dots, x_n = 2$.
2. If $\frac{b-a}{n} = \frac{2}{n}$, $x_i = \frac{2i}{n}$, and $f(x) = 3x - x^2$, find:
   (a) $f(x_i)$
   (b) $f(x_i)\frac{b-a}{n}$
   (c) $\sum_{i=1}^{n} f(x_i)\frac{b-a}{n}$, and simplify
   (d) $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\frac{b-a}{n}$

EXAMPLE 3

To approximate the area under the graph of $y = \sqrt{x}$ on the interval [0, 4], do the following.

(a) Use a calculator or computer to find S_L and S_R for $n = 8$ on the interval [0, 4].
(b) Predict the area under the curve in the graph by evaluating S_L and S_R for larger values of n.

Solution

Figure 13.6(a) shows the graph of $y = \sqrt{x}$ with 8 rectangles whose heights are determined by evaluating the function at the left-hand endpoint of each interval (the first of these rectangles has height 0). Figure 13.6(b) shows the same graph with 8 rectangles whose heights are determined at the right-hand endpoints.

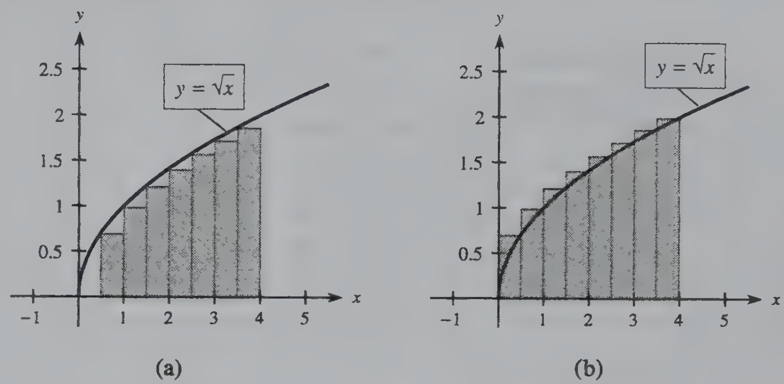


Figure 13.6

(a) The formula for the sum of the areas of n rectangles over the interval [0, 4], using left-hand endpoints, is

$$S_L = \sum_{i=1}^{n} \sqrt{\frac{4(i-1)}{n}}\,\frac{4}{n} = \frac{8}{n^{3/2}} \sum_{i=1}^{n} \sqrt{i-1}$$

and the formula for the sum of the areas of n rectangles over the interval [0, 4], using right-hand endpoints, is

$$S_R = \sum_{i=1}^{n} \sqrt{\frac{4(i)}{n}}\,\frac{4}{n} = \frac{8}{n^{3/2}} \sum_{i=1}^{n} \sqrt{i}$$

We have no formula to write either of these summations in a simpler form, but we can use a graphing calculator, computer program, or spreadsheet to compute the sum for any given n. For $n = 8$, the sums are

$$S_L = \frac{1}{\sqrt{8}} \sum_{i=1}^{8} \sqrt{i-1} \qquad \text{and} \qquad S_R = \frac{1}{\sqrt{8}} \sum_{i=1}^{8} \sqrt{i}$$

The following spreadsheet output shows the sum for $n = 8$ rectangles.

	A	B	C	D
1	Area for	n=8		
2		i	SL	SR
3		1	0	.35355
4		2	.35355	.5
5		3	.5	.61237
6		4	.61237	.70711
7		5	.70711	.79057
8		6	.79057	.86603
9		7	.86603	.93541
10		8	.93541	1
11	Total		4.76504	5.76504

As line 11 shows, $S_L = 4.76504$ and $S_R = 5.76504$ for $n = 8$.

(b) Larger values of n give better approximations of the area under the curve (try some). For example, $n = 1000$ gives $S_L = 5.3293$ and $S_R = 5.3373$, and because the area under the curve is between these values, the area is approximately 5.33, to two decimal places. By using values of n larger than 1000, we can get better approximations of the area.

CHECKPOINT SOLUTIONS

1. All parts are true.

2. (a) $f(x_i) = f\left(\frac{2i}{n}\right) = 3\left(\frac{2i}{n}\right) - \left(\frac{2i}{n}\right)^2 = \frac{6i}{n} - \frac{4i^2}{n^2}$

(b) $f(x_i)\frac{b-a}{n} = \left(\frac{6i}{n} - \frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right) = \frac{12i}{n^2} - \frac{8i^2}{n^3}$

(c)
$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\frac{b-a}{n} &= \sum_{i=1}^{n}\left(\frac{12i}{n^2} - \frac{8i^2}{n^3}\right)\\ &= \sum_{i=1}^{n}\frac{12i}{n^2} - \sum_{i=1}^{n}\frac{8i^2}{n^3}\\ &= \frac{12}{n^2}\sum_{i=1}^{n} i - \frac{8}{n^3}\sum_{i=1}^{n} i^2\\ &= \frac{12}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\\ &= \frac{6(n+1)}{n} - \frac{4(n+1)(2n+1)}{3n^2}\end{aligned}$$

(d) $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\frac{b-a}{n} = \lim_{n\to\infty}\left(\frac{6n+6}{n} - \frac{8n^2+12n+4}{3n^2}\right) = 6 - \frac{8}{3} = \frac{10}{3}$

EXERCISE 13.1

In Problems 1–4, approximate the area under each curve over the interval specified by using the indicated number of subintervals (or rectangles) and evaluating the function at the *right-hand* endpoints of the subintervals.

1. $f(x) = 4x - x^2$ from $x = 0$ to $x = 2$; 2 subintervals
2. $f(x) = x^3$ from $x = 0$ to $x = 3$; 3 subintervals
3. $f(x) = 9 - x^2$ from $x = -1$ to $x = 3$; 4 subintervals
4. $f(x) = x^2 + x + 1$ from $x = -1$ to $x = 1$; 4 subintervals

In Problems 5–8, approximate the area under each curve by evaluating the function at the *left-hand* endpoints of the subintervals.

5. $f(x) = 4x - x^2$ from $x = 0$ to $x = 2$; 2 subintervals
6. $f(x) = x^3$ from $x = 0$ to $x = 3$; 3 subintervals
7. $f(x) = 9 - x^2$ from $x = -1$ to $x = 3$; 4 subintervals
8. $f(x) = x^2 + x + 1$ from $x = -1$ to $x = 1$; 4 subintervals

When the area under $f(x) = x^2 + x$ from $x = 0$ to $x = 2$ is approximated, the formulas for the sum of n rectangles using *left-hand* endpoints and *right-hand* endpoints are

Left-hand endpoints: $S_L = \frac{14}{3} - \frac{6}{n} + \frac{4}{3n^2}$

Right-hand endpoints: $S_R = \frac{14n^2 + 18n + 4}{3n^2}$

Use these formulas to answer Problems 9–13.

9. Find $S_L(10)$ and $S_R(10)$.
10. Find $S_L(100)$ and $S_R(100)$.
11. Find $\lim_{n\to\infty} S_L$ and $\lim_{n\to\infty} S_R$.
12. Compare the right-hand and left-hand values by finding $S_R - S_L$ for $n = 10$, for $n = 100$, and as $n \to \infty$. (Use Problems 9–11.)
13. Because $f(x) = x^2 + x$ is increasing over the interval from $x = 0$ to $x = 2$, functional values at the right-

hand endpoints are maximum values for each subinterval, and functional values at the left-hand endpoints are minimum values for each subinterval. How would the approximate area using $n = 10$ and *any* other point within each subinterval compare with $S_L(10)$ and $S_R(10)$? What would happen to the area result as $n \to \infty$ if any other point in each subinterval were used?

In Problems 14–23, find the value of each sum.

14. $\sum_{k=1}^{3} x_k$, if $x_1 = 1, x_2 = 3, x_3 = -1, x_4 = 5$

15. $\sum_{i=1}^{4} x_i$, if $x_1 = 3, x_2 = -1, x_3 = 3, x_4 = -2$

16. $\sum_{i=3}^{5} (i^2 + 1)$

17. $\sum_{j=2}^{5} (j^2 - 3)$

18. $\sum_{i=4}^{7} \left(\frac{i-3}{i^2}\right)$

19. $\sum_{j=0}^{4} (j^2 - 4j + 1)$

20. $\sum_{k=1}^{50} 1$

21. $\sum_{j=1}^{60} 3$

22. $\sum_{k=1}^{50} (6k^2 + 5)$

23. $\sum_{k=1}^{30} (k^2 + 4k)$

In Problems 24 and 25, use the Sum Formulas I–V to express each of the following without the summation symbol.

24. $\sum_{i=1}^{n} \left(1 - \frac{i^2}{n^2}\right)\left(\frac{2}{n}\right)$

25. $\sum_{i=1}^{n} \left(1 - \frac{2i}{n} + \frac{i^2}{n^2}\right)\left(\frac{3}{n}\right)$

Use the function $y = x$ from $x = 0$ to $x = 1$ and n equal subintervals with the function evaluated at the *left-hand* endpoint of each subinterval for Problems 26–27.

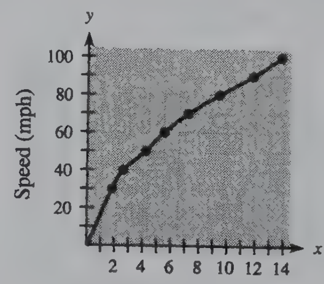
26. What is the area of the
    (a) first rectangle? (b) second rectangle?
    (c) ith rectangle?
27. (a) Find a formula for the sum of the areas of the n rectangles (call this S). Then find
    (b) $S(10)$. (c) $S(100)$. (d) $S(1000)$. (e) $\lim_{n\to\infty} S$.
28. How do your answers to Problems 27(a)–(e) compare with the corresponding calculations in the discussion (after Example 1) of the area under $y = x$ using *right-hand* endpoints?

For Problems 29(a)–(e), use the function $y = x^2$ from $x = 0$ to $x = 1$ and n equal subintervals with the function evaluated at the *right-hand* endpoints.

29. (a) Find a formula for the sum of the areas of the n rectangles (call this S). Then find
    (b) $S(10)$. (c) $S(100)$. (d) $S(1000)$. (e) $\lim_{n\to\infty} S$.
30. How do your answers to Problems 29(a)–(e) compare with the corresponding calculations in Example 2?
31. Use rectangles to find the area between $y = x^2 - 6x + 8$ and the x-axis from $x = 0$ to $x = 2$. Divide the interval $[0, 2]$ into n equal subintervals so that each subinterval has length $2/n$.
32. Use rectangles to find the area between $y = 4x - x^2$ and the x-axis from $x = 0$ to $x = 4$. Divide the interval $[0, 4]$ into n equal subintervals so that each subinterval has length $4/n$.

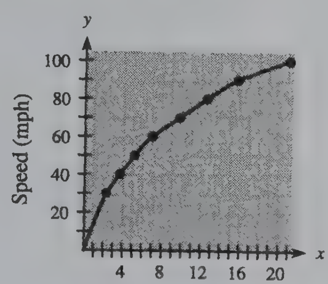
Applications

33. ***Speed trials*** The graph in the figure gives the times that it takes a 1994 Porsche 911 to reach speeds from 0 mph to 100 mph, in increments of 10 mph, with a curve connecting them. The area under this curve from $t = 0$ seconds to $t = 14$ seconds represents the total amount of distance traveled over the 14-second period. Count the squares under the curve to estimate this distance. This car will travel 1/4 mile in 14 seconds, to a speed of 100.2 mph. Is your estimate close to this result? (Be careful with time units.)



Source: *Motor Trend*, January 1994

34. ***Speed trials*** The graph in the figure gives the times that it takes a 1995 Mitsubishi Eclipse GSX to reach speeds from 0 mph to 100 mph, in increments of 10 mph, with a curve connecting them. The area under this curve from $t = 0$ seconds to $t = 21.1$ seconds represents the total amount of distance traveled over the 21.1-second period. Count the squares under the curve to estimate this distance. This car will travel 1/4 mile in 15.4 seconds, to a speed of 89.0 mph, so your estimate should be more than 1/4 mile. Is it? (Be careful with time units.)

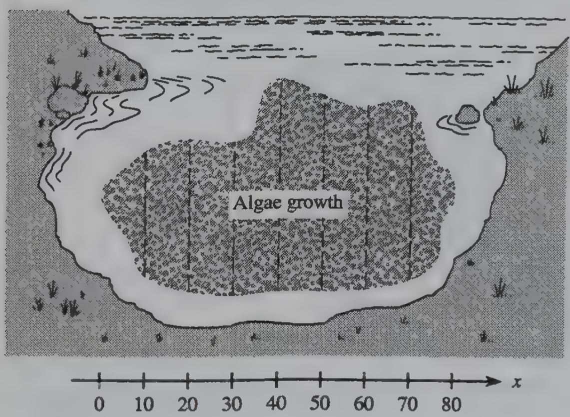


Source: *Road & Track*, June 1994

35. ***Pollution monitoring*** Suppose the presence of phosphates in certain waste products dumped into a lake promotes the growth of algae. Rampant growth of algae affects the oxygen supply in the water, so an environmental group wishes to estimate the area of algae growth. The group measures the length across the algae growth (see the figure) and obtains the following data (in feet).

x	*Length*	x	*Length*
0	0	50	27
10	15	60	24
20	18	70	23
30	18	80	0
40	30		

Use 8 rectangles with bases of 10 feet and lengths measured at the left-hand endpoints to approximate the area of the algae growth.



36. ***Drug levels in the blood*** The manufacturer of a medicine wants to test how a new 300-milligram capsule is released into the bloodstream. After a volunteer is given a capsule, blood samples are drawn every half-hour, and the number of milligrams of the drug in the bloodstream is calculated. The results obtained follow.

Time t (*hr*)	$N(t)$ (*mg*)	*Time t* (*hr*)	$N(t)$ (*mg*)
0	0	2.0	178.3
0.5	247.3	2.5	113.9
1.0	270	3.0	56.2
1.5	236.4	3.5	19.3

Use 7 rectangles, each with height $N(t)$ and with width 0.5 hr, to estimate the area under the graph representing these data. Divide this area by 3.5 hr to estimate the average drug level over this time period.

13.2 The Definite Integral; The Fundamental Theorem of Calculus

OBJECTIVES

- *To evaluate definite integrals using the Fundamental Theorem of Calculus*
- *To use definite integrals to find the area under a curve*

APPLICATION PREVIEW

Suppose that money flows continuously into a slot machine at a casino and grows at a rate given by

$$A'(t) = 100e^{0.1t}$$

where t is the time in hours and $0 \le t \le 10$. Then the total amount of money that accumulates over the 10-hour period, if no money is paid out, is given by the **definite integral**

$$\int_0^{10} 100e^{0.1t}\,dt$$

In the previous section, we used the sum of areas of rectangles to approximate the area under curves. In this section, we will see how this sum is related to the definite integral and how to evaluate definite integrals. In addition, we will see how definite integrals will be used to solve several types of applied problems.

In the previous section, we saw that we could determine the area under a curve using equal subintervals and the functional values at either the left-hand endpoints or the right-hand endpoints of the subintervals. In fact, we can use subintervals that are not of equal length, and we can use any point within each subinterval to determine the height of each rectangle. Suppose that we wish to find the area above the x-axis and under the curve $y = f(x)$ over a closed interval $[a, b]$. We can divide the interval into n subintervals (not necessarily equal), with the endpoints of these intervals at $x_0 = a, x_1, x_2 \ldots, x_n = b$. We now choose a point (*any* point) in each subinterval, and denote the points $x_1^*, x_2^*, \ldots, x_i^*, \ldots, x_n^*$. Then the ith rectangle (for any i) has height $f(x_i^*)$ and width $x_i - x_{i-1}$, so its area is $f(x_i^*)(x_i - x_{i-1})$. Then the sum of the areas of the n rectangles is

$$
\begin{aligned}
S &= \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} f(x_i^*)\Delta x_i, \qquad \text{where } \Delta x_i = x_i - x_{i-1}
\end{aligned}
$$

Because the points in the subinterval may be chosen anywhere in the subinterval, we cannot be sure whether the rectangles will underestimate or overestimate the area under the curve. But increasing the number of subintervals (increasing n) and making sure that every interval becomes smaller (just increasing n will not guarantee this if the subintervals are unequal) will in the long run improve the estimation. Thus for any subdivision of $[a, b]$ and any x_i^*, the area is given by

$$A = \lim_{\substack{n\to\infty \\ \max \Delta x_i \to 0}} \sum_{i=1}^{n} f(x_i^*)\Delta x_i, \qquad \text{provided that this limit exists}$$

The preceding discussion takes place in the context of finding the area under a curve. However, if f is any function (not necessarily nonnegative) defined on $[a, b]$, then for each subdivision of $[a, b]$ and each choice of x_i^*, we define the sum S above as the **Riemann sum** of f for the subdivision of $[a, b]$. In addition, the limit of the Riemann sum (as max $\Delta x_i \to 0$) has other important applications and is called the **definite integral** of $f(x)$ over interval $[a, b]$.

Definite Integral If f is a function on the interval $[a, b]$, then the *definite integral* of f from a to b is

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \max \Delta x_i \to 0}} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

If the limit exists, then the definite integral exists and we say that f is integrable on $[a, b]$.

The obvious question is how this definite integral is related to the indefinite integral (antiderivative) we have been studying. The answer to this question is given by the **Fundamental Theorem of Calculus.**

Fundamental Theorem of Calculus

Let f be a continuous function on the closed interval $[a, b]$; then the definite integral of f exists on this interval, and

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any function such that $F'(x) = f(x)$ for all x in $[a, b]$.

Stated differently, the theorem says that if the function F is an indefinite integral of a function f that is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

We denote $F(b) - F(a)$ by $F(x)\Big|_a^b$.

EXAMPLE 1

Evaluate $\int_2^4 (x^3 + 4)\,dx$.

Solution

$$\begin{aligned}\int_2^4 (x^3 + 4)\,dx &= \frac{x^4}{4} + 4x + C\Big|_2^4 \\ &= \left[\frac{(4)^4}{4} + 4(4) + C\right] - \left[\frac{(2)^4}{4} + 4(2) + C\right] \\ &= (64 + 16 + C) - (4 + 8 + C) \\ &= 68 \qquad \text{(Note that the } C\text{'s subtract out.)}\end{aligned}$$

Note that the Fundamental Theorem states that F can be *any* indefinite integral of f, so we need not add the constant of integration to the integral.

EXAMPLE 2

Evaluate $\int_1^3 (3x^2 + 6x)\,dx$.

Solution

$$\begin{aligned}\int_1^3 (3x^2 + 6x)\,dx &= x^3 + 3x^2\Big|_1^3 \\ &= (3^3 + 3 \cdot 3^2) - (1^3 + 3 \cdot 1^2) \\ &= 54 - 4 = 50\end{aligned}$$

The properties of definite integrals given below follow from properties of summations.

1. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
2. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx,$ where k is a constant

The following example uses both of these properties.

EXAMPLE 3

Evaluate $\displaystyle\int_3^5 (\sqrt{x^2-9}+2)x\,dx.$

Solution

$$\begin{aligned}\int_3^5 (\sqrt{x^2-9}+2)x\,dx &= \int_3^5 \sqrt{x^2-9}(x\,dx) + \int_3^5 2x\,dx\\ &= \frac{1}{2}\int_3^5 (x^2-9)^{1/2}(2x\,dx) + \int_3^5 2x\,dx\\ &= \frac{1}{2}\left[\frac{2}{3}(x^2-9)^{3/2}\right]\Bigg|_3^5 + x^2\Bigg|_3^5\\ &= \frac{1}{3}[(16)^{3/2}-(0)^{3/2}] + (25-9)\\ &= \frac{64}{3}+16 = \frac{64}{3}+\frac{48}{3} = \frac{112}{3}\end{aligned}$$

In the integral $\int_a^b f(x)\,dx$, we call a the *lower limit* and b the *upper limit* of integration. Although we developed the definite integral with the assumption that the lower limit was less than the upper limit, the following properties permit us to evaluate the definite integral even when that is not the case.

3. $\displaystyle\int_a^a f(x)\,dx = 0$
4. If f is integrable on $[a, b]$, then

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

The following examples illustrate these properties.

EXAMPLE 4

Evaluate $\displaystyle\int_4^4 x^2\,dx.$

Solution

$$\int_4^4 x^2\,dx = \frac{x^3}{3}\Bigg|_4^4 = \frac{4^3}{3}-\frac{4^3}{3} = 0$$

Note that because the limits of integration are the same, it is not necessary to integrate to see that the value of the integral is 0.

EXAMPLE 5

Compare $\int_2^4 3x^2\,dx$ and $\int_4^2 3x^2\,dx$.

Solution

$$\int_2^4 3x^2\,dx = x^3\Big|_2^4 = 4^3 - 2^3 = 56$$
$$\int_4^2 3x^2\,dx = x^3\Big|_4^2 = 2^3 - 4^3 = -56$$

Thus

$$\int_4^2 3x^2\,dx = -\int_2^4 3x^2\,dx$$

Another property of definite integrals is called the additive property.

5. If f is continuous on some interval containing a, b, and c,* then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

EXAMPLE 6

Show that $\int_2^3 4x\,dx + \int_3^5 4x\,dx = \int_2^5 4x\,dx$.

Solution

$$\int_2^3 4x\,dx = 2x^2\Big|_2^3 = 18 - 8 = 10$$
$$\int_3^5 4x\,dx = 2x^2\Big|_3^5 = 50 - 18 = 32$$
$$\int_2^5 4x\,dx = 2x^2\Big|_2^5 = 50 - 8 = 42$$

Thus

$$\int_2^3 4x\,dx + \int_3^5 4x\,dx = \int_2^5 4x\,dx$$

Let us now return to area problems, to see the relationship between the definite integral and the area under a curve. By the formula for the area of a triangle and by summing areas of rectangles, we found the area under the curve (line) $y = x$ from $x = 0$ to $x = 1$ to be $\frac{1}{2}$ (see Figure 13.7a). Using the definite integral to find the area gives

*Note that c need not be between a and b.

$$A = \int_0^1 x\,dx = \frac{x^2}{2}\bigg|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

In Example 2 of the previous section, we used rectangles to find that the area under $y = x^2$ from $x = 0$ to $x = 1$ was $\frac{1}{3}$ (see Figure 13.7b). Using the definite integral, we get

$$A = \int_0^1 x^2\,dx = \frac{x^3}{3}\bigg|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

which agrees with the answer obtained in Example 2.

However, not every definite integral represents the area between the curve and the x-axis over an integral. For example,

$$\int_0^2 (x-2)\,dx = \frac{x^2}{2} - 2x\bigg|_0^2 = (2-4) - (0) = -2$$

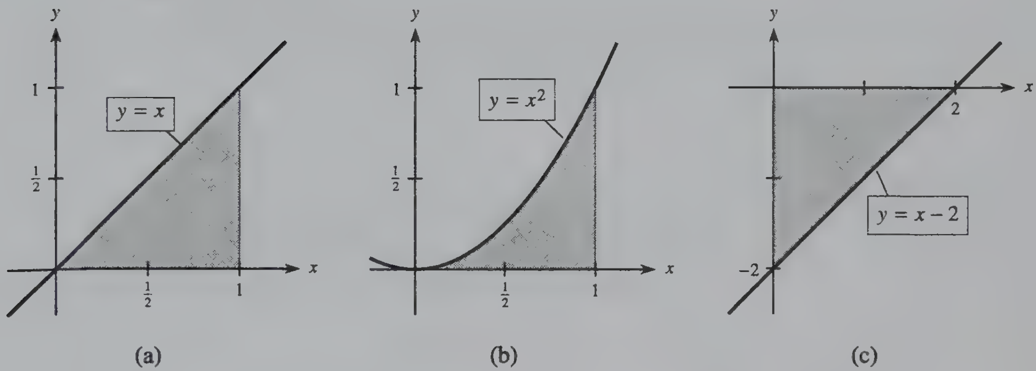


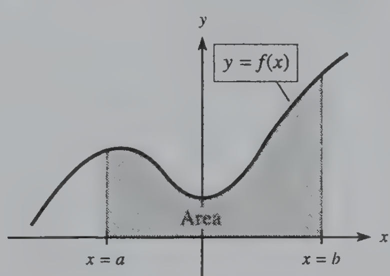
Figure 13.7

This would indicate that the area between the curve and the x-axis is negative, but area must be positive. A look at the graph of $y = x - 2$ (see Figure 13.7c) shows us what is happening. The region bounded by $y = x - 2$ and x-axis between $x = 0$ and $x = 2$ is a triangle whose base is 2 and height is 2, so its area is $\frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$. The integral has value -2 because $y = x - 2$ lies below the x-axis from $x = 0$ to $x = 2$, and the functional values over the interval $[0, 2]$ are negative. Thus the value of the definite integral over *this* interval does not represent the area between the curve and the x-axis.

In general, the definite integral will give the area under the curve and above the x-axis only when $f(x) \geq 0$ for all x in $[a, b]$.

Area Under a Curve If f is a continuous function on $[a, b]$ and $f(x) \geq 0$ on $[a, b]$, then the exact area between $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is given by

$$\begin{matrix}\text{Area}\\ \text{(shaded)}\end{matrix} = \int_a^b f(x)\,dx$$



CHECKPOINT

1. True or false:
   (a) For any integral, we can omit the constant of integration (the $+C$).
   (b) $-\int_{-1}^{3} f(x)\,dx = \int_{3}^{-1} f(x)\,dx$, if f is integrable on $[-1, 3]$.
   (c) The area between $f(x)$ and the x-axis on the interval $[a, b]$ is given by $\int_a^b f(x)\,dx$.
2. Evaluate:
   (a) $\int_0^3 (x^2+1)\,dx$
   (b) $\int_0^3 (x^2+1)^4 x\,dx$

If the rate of growth of some function with respect to time t is $f'(t)$, then the total growth of the function during the period from $t = 0$ to $t = k$ can be found by evaluating the definite integral

$$\int_0^k f'(t)\,dt = f(t)\Big|_0^k = f(k) - f(0)$$

For nonnegative rates of growth, this definite integral (and thus growth) is the same as the area under the graph of $f'(t)$ from $t = 0$ to $t = k$.

We now return to the problem introduced in the Application Preview.

EXAMPLE 7

Suppose that money flows continuously into a slot machine at a casino and grows at a rate given by

$$A'(t) = 100e^{0.1t}$$

where t is time in hours and $0 \le t \le 10$. Find the total amount that accumulates in the machine during the 10-hour period, if no money is paid out.

Solution

The total amount is given by

$$\begin{aligned} A = \int_0^{10} 100e^{0.1t}\,dt &= \frac{100}{0.1}\int_0^{10} e^{0.1t}(0.1)\,dt \\ &= 1000e^{0.1t}\Big|_0^{10} \\ &= 1000e - 1000 \\ &\approx 1718.28 \quad \text{(dollars)} \end{aligned}$$

In Section 8.5, "Normal Probability Distribution," we stated that the total area under the normal curve is 1 and that the area under the curve from value x_1 to value x_2 represents the probability that a score chosen at random will lie between x_1 and x_2.

The normal distribution is an example of a **continuous distribution** because the values of the random variable are considered over intervals rather than at discrete values. The above statements relating probability and area under the graph apply to other continuous probability distributions determined by **probability density functions.** Thus we can use the definite integral to find the probability that a random variable lies within a given interval. Consider the following example.

EXAMPLE 8

Suppose the probability density function for the life of a computer component is $f(x) = 0.10e^{-0.10x}$, where $x \geq 0$ is the number of years the component is in use. Find the probability that the component will last between 3 and 5 years.

Solution

The probability that the component will last between 3 and 5 years is the area under the graph of the function between $x = 3$ and $x = 5$. The probability is given by the integral

$$\begin{aligned}\int_3^5 0.10e^{-0.10x}\,dx &= -e^{-0.10x}\Big|_3^5 \\ &= -e^{-0.5} + e^{-0.3} \\ &\approx -0.6065 + 0.7408 \\ &= 0.1343\end{aligned}$$

Graphing Utilities

Most graphing calculators and graphing utilities have a numerical integration feature that can be used to get very accurate approximations of definite integrals. This feature can be used to evaluate definite integrals directly or to check those done with the Fundamental Theorem. Figure 13.8 shows the numerical integration feature applied to the integral in Example 8. Note that when this answer is rounded to four decimal places, the results agree.

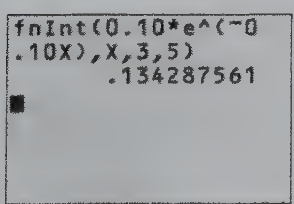


Figure 13.8

CHECKPOINT SOLUTIONS

1. (a) False. We can omit the constant of integration ($+C$) only for definite integrals.
   (b) True
   (c) False. Only if $f(x) \geq 0$ on $[a, b]$ is this true.
2. (a) $\displaystyle\int_0^3 (x^2+1)\,dx = \frac{x^3}{3} + x\Big|_0^3 = \left(\frac{27}{3}+3\right) - (0+0) = 12$

   (b) $$\begin{aligned}\int_0^3 (x^2+1)^4 x\,dx &= \frac{1}{2}\int_0^3 (x^2+1)^4(2x\,dx)\\ &= \frac{1}{2}\cdot\frac{(x^2+1)^5}{5}\Big|_0^3\\ &= \frac{1}{10}[(3^2+1)^5 - (1)^5]\\ &= \frac{1}{10}(10^5 - 1) = 9999.9\end{aligned}$$

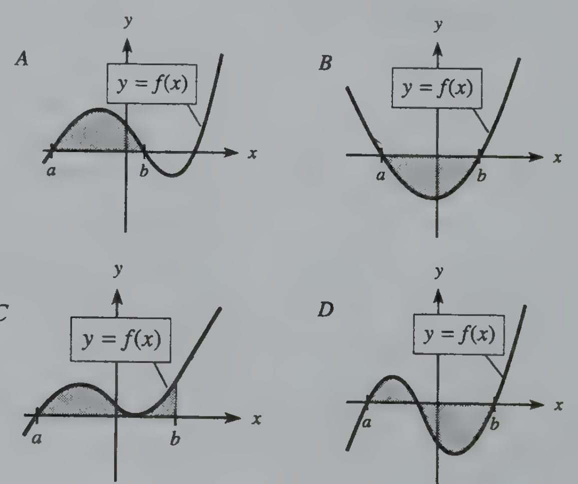
EXERCISE 13.2

Evaluate the definite integrals in Problems 1–26.

1. $\displaystyle\int_0^3 4x\,dx$
2. $\displaystyle\int_0^1 8x\,dx$
3. $\displaystyle\int_2^4 dx$
4. $\displaystyle\int_1^5 2\,dy$
5. $\displaystyle\int_2^4 x^3\,dx$
6. $\displaystyle\int_0^5 x^2\,dx$
7. $\displaystyle\int_0^5 4\sqrt[3]{x^2}\,dx$
8. $\displaystyle\int_2^4 3\sqrt{x}\,dx$
9. $\displaystyle\int_2^4 (4x^3 - 6x^2 - 5x)\,dx$
10. $\displaystyle\int_0^2 (x^4 - 5x^3 + 2x)\,dx$
11. $\displaystyle\int_2^3 (x-4)^2\,dx$
12. $\displaystyle\int_{-1}^3 (x+2)^3\,dx$
13. $\displaystyle\int_2^4 (x^2+2)^3\,x\,dx$
14. $\displaystyle\int_0^3 (2x - x^2)^4(1-x)\,dx$
15. $\displaystyle\int_{-1}^2 (x^3 - 3x^2)^3(x^2 - 2x)\,dx$
16. $\displaystyle\int_0^4 (3x^2 - 2)^4\,x\,dx$
17. $\displaystyle\int_2^3 x\sqrt{x^2+3}\,dx$
18. $\displaystyle\int_{-1}^2 x\sqrt[3]{x^2-5}\,dx$
19. $\displaystyle\int_1^3 \frac{3}{y^2}\,dy$
20. $\displaystyle\int_1^2 \frac{5}{z^3}\,dz$
21. $\displaystyle\int_0^1 e^{3x}\,dx$
22. $\displaystyle\int_0^2 e^{4x}\,dx$
23. $\displaystyle\int_1^e \frac{4}{z}\,dz$
24. $\displaystyle\int_1^e 3y^{-1}\,dy$
25. $\displaystyle\int_4^4 \sqrt{x^2-2}\,dx$
26. $\displaystyle\int_2^2 (x^3 - 4x)\,dx$

In Problems 27–30, evaluate each integral (a) with the Fundamental Theorem and (b) with a graphing utility (as a check).

27. $\displaystyle\int_3^6 \frac{x}{3x^2+4}\,dx$
28. $\displaystyle\int_0^2 \frac{x}{x^2+4}\,dx$
29. $\displaystyle\int_1^2 \frac{x^2+3}{x}\,dx$
30. $\displaystyle\int_1^4 \frac{4\sqrt{x}+5}{\sqrt{x}}\,dx$
31. In the figure, which of the shaded regions (A, B, C, or D) has the area given by
    (a) $\displaystyle\int_a^b f(x)\,dx$? (b) $\displaystyle-\int_a^b f(x)\,dx$?

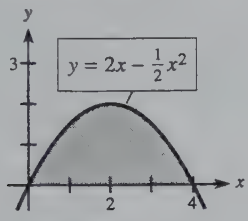


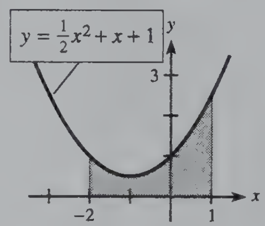
32. For which of the following functions $f(x)$ does

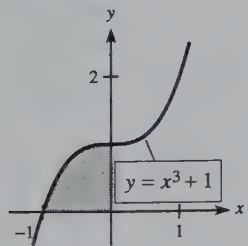
$$\int_0^2 f(x)\,dx$$

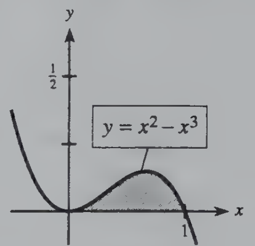
give the area between the graph of $f(x)$ and the x-axis from $x = 0$ to $x = 2$?
(a) $f(x) = x^2 + 1$
(b) $f(x) = -x^2$
(c) $f(x) = x - 1$

In Problems 33–36,
(a) write the integral that describes the area of the shaded region.
(b) find the area.

33.


34.


35.


36.


37. Find the area between the curve $y = -x^2 + 3x - 2$ and the x-axis from $x = 1$ to $x = 2$.
38. Find the area between the curve $y = x^2 + 3x + 2$ and the x-axis from $x = -1$ to $x = 3$.
39. Find the area between the curve $y = xe^{x^2}$ and the x-axis from $x = 1$ to $x = 3$.
40. Find the area between the curve $y = e^{-x}$ and the x-axis from $x = -1$ to $x = 1$.
41. How does $\int_{-1}^{-3} x\sqrt{x^2+1}\,dx$ compare with $\int_{-3}^{-1} x\sqrt{x^2+1}\,dx$?
42. If $\int_{-1}^{0} x^3\,dx = -\frac{1}{4}$ and $\int_0^1 x^3\,dx = \frac{1}{4}$, what does $\int_{-1}^{1} x^3\,dx$ equal?
43. If $\int_1^2 (2x - x^2)\,dx = \frac{2}{3}$ and $\int_2^4 (2x - x^2)\,dx = -\frac{20}{3}$, what does $\int_1^4 (x^2 - 2x)\,dx$ equal?
44. If $\int_1^2 (2x - x^2)\,dx = \frac{2}{3}$, what does $\int_1^2 6(2x - x^2)\,dx$ equal?

Applications

45. ***Total income*** The income from an oil change service chain can be considered as flowing continuously at an annual rate given by

$$f(t) = 10{,}000e^{0.02t} \qquad \text{(dollars/year)}$$

Find the total income for this chain over the first 2 years (from $t = 0$ to $t = 2$).

46. ***Total income*** Suppose that a vending machine service company models its income by assuming that money flows continuously into the machines, with the annual rate of flow given by

$$f(t) = 120e^{0.01t}$$

in thousands of dollars. Find the total income from the machines over the first 3 years.

Velocity of blood In Problems 47 and 48, the velocity of blood through a vessel is given by $v = K(R^2 - r^2)$, where K is the (constant) maximum velocity of the blood, R the (constant) radius of the vessel, and r the distance of the particular corpuscle from the center of the vessel. The rate of flow can be found by measuring the volume of blood that flows past a point in a given time period. This volume, V, is given by

$$V = \int_0^R v(2\pi r\,dr)$$

47. If $R = 0.30$ cm and $v = (0.30 - 3.33r^2)$ cm/s, find the volume.
48. Develop a general formula for V by evaluating

$$V = \int_0^R v(2\pi r\,dr)$$

using $v = K(R^2 - r^2)$.

Production In Problems 49 and 50, the rate of production of a new line of products is given by

$$\frac{dx}{dt} = 200\left[1 + \frac{400}{(t+40)^2}\right]$$

where x is the number of items produced and t is the number of weeks the products have been in production.
49. How many units were produced in the first 5 weeks?
50. How many units were produced in the sixth week?

51. ***Depreciation*** If the rate of depreciation of a building is given by $D'(t) = 3000(20 - t)$, $0 \le t \le 20$, what is the total depreciation of the building over the first 10 years ($t = 0$ to $t = 10$)?

52. ***Depreciation*** What is the total depreciation of the building in Problem 51 during the next 10 years ($t = 10$ to $t = 20$)?

53. ***Sales and advertising*** A store finds that its sales change at a rate given by

$$S'(t) = -3t^2 + 300t$$

where t is the number of days after an advertising campaign ends and $0 \le t \le 30$.
(a) Find the total sales for the first week after the campaign ends ($t = 0$ to $t = 7$).
(b) Find the total sales for the second week after the campaign ends ($t = 7$ to $t = 14$).

54. ***Health care costs*** The annual health care costs in the United States for selected years are given in the table below. The equation

$$y = 1.094x^2 - 11.103x + 49.562$$

models the annual health care costs, y (in billions of dollars), as a function of the years past 1960, x. Use a definite integral and this model to find the total cost of health care over the period 1960–1995.

Year	1960	1965	1970	1975	1980	1985
Cost	27.1	41.1	73.2	130.7	247.2	428.2

Year	1990	1992	1993	1994	1995
Cost ($ million)	697.5	834.2	892.1	937.1	988.5

Source: *World Almanac*, 1998

55. ***Dodge Viper acceleration*** Table 13.1(a) shows the time in seconds that a 1996 Dodge Viper GTS requires to reach various speeds up to 100 mph. Table 13.1(b) shows the same data, but with speeds in miles per second.
(a) Fit a power model to the data in Table 13.1(b).
(b) Use a definite integral from 0 to 9.2 of the function you found in (a) to find the distance traveled by the Viper as it went from 0 mph to 100 mph in 9.2 seconds.

TABLE 13.1

(a) Time (seconds)	(a) Speed (mph)	(b) Time (seconds)	(b) Speed (mi/s)
1.7	30	1.7	0.00833
2.4	40	2.4	0.01111
3.2	50	3.2	0.01389
4.1	60	4.1	0.01667
5.8	70	5.8	0.01944
6.2	80	6.2	0.02222
7.8	90	7.8	0.02500
9.2	100	9.2	0.02778

Source: *Motor Trend*, April 1998

13.3 Area Between Two Curves

OBJECTIVES

- *To find the area between two curves*
- *To find the average value of a function*

APPLICATION PREVIEW

In economics, the **Lorenz curve** is used to represent the inequality of income distribution among different groups in the population of a country. The curve is constructed by plotting the cumulative percentage of families at or below a given income level and the cumulative percentage of total personal income received by these families. For example, the table shows the coordinates of some points on the Lorenz curve $y = L(x)$ that divide the income (for the United States in 1996) into 5 equal income levels (quintiles). The point (0.40, 0.142) is on the Lorenz curve because the families with income in the bottom 40% of the country received 14.2% of the total income in 1996. The graph of the Lorenz curve $y = L(x)$ is shown in Figure 13.9.

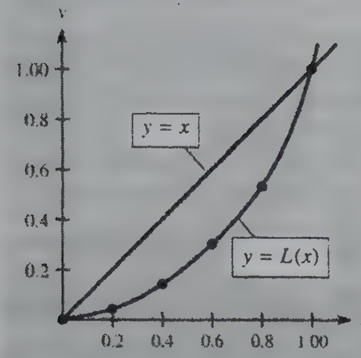


Figure 13.9

U.S. Income Distribution for 1996 (Points on the Lorenz Curve)

x, Cumulative Proportion of Families Below Income Level	*y = L(x), Cumulative Proportion of Total Income*
0	0
0.20	0.042
0.40	0.142
0.60	0.302
0.80	0.532
1	1

Source: *Statistical Abstract of the United States*, 1997

Equality of income would result if each family received an equal proportion of the total income, so that the bottom 20% would receive 20% of the total income, the bottom 40% would receive 40%, and so on. The Lorenz curve representing this would have the equation $y = x$.

The inequality of income distribution is measured by the **Gini coefficient** of income, which measures how far the Lorenz curve falls below $y = x$. It is defined as

$$\frac{\text{Area between } y = x \text{ and } y = L(x)}{\text{Area below } y = x}$$

Because the area of the triangle below $y = x$ and above the x-axis from $x = 0$ to $x = 1$ is $1/2$, the Gini coefficient of income is

$$\frac{\text{Area between } y = x \text{ and } y = L(x)}{1/2} = 2 \cdot [\text{area between } y = x \text{ and } y = L(x)]$$

In this section we will use the definite integral to find the area between two curves. We will use the area between two curves to find the Gini coefficient of income and to find average cost, average revenue, average profit, and average inventory.

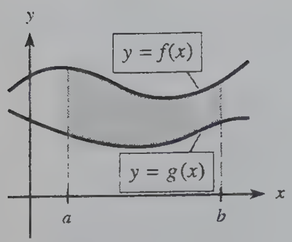


Figure 13.10

We have used the definite integral to find the area of the region between a curve and the x-axis over an interval where the curve lies above the x-axis. We can easily extend this technique to finding the area between two curves over an interval where one curve lies above the other. (See Figure 13.10.)

Suppose that the graphs of both $y = f(x)$ and $y = g(x)$ lie above the x-axis and that the graph of $y = f(x)$ lies above $y = g(x)$ throughout the interval from $x = a$ to $x = b$; that is, $f(x) \geq g(x)$ on $[a, b]$.

Then $\int_a^b f(x)\,dx$ gives the area between the graph of $y = f(x)$ and the x-axis (see Figure 13.11a), and $\int_a^b g(x)\,dx$ gives the area between the graph of $y = g(x)$ and the x-axis (see Figure 13.11b). As Figure 13.11(c) shows, the area of the region between the graphs of $y = f(x)$ and $y = g(x)$ is the difference of these two areas. That is,

$$\text{Area between the curves} = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

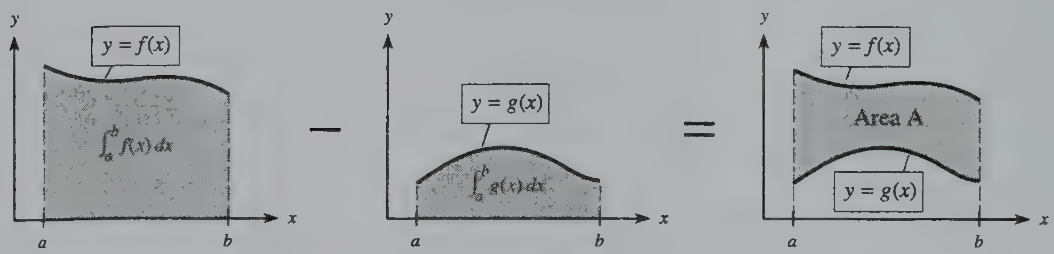


Figure 13.11 (a) (b) (c)

Although Figure 13.11(c) shows the graphs of both $y = f(x)$ and $y = g(x)$ lying above the x-axis, this difference of their integrals will always give the area between their graphs if both functions are continuous and if $f(x) \geq g(x)$ on the interval $[a, b]$. Using the fact that

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$$

we have the following result.

Area Between Two Curves If f and g are continuous functions on $[a, b]$ and if $f(x) \geq g(x)$ on $[a, b]$, then the area of the region bounded by $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

EXAMPLE 1

Find the area of the region bounded by $y = x^2 + 4$, $y = x$, $x = 0$, and $x = 3$.

Solution

We first sketch the graphs of the functions. The graphs of the region is shown in Figure 13.12. Because $y = x^2 + 4$ lies above $y = x$ in the interval from $x = 0$ to $x = 3$, the area is

$$\begin{aligned} A &= \int_0^3 [(x^2+4) - x]\,dx = \frac{x^3}{3} + 4x - \frac{x^2}{2}\bigg|_0^3 \\ &= \left(9 + 12 - \frac{9}{2}\right) - (0 + 0 - 0) \\ &= 16\frac{1}{2} \text{ square units} \end{aligned}$$

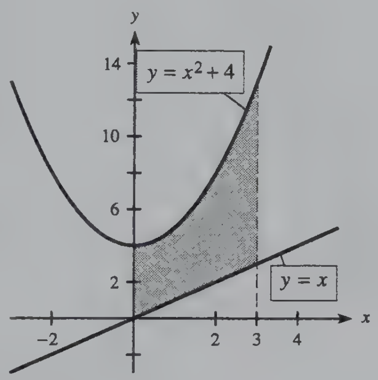


Figure 13.12

We are sometimes asked to find the area enclosed by two curves. In this case, we find the points of intersection of the curves to determine a and b.

EXAMPLE 2

Find the area enclosed by $y = x^2$ and $y = 2x + 3$.

Solution

We first find a and b by finding the x-coordinates of the points of intersection of the graphs. Setting the y-values equal gives

$$\begin{aligned} x^2 &= 2x + 3 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \\ x = 3, \quad x &= -1 \end{aligned}$$

Thus $a = -1$ and $b = 3$.

We next sketch the graphs of these functions on the same set of axes. Because the graphs do not intersect on the interval $(-1, 3)$, we can determine which function is larger on this interval by evaluating $2x + 3$ and x^2 at any value c where $-1 < c < 3$. Figure 13.13 on the next page shows the region between the graphs, with $2x + 3 \geq x^2$ from $x = -1$ to $x = 3$. The area of the enclosed region is

$$A = \int_{-1}^{3} [(2x+3) - x^2]\, dx = x^2 + 3x - \frac{x^3}{3}\Bigg|_{-1}^{3}$$

$$= (9 + 9 - 9) - \left(1 - 3 + \frac{1}{3}\right)$$

$$= 10\tfrac{2}{3} \text{ square units}$$

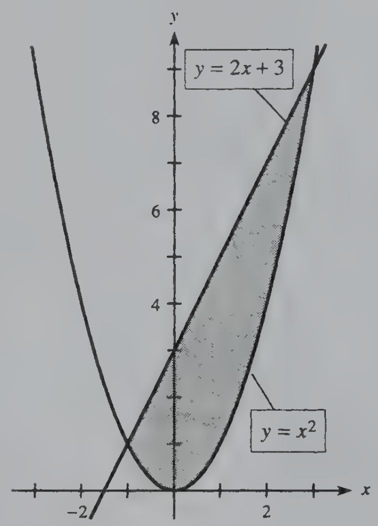


Figure 13.13

Some graphs enclose two or more regions because they have more than two points of intersection.

EXAMPLE 3

Find the area of the region enclosed by the graphs of

$$y = f(x) = x^3 - x^2 \quad \text{and} \quad y = g(x) = 2x$$

Solution

To find the points of intersection of the graphs, we set the y-values equal and solve for x.

$$x^3 - x^2 = 2x$$
$$x^3 - x^2 - 2x = 0$$
$$x(x-2)(x+1) = 0$$
$$x = 0, \quad x = 2, \quad x = -1$$

Graphing these functions between $x = -1$ and $x = 2$, we see that for any x-value in the interval $(-1, 0)$, $f(x) \geq g(x)$, so $f(x) \geq g(x)$ for the region enclosed by the curves from $x = -1$ to $x = 0$. But evaluating the functions for any x-value in the interval $(0, 2)$ shows that $f(x) \leq g(x)$ for the region enclosed by the curves from $x = 0$ to $x = 2$. See Figure 13.14.

Thus we need one integral to find the area of the region from $x = -1$ to $x = 0$ and a second integral to find the area from $x = 0$ to $x = 2$. The area is found by summing these two integrals.

$$A = \int_{-1}^{0} [(x^3 - x^2) - (2x)]\,dx + \int_{0}^{2} [(2x) - (x^3 - x^2)]\,dx$$

$$= \int_{-1}^{0} (x^3 - x^2 - 2x)\,dx + \int_{0}^{2} (2x - x^3 + x^2)\,dx$$

$$= \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Big|_{-1}^{0} + \left(x^2 - \frac{x^4}{4} + \frac{x^3}{3}\right)\Big|_{0}^{2}$$

$$= \left[(0) - \left(\frac{1}{4} - \frac{-1}{3} - 1\right)\right] + \left[\left(4 - \frac{16}{4} + \frac{8}{3}\right) - (0)\right] = \frac{37}{12}$$

Thus the area between the curves is $\frac{37}{12}$ square units.

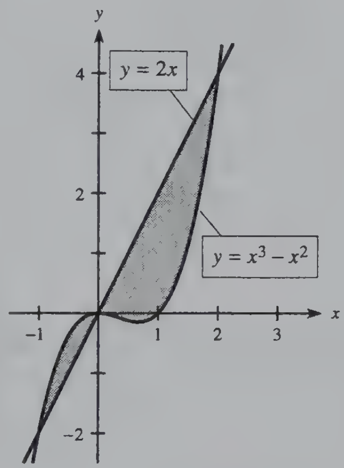


Figure 13.14

Graphing Utilities Most graphing utilities have a numerical integration feature, and some can perform both symbolic and numerical integration.

A graphing utility could be used to find the area enclosed by the graphs of $y = f(x) = x^3 - x^2$ and $y = g(x) = 2x$, found in Example 3. By using SOLVER, INTERSECT, or ZOOM and TRACE, we can find that the curves intersect at $x = -1$, $x = 0$, and $x = 2$. From the graph, shown in Figure 13.13, we see that the curves enclose two regions, with $f(x) > g(x)$ for $-1 < x < 0$ and $g(x) > f(x)$ for $0 < x < 2$. Thus the area is given by

$$\int_{-1}^{0} [(x^3 - x^2) - (2x)]\,dx + \int_{0}^{2} [(2x) - (x^3 - x^2)]\,dx$$

By using the numerical integration feature of a graphing utility, we can find the integral (and the area) to be $0.4166666667 + 2.666666667 = 3.083333333$. This is the same (to nine decimal places) as the decimal representation of 37/12, found in Example 3.

CHECKPOINT

1. True or false:
   (a) Over the interval $[a, b]$, the area between the continuous functions $f(x)$ and $g(x)$ is
   $$\int_a^b [f(x) - g(x)]\, dx$$
   (b) If $f(x) \geq g(x)$ and the area between $f(x)$ and $g(x)$ is given by
   $$\int_a^b [f(x) - g(x)]\, dx$$
   then $x = a$ and $x = b$ represent the left and right boundaries, respectively, of the region.
   (c) To find points of intersection of $f(x)$ and $g(x)$, solve $f(x) = g(x)$.
2. Consider the functions $f(x) = x^2 + 3x - 9$ and $g(x) = \frac{1}{4}x^2$.
   (a) Find the points of intersection of $f(x)$ and $g(x)$.
   (b) Determine which function is greater than the other between the points found in (a).
   (c) Set up the integral used to find the area between the curves in the interval between the points found in (a).
   (d) Find the area.

EXAMPLE 4

As we noted in the Application Preview, the inequality of income distribution is measured by the Gini coefficient of income, which is defined as

$$\frac{\text{Area between } y = x \text{ and } y = L(x)}{\text{Area below } y = x} = \frac{\int_0^1 [x - L(x)]\, dx}{1/2}$$
$$= 2\int_0^1 [x - L(x)]\, dx$$

We can use the data in the Application Preview and a graphing utility to model the Lorenz curve for United States income in 1996. By using a power model, we obtain $L(x) = 0.8743x^{1.9262}$.

(a) Use this $L(x)$ to find the Gini coefficient of income for 1996.
(b) If the Gini coefficient of income for 1991 is 0.374, during which year is the distribution of income more nearly equal?

Solution

(a) The Gini coefficient of income for 1996 is

$$2\int_0^1 (x - L(x))\, dx = 2\int_0^1 (x - 0.8743x^{1.9262})\, dx$$
$$= 2\left[\frac{x^2}{2} - \frac{0.8743x^{2.9262}}{2.9262}\right]\Bigg|_0^1 = 2\left[\frac{1}{2} - 0.2988\right] = 0.4024$$

(b) Absolute equality of income would occur if the Gini coefficient of income were 0; and smaller coefficients indicate more nearly equal incomes. Thus the distribution of income was more nearly equal in 1991.

If the graph of $y = f(x)$ lies on or above the x-axis from $x = a$ to $x = b$, then the area between the graph and the x-axis is

$$A = \int_a^b f(x)\,dx \qquad \text{(See Figure 13.15a.)}$$

The area A is also the area of a rectangle with base equal to $b - a$ and height equal to the *average value* (or average height) of the function $y = f(x)$ (see Figure 13.15b).

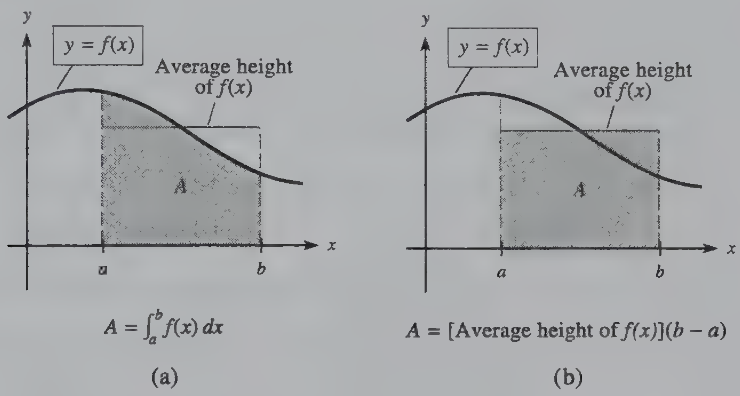


Figure 13.15

Thus the average value of the function is

$$\frac{A}{b-a} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Even if $f(x) \le 0$ on all or part of the interval $[a, b]$, we can find the average value by using the integral. Thus we have the following.

Average Value The average value of a continuous function $y = f(x)$ over the interval $[a, b]$ is

$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

EXAMPLE 5

Suppose that the cost function for a product is $C(x) = 400 + x + 0.3x^2$.

(a) What is the average value of $C(x)$ for $x = 10$ to $x = 20$ units?
(b) Find the average cost per unit if 40 units are produced.

Solution

(a) The average value of $C(x)$ is

$$\begin{aligned}\frac{1}{20-10}\int_{10}^{20} (400 + x + 0.3x^2)\,dx &= \frac{1}{10}\left(400x + \frac{x^2}{2} + 0.1x^3\right)\Bigg|_{10}^{20} \\ &= \frac{1}{10}[(8000 + 200 + 800) - (4000 + 50 + 100)] \\ &= 485 \quad \text{(dollars)}\end{aligned}$$

(b) The average cost per unit if 40 units are produced is the average cost function evaluated at $x = 40$. The average cost function is

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{400}{x} + 1 + 0.3x$$

Thus the average cost per unit if 40 units are produced is

$$\overline{C}(40) = \frac{400}{40} + 1 + 0.3(40) = 23 \quad \text{(dollars)}$$

CHECKPOINT

3. Find the average value of $f(x) = x^2 - 4$ over $[-1, 3]$.

EXAMPLE 6

Consider the functions $f(x) = x^2 - 4$ and $g(x) = x^3 - 4x$. For each function, do the following.

(a) Graph the function on the interval $[-2, 2]$.
(b) On the graph, "eyeball" the average value (height) of each function on $[-2, 2]$.
(c) Compute the average value of the function over the interval $[-2, 2]$.

Solution

For $f(x) = x^2 - 4$.

(a) The graph of $f(x) = x^2 - 4$ is shown in Figure 13.16(a).
(b) The average height of $f(x)$ may be near -2.
(c) The average value of $f(x)$ over the integral is given by

$$\begin{aligned}\frac{1}{2-(-2)}\int_{-2}^{2}(x^2-4)\,dx &= \frac{1}{4}\left[\frac{x^3}{3} - 4x\right]_{-2}^{2} \\ &= \left(\frac{8}{12} - 2\right) - \left(-\frac{8}{12} + 2\right) \\ &= \frac{4}{3} - 4 = -\frac{8}{3} = -2\tfrac{2}{3}\end{aligned}$$

For $g(x) = x^3 - 4x$.

(a) The graph of $g(x) = x^3 - 4x$ is shown in Figure 13.16(b).
(b) The average height of $g(x)$ graph may be approximately 0.
(c) The average value of $g(x)$ is given by

$$\begin{aligned}\frac{1}{2-(-2)}\int_{-2}^{2}(x^3-4x)\,dx &= \frac{1}{4}\left[\frac{x^4}{4} - \frac{4x^2}{2}\right]_{-2}^{2} \\ &= \left(\frac{16}{16} - \frac{16}{8}\right) - \left(\frac{16}{16} - \frac{16}{8}\right) \\ &= -1 + 1 = 0\end{aligned}$$

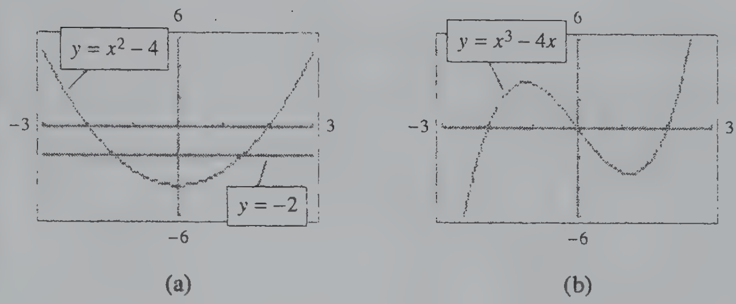


Figure 13.16

CHECKPOINT SOLUTIONS

1. (a) False. This is true only if $f(x) \geq g(x)$ over $[a, b]$.

   (b) True (c) True

2. (a) Solve $f(x) = g(x)$, or $x^2 + 3x - 9 = \frac{1}{4}x^2$.

$$\begin{aligned} \frac{3}{4}x^2 + 3x - 9 &= 0 \\ 3x^2 + 12x - 36 &= 0 \\ x^2 + 4x - 12 &= 0 \\ (x+6)(x-2) &= 0 \\ x = -6 \quad &| \quad x = 2 \end{aligned}$$

   The points of intersection are $(-6, 9)$ and $(2, 1)$.

   (b) Evaluating $g(x)$ and $f(x)$ at any point in the interval $(-6, 2)$ shows that $g(x) > f(x)$, so $g(x) \geq f(x)$ on $[-6, 2]$.

   (c) $A = \displaystyle\int_{-6}^{2} \left[\frac{1}{4}x^2 - (x^2 + 3x - 9)\right] dx$

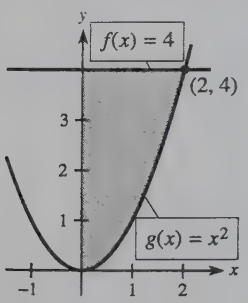
   (d) $A = \left.\dfrac{x^3}{12} - \dfrac{x^3}{3} - \dfrac{3x^2}{2} + 9x\right|_{-6}^{2} = 64$ square units

3. $$\begin{aligned} \frac{1}{3-(-1)} \int_{-1}^{3} (x^2 - 4)\, dx &= \frac{1}{4}\left(\frac{x^3}{3} - 4x\right)\Bigg|_{-1}^{3} \\ &= \frac{1}{4}\left[(9 - 12) - \left(-\frac{1}{3} + 4\right)\right] \\ &= \frac{1}{4}\left(\frac{-20}{3}\right) = -\frac{5}{3} \end{aligned}$$

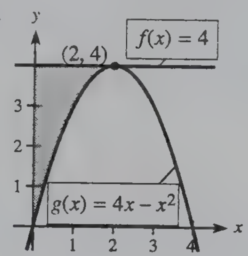
EXERCISE 13.3

For each shaded region in Problems 1–6, (a) form the integral that represents the area of the shaded region and (b) find the area of the region.

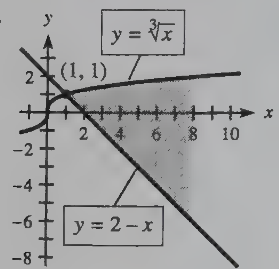
1.



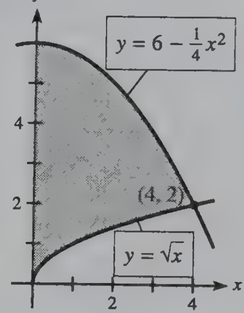
2.



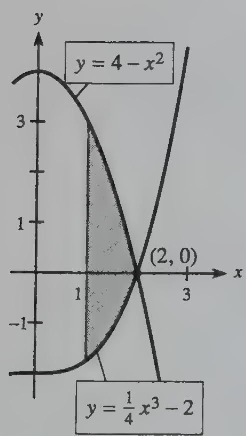
3.



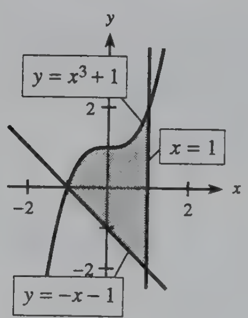
4.



5.

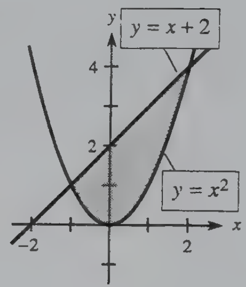


6.

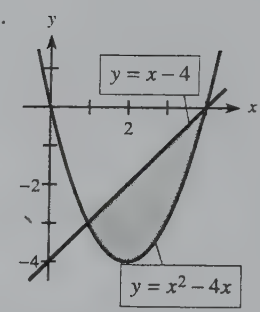


For each shaded region in Problems 7–12, (a) find the points of intersection of the curves, (b) form the integral that represents the area of the shaded region, and (c) find the area of the shaded region.

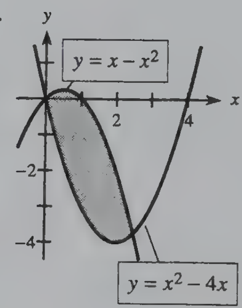
7.



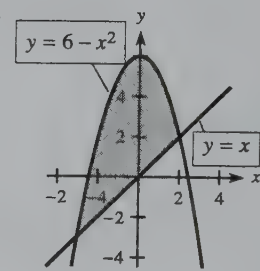
8.



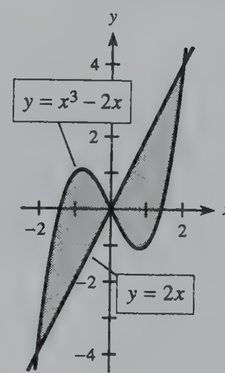
9.



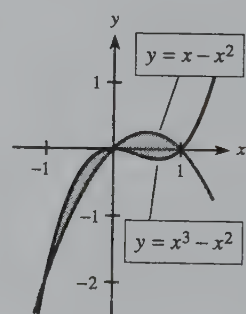
10.



11.



12.



In Problems 13–26, equations are given whose graphs enclose a region. In each problem, find the area of the region.

13. $f(x) = x^2 + 2$; $g(x) = -x^2$; $x = 0$; $x = 2$
14. $f(x) = x^2$; $g(x) = -\frac{1}{10}(10 + x)$; $x = 0$; $x = 3$
15. $y = x^3 - 1$; $y = x - 1$; to the right of the y-axis
16. $y = x^2 - 2x + 1$; $y = x^2 - 5x + 4$; $x = 2$
17. $y = \frac{1}{2}x^2$; $y = x^2 - 2x$
18. $y = x^2$; $y = 4x - x^2$
19. $h(x) = x^2$; $k(x) = \sqrt{x}$
20. $g(x) = 1 - x^2$; $h(x) = x^2 + x$
21. $f(x) = x^3$; $g(x) = x^2 + 2x$
22. $f(x) = x^3$; $g(x) = 2x - x^2$
23. $f(x) = \dfrac{3}{x}$; $g(x) = 4 - x$

24. $f(x) = \frac{6}{x}$; $g(x) = -x - 5$

25. $y = \sqrt{x+3}$; $x = -3$; $y = 2$

26. $y = \sqrt{4-x}$; $x = 4$; $y = 3$

In Problems 27–32, find the average value of each function over the given interval.

27. $f(x) = 9 - x^2$ over $[0, 3]$

28. $f(x) = 2x - x^2$ over $[0, 2]$

29. $f(x) = x^3 - x$ over $[-1, 1]$

30. $f(x) = \frac{1}{2}x^3 + 1$ over $[-2, 0]$

31. $f(x) = \sqrt{x} - 2$ over $[1, 4]$

32. $f(x) = \sqrt[3]{x}$ over $[-8, -1]$

33. Use a graphing calculator or computer to find the area between the curves $y = f(x) = x^3 - 4x$ and $y = g(x) = x^2 - 4$.

34. Use a graphing calculator or computer software to find the area between the curves $f(x) = \sqrt[3]{x}$ and $g(x) = x^3 - x$.

Applications

35. ***Income distribution*** Changes in tax laws in the United States during the 1980s were widely thought to help the wealthy at the expense of the poor. The Lorenz curves for the income distribution in 1980 and in 1990 are given below. Find the Gini coefficient of income for both years and determine whether the distribution of income is more or less nearly equal in 1990 than it was in 1980. What was the effect of the tax laws?

1980	$y = 0.916x^{1.821}$
1990	$y = 0.896x^{1.878}$

36. ***Income distribution*** The Lorenz curves for the income distribution in the United States in 1950 and in 1970 are given below. Find the Gini coefficient of income for both years and compare the distributions of income for these years.

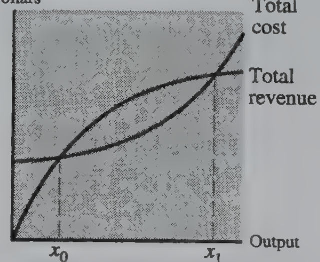
1950	$y = 0.925x^{1.891}$
1970	$y = 0.920x^{1.783}$

37. ***Income distribution*** Data from the U.S. Bureau of the Census yields the Lorenz curves given below for income distribution among blacks and among whites in the United States for 1996. Find the Gini coefficient of income for both groups and determine in which the group income is more nearly equally distributed.

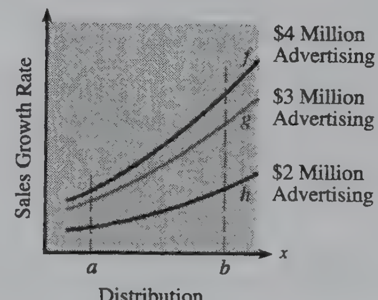
Whites	$y = 0.8693x^{1.8556}$
Blacks	$y = 0.8693x^{2.0982}$

38. ***Income distribution*** In an effort to make the distribution of income more nearly equal, the government of a country passes a tax law that changes the Lorenz curve for one year from $y = 0.99x^{2.1}$ to $y = 0.32x^2 + 0.68x$ for the next year. Find the Gini coefficient of income for both years and compare the distributions of income before and after the tax law is passed. Interpret the result.

39. ***Average profit*** For the product whose total cost and total revenue are shown in the figure, represent total revenue by $R(x)$ and total cost by $C(x)$ and write an integral that gives the average profit for the product over the interval from x_0 to x_1.

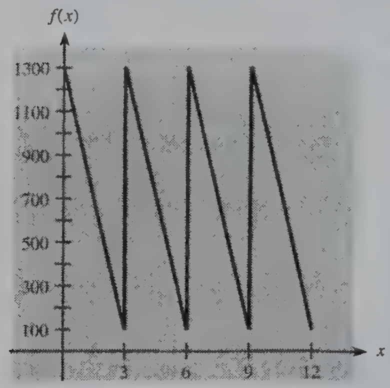


40. ***Sales and advertising*** The figure shows the sales growth rates under different levels of distribution and advertising from a to b. Set up an integral to determine the extra sales growth if \$4 million is used in advertising rather than \$2 million.



41. ***Cost*** The cost of producing x units of a certain item is $C(x) = x^2 + 400x + 2000$.
   (a) Use $C(x)$ to find the average cost of producing 1000 units.
   (b) Find the average value of the cost function $C(x)$ over the interval from 0 to 1000.

42. ***Inventory management*** The figure shows how an inventory of a product is depleted each quarter of a given year. What is the average inventory per month for the first 3 months for this product? (Assume that the graph is a line joining (0, 1300) and (3,100).)



43. ***Sales and advertising*** The number of daily sales of a product was found to be given by

$$S = 100xe^{-x^2} + 100$$

x days after the start of an advertising campaign for this product.
(a) Find the average daily sales during the first 20 days of the campaign—that is, from $x = 0$ to $x = 20$.
(b) If no new advertising campaign is begun, what is the average number of sales per day for the next 10 days (from $x = 20$ to $x = 30$)?

44. ***Demand*** The demand function for a certain product is given by

$$p = 500 + \frac{1000}{q + 1}$$

where p is the price and q is the number of units demanded. Find the average price as demand ranges from 49 to 99 units.

45. ***Interest rates*** The equation $y = 0.00144100x^4 - 0.0593608x^3 + 0.740741x^2 - 2.573133x + 6.941375$ describes interest rates for the years 1970 to 1989, where $x = 0$ in 1970. Use a definite integral from 0 to 19 to compute the average interest rate over the period.

46. ***Total income*** Suppose that the income from a slot machine in a casino flows continuously at a rate

$$f(t) = 100e^{0.1t}$$

where t is the time in hours since the casino opened. Then the total income during the first 10 hours is given by

$$\int_0^{10} 100e^{0.1t}\, dt$$

Find the average income over the first 10 hours.

47. ***Drug levels in the blood*** A drug manufacturer has developed a time-release capsule with the number of milligrams of the drug in the bloodstream given by

$$S = 30x^{18/7} - 240x^{11/7} + 480x^{4/7}$$

where x is in hours and $0 \le x \le 4$. Find the average number of milligrams of the drug in the bloodstream for the first 4 hours after a capsule is taken.

13.4 Applications of Definite Integrals in Business and Economics

OBJECTIVES

- *To use definite integrals to find total income, present value, and future value of continuous income streams*
- *To use definite integrals to find the consumer's surplus*
- *To use definite integrals to find the producer's surplus*

APPLICATION PREVIEW

The definite integral can be used in a number of applications in business and economics. For example, the definite integral can be used to find the total income over a fixed number of years from a **continuous income stream.** The definite integral can also be used to find the **present value** and the **future value** of a continuous income stream. And it can be used to find the **consumer's surplus** and **producer's surplus** when the demand function and the supply function for a product are known.

Continuous Income Streams

An oil company's profits depend on the amount of oil that can be pumped from a well. Thus we can consider a pump at an oil field as producing a **continuous stream of income** for the owner. Because both the pump and the oil field "wear out" with time, the continuous stream of income is a function of time. Suppose $f(t)$ is the (annual) *rate* of flow of income from this pump; then we can find the total income from the rate of income by using integration. In particular, the total income for k years is given by

$$\text{Total income} = \int_0^k f(t)\,dt$$

EXAMPLE 1

A small oil company considers the continuous pumping of oil from a well as a continuous income stream with its annual rate of flow at time t given by

$$f(t) = 600e^{-0.2t}$$

in thousands of dollars. Find an estimate of the total income from this well over the next 10 years.

Solution

$$\begin{aligned}\text{Total income} &= \int_0^{10} f(t)\,dt = \int_0^{10} 600e^{-0.2t}\,dt \\ &= \frac{600}{-0.2}e^{-0.2t}\Big|_0^{10} \\ &\approx 2594 \quad \text{(to the nearest integer)}\end{aligned}$$

Thus the total income is approximately \$2,594,000.

In addition to the total income from a continuous income stream, the **present value** of the stream is also important. The present value is the value today of a continuous income stream that will be providing income in the future. The present value is useful in deciding when to replace machinery (such as the oil pump in the example) or what new equipment to select.

To find the present value of a continuous stream of income with rate of flow $f(t)$, we first graph the function $f(t)$ and divide the time interval from 0 to k into n subintervals of width Δt_i, $i = 1$ to n.

The total amount of income is the area under this curve between $t = 0$ and $t = k$. We can approximate the amount of income in each subinterval by finding the area of the rectangle in that subinterval. (See Figure 13.17.)

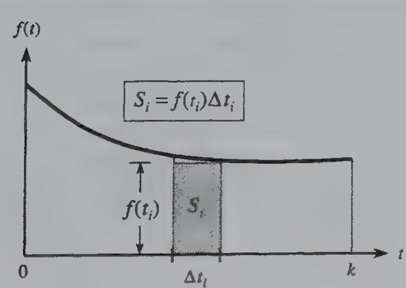


Figure 13.17

We have shown that the future value S that accrues if $\$P$ is invested for t years at an annual rate r, compounded continuously, is

$$S = Pe^{rt}$$

Thus the present value of the investment that yields the single payment of $\$S$ after t years is

$$P = \frac{S}{e^{rt}} = Se^{-rt}$$

The contribution to S in the ith subinterval is $S_i = f(t_i)\,\Delta t_i$, and the present value of this amount is

$$P_i = f(t_i)\,\Delta t_i e^{-rt_i}$$

Thus the total present value of S can be approximated by

$$\sum_{i=1}^{n} f(t_i)\,\Delta t_i e^{-rt_i}$$

This approximation improves as $\Delta t_i \to 0$ with the present value given by

$$\lim_{\Delta t_i \to 0} \sum_{i=1}^{n} f(t_i)\,\Delta t_i e^{-rt_i}$$

This limit gives the present value as a definite integral.

Present Value of a Continuous Income Stream

If $f(t)$ is the rate of continuous income flow earning interest at rate r, compounded continuously, then the present value of the continuous income stream is

$$\text{Present value} = \int_0^k f(t)e^{-rt}\,dt$$

where $t = 0$ to $t = k$ is the time interval.

EXAMPLE 2

Suppose that the oil company in Example 1 is planning to sell the well because of its remote location. Suppose further that the company wants to use the present value of the well over the next 10 years to help establish its selling price. If the company determines that the annual rate of flow is

$$f(t) = 600e^{-0.2(t+5)}$$

in thousands of dollars, and if money is worth 10%, compounded continuously, find this present value.

Solution

$$\begin{aligned}
\text{Present value} &= \int_0^{10} f(t)e^{-rt}\,dt \\
&= \int_0^{10} 600e^{-0.2(t+5)}e^{-0.1t}\,dt \\
&= \int_0^{10} 600e^{-0.3t-1}\,dt \\
&= \frac{600}{-0.3}e^{-0.3t-1}\Big|_0^{10} \\
&= -2000(e^{-4} - e^{-1}) \\
&\approx 699 \quad \text{(to the nearest integer)}
\end{aligned}$$

Thus the present value is \$699,000.

Future Value of a Continuous Income Stream

If $f(t)$ is the rate of continuous income flow for k years earning interest at rate r, compounded continuously, then the future value of the continuous income stream is

$$FV = e^{rk}\int_0^k f(t)e^{-rt}\,dt$$

EXAMPLE 3

If the rate of flow of income from an asset is $1000e^{0.02t}$ and if the income is invested at 6%, compounded continuously, find the future value of the asset 4 years from now.

Solution

The future value is given by

$$\begin{aligned}
FV &= e^{rk}\int_0^k f(t)e^{-rt}\,dt \\
&= e^{(0.06)4}\int_0^4 1000e^{0.02t}e^{-0.06t}\,dt = e^{0.24}\int_0^4 1000e^{-0.04t}\,dt \\
&= e^{0.24}\left[-25000e^{-0.04t}\right]_0^4 = -25000e^{0.24}\left[e^{-0.16} - 1\right] \\
&\approx 4699.05
\end{aligned}$$

CHECKPOINT

1. Suppose that a continuous income stream has an annual rate of flow given by $f(t) = 5000e^{-0.01t}$, and suppose that money is worth 7%, compounded continuously. Create the integral used to find
   (a) the total income for the next 5 years.
   (b) the present value for the next 5 years.
   (c) the future value 5 years from now.

Consumer's Surplus

Suppose that the demand for a product is given by $p = f(x)$ and that the supply of the product is described by $p = g(x)$. The price p_1 where the graphs of these functions intersect is the **equilibrium price** (see Figure 13.18). As the demand curve shows, some consumers (but not all) would be willing to pay more than $\$p_1$ for the product.

For example, some consumers would be willing to buy x_3 units if the price were $\$p_3$. Those consumers willing to pay more than $\$p_1$ are benefiting from the lower price. The total gain for all those consumers willing to pay more than $\$p_1$ is called the **consumer's surplus,** and under proper assumptions the area of the shaded region in Figure 13.18 represents this consumer's surplus.

Looking at Figure 13.19, we see that if the demand curve has equation $p = f(x)$, the consumer's surplus is given by the area between $f(x)$ and the x-axis from 0 to x_1, *minus* the area of the rectangle denoted *TR:*

$$CS = \int_0^{x_1} f(x)\,dx - p_1x_1$$

Note that p_1x_1 is the area of the rectangle that represents the total revenue (see Figure 13.19).

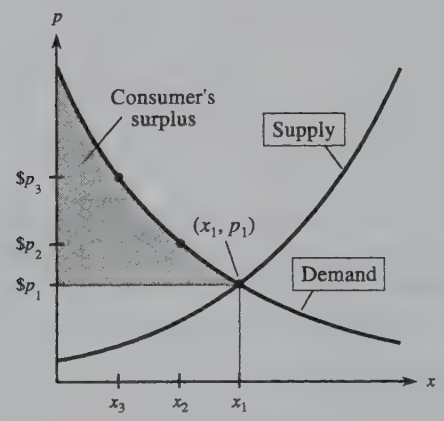


Figure 13.18

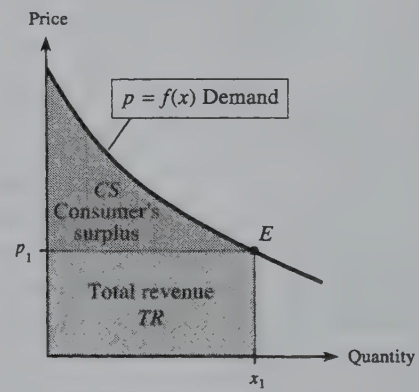


Figure 13.19

EXAMPLE 4

The demand function for a product is $p = 100/(x + 1)$. If the equilibrium price is \$20, what is the consumer's surplus?

Solution

We must first find the quantity that will be purchased at this price. Letting $p = 20$ and solving for x, we get

$$20 = \frac{100}{x+1}$$
$$20(x + 1) = 100$$
$$x + 1 = 5$$
$$x = 4$$

Thus the equilibrium point is (4, 20). The consumer's surplus is given by the formula

$$
\begin{aligned}
CS = \int_0^{x_1} f(x)\,dx - p_1x_1 &= \int_0^4 \frac{100}{x+1}\,dx - 20 \cdot 4 \\
&= 100 \ln|x+1| \Big|_0^4 - 80 \\
&= 100(\ln 5 - \ln 1) - 80 \\
&\approx 100(1.6094 - 0) - 80 \\
&= 160.94 - 80 \\
&= 80.94
\end{aligned}
$$

The consumer's surplus is \$80.94.

EXAMPLE 5

A product's demand function is $p = \sqrt{49 - 6x}$ and its supply function is $p = x + 1$. Find the equilibrium point and the consumer's surplus there.

Solution

We can determine the equilibrium point by solving the two equations simultaneously.

$$
\begin{aligned}
\sqrt{49 - 6x} &= x + 1 \\
49 - 6x &= (x+1)^2 \\
0 &= x^2 + 8x - 48 \\
0 &= (x+12)(x-4) \\
x = 4 \quad &\text{or} \quad x = -12
\end{aligned}
$$

Thus the equilibrium quantity is 4 and the equilibrium price is \$5 (because $x = -12$ is not a solution). The graphs of the supply and demand functions are shown in Figure 13.20.

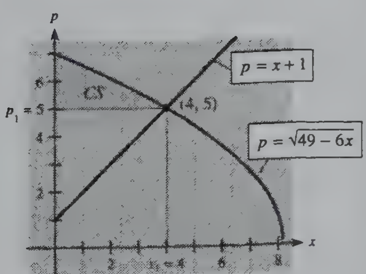


Figure 13.20

The consumer's surplus is given by

$$
\begin{aligned}
CS &= \int_0^4 f(x)\,dx - p_1x_1 \\
&= \int_0^4 \sqrt{49 - 6x}\,dx - 5 \cdot 4 \\
&= -\frac{1}{6}\int_0^4 \sqrt{49 - 6x}\,(-6\,dx) - 20 \\
&= -\frac{1}{9}(49 - 6x)^{3/2}\Big|_0^4 - 20 \\
&= -\frac{1}{9}[(25)^{3/2} - (49)^{3/2}] - 20 \\
&= -\frac{1}{9}(125 - 343) - 20 \\
&\approx 24.22 - 20 = 4.22
\end{aligned}
$$

The consumer's surplus is \$4.22.

EXAMPLE 6

If a monopoly has a total cost function $C(x) = 60 + 2x^2$ for a product whose demand is given by $p = 30 - x$, find the consumer's surplus at the point where the monopoly has maximum profit.

Solution

We must first find the point where the profit function is maximized. Because the demand for x units is $p = 30 - x$, the total revenue is

$$R(x) = (30 - x)x = 30x - x^2$$

Thus the profit function is

$$P(x) = R(x) - C(x)$$
$$P(x) = 30x - x^2 - (60 + 2x^2)$$
$$P(x) = 30x - 60 - 3x^2$$

Then $P'(x) = 30 - 6x$. So $0 = 30 - 6x$ has the solution $x = 5$.

Because $P''(5) = -6 < 0$, the profit for the monopolist is maximized when $x = 5$ units are sold at price $p = 30 - x = 25$.

The consumer's surplus at $x = 5, p = 25$ is given by

$$CS = \int_0^5 f(x)\,dx - 5 \cdot 25$$

where $f(x)$ is the demand function.

$$\begin{aligned} CS &= \int_0^5 (30 - x)\,dx - 125 \\ &= 30x - \frac{x^2}{2}\Big|_0^5 - 125 \\ &= \left(150 - \frac{25}{2}\right) - 125 \\ &= \frac{25}{2} = 12.50 \end{aligned}$$

The consumer's surplus is \$12.50.

Producer's Surplus

When a product is sold at the equilibrium price, some producers will also benefit, for they would have sold the product at a lower price. The area between the line $p = p_1$ and the supply curve (from $x = 0$ to $x = x_1$) gives the producer's surplus (see Figure 13.21).

If the supply function is $p = g(x)$, the **producer's surplus** is given by the area between the graph of $p = g(x)$ and the x-axis from 0 to x_1 *subtracted from* the area of the rectangle $0x_1Ep_1$.

$$PS = p_1x_1 - \int_0^{x_1} g(x)\,dx$$

Note that p_1x_1 represents the total revenue at the equilibrium point.

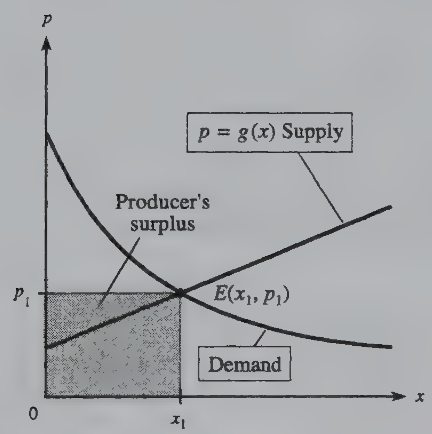


Figure 13.21

EXAMPLE 7

Suppose that the supply function for a product is $p = x^2 + x$. If the equilibrium price is \$20, what is the producer's surplus?

Solution

Because $p = 20$, we can find x as follows:

$$\begin{aligned} 20 &= x^2 + x \\ 0 &= x^2 + x - 20 \\ 0 &= (x + 5)(x - 4) \\ x &= -5, \quad x = 4 \end{aligned}$$

The equilibrium point is $x = 4$, $p = 20$. The producer's surplus is given by

$$\begin{aligned} PS &= 20 \cdot 4 - \int_0^4 (x^2 + x)\,dx \\ &= 80 - \left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_0^4 \\ &= 80 - \left(\frac{64}{3} + 8\right) \\ &\approx 50.67 \end{aligned}$$

The producer's surplus is \$50.67. See Figure 13.22.

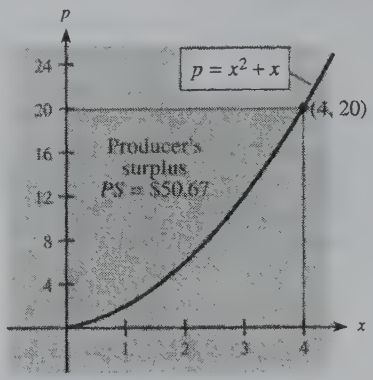


Figure 13.22

EXAMPLE 8

The demand function for a product is $p = \sqrt{49 - 6x}$ and the supply function is $p = x + 1$. Find the producer's surplus.

Solution

We found the equilibrium point for these functions to be (4, 5) in Example 5 (see Figure 13.20). The producer's surplus is

$$PS = 5 \cdot 4 - \int_0^4 (x + 1)\, dx$$
$$= 20 - \left(\frac{x^2}{2} + x\right)\Bigg|_0^4$$
$$= 20 - (8 + 4) = 8$$

The producer's surplus is \$8.

CHECKPOINT

2. Suppose that for a certain product, the supply function is $p = f(x)$, the demand function is $p = g(x)$, and the equilibrium point is (x_1, p_1). Decide whether the following are true or false.
   (a) $CS = \int_0^{x_1} f(x)\, dx - p_1x_1$ (b) $PS = \int_0^{x_1} f(x)dx - p_1x_1$
3. If demand is $p = \frac{100}{x + 1}$, supply is $p = x + 1$, and the market equilibrium is (9, 10), create the integral used to find the
   (a) consumer's surplus. (b) producer's surplus.

EXAMPLE 9

Suppose that for a certain product, the demand function is $p = 200e^{-0.01x}$ and the supply function is $p = \sqrt{200x + 49}$.

(a) Use a calculator or computer to find the market equilibrium point.
(b) Find the consumer's surplus.
(c) Find the producer's surplus.

Solution

(a) Solving $200e^{-0.01x} = \sqrt{200x + 49}$ requires solving

$$40000e^{-0.02x} = 200x + 49$$

which is very difficult using algebraic techniques. Using SOLVER or TRACE gives $x = 60$, to the nearest unit, with a price of \$109.76. (See Figure 13.23a.)

(b) The consumer's surplus is

$$\int_0^{60} 200e^{-0.01x}\,dx - 109.76(60) = \left[-20{,}000e^{-0.01x}\right]_0^{60} - 6585.60$$
$$= -20{,}000e^{-0.6} + 20{,}000 - 6585.60$$
$$= 2438.17$$

(c) The producer's surplus is

$$60(109.76) - \int_0^{60} \sqrt{200x + 49}\,dx$$
$$= 6585.60 - \frac{1}{200}\left[\frac{(200x + 49)^{3/2}}{3/2}\right]_0^{60}$$
$$= 6585.60 - \frac{1}{300}[(12049^{3/2} - 49^{3/2})]$$
$$\approx 2178.10$$

Note that we also could have evaluated these definite integrals with the numerical integration feature of a graphing utility, and we would have obtained the same results.

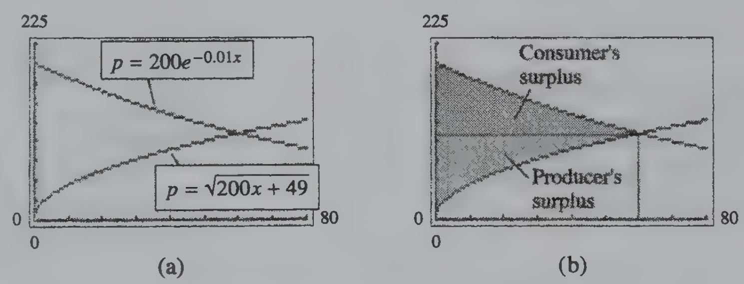


Figure 13.23

CHECKPOINT SOLUTIONS

1. (a) $\int_0^5 5000e^{-0.01t}\,dt$

   (b) $\int_0^5 (5000e^{-0.01t})(e^{-0.07t})\,dt = \int_0^5 5000e^{-0.08t}\,dt$

   (c) $e^{(0.07)(5)} \int_0^5 (5000e^{-0.01t})(e^{-0.07t})\,dt = e^{0.35} \int_0^5 5000e^{-0.08t}\,dt$

2. (a) False. Consumer's surplus uses the demand function, so

$$CS = \int_0^{x_1} g(x)\,dx - p_1x_1$$

(b) False. Producer's surplus uses the supply function, but the formula is

$$PS = p_1x_1 - \int_0^{x_1} f(x)\,dx$$

3. (a) $CS = \int_0^9 \frac{100}{x+1}\,dx - 90$ (b) $PS = 90 - \int_0^9 (x+1)\,dx$

EXERCISE 13.4

Continuous Income Streams

1. Find the total income over the next 10 years from a continuous income stream that has an annual rate of flow at time t given by $f(t) = 12{,}000$ (dollars).
2. Find the total income over the next 8 years from a continuous income stream with an annual rate of flow at time t given by $f(t) = 8500$ (dollars).
3. Suppose that a steel company views the production of its continuous caster as a continuous income stream with a monthly rate of flow at time t given by

$$f(t) = 24{,}000e^{0.03t} \quad \text{(dollars)}$$

Find the total income from this caster in the first year.

4. Suppose that the Quick-Fix Car Service franchise finds that the income generated by its stores can be modeled by assuming that the income is a continuous stream with a monthly rate of flow at time t given by

$$f(t) = 10{,}000e^{0.02t} \quad \text{(dollars)}$$

Find the total income from a Quick-Fix store for the first 2 years of operation.

5. A small brewery considers the output of its bottling machine as a continuous income stream with an annual rate of flow at time t given by

$$f(t) = 80e^{-0.1t}$$

in thousands of dollars. Find the income from this stream for the next 10 years.

6. A company that services a number of vending machines considers its income as a continuous stream with an annual rate of flow at time t given by

$$f(t) = 120e^{-0.4t}$$

in thousands of dollars. Find the income from this stream over the next 5 years.

7. A franchise models the profit from its store as a continuous income stream with a monthly rate of flow at time t given by

$$f(t) = 3000e^{0.004t} \quad \text{(dollars)}$$

When a new store opens, its manager is judged against the model, with special emphasis on the second half of the first year. Find the total profit for the second 6-month period ($t = 6$ to $t = 12$).

8. The Quick-Fix Car Service franchise has a continuous income stream with a monthly rate of flow modeled by $f(t) = 10{,}000e^{0.02t}$ (dollars). Find the total income for years 2 through 5.
9. A continuous income stream has an annual rate of flow at time t given by

$$f(t) = 12{,}000e^{0.04t} \quad \text{(dollars)}$$

If money is worth 8%, compounded continuously, find the present value of this stream for the next 8 years.

10. A continuous income stream has an annual rate of flow at time t given by

$$f(t) = 9000e^{0.12t} \quad \text{(dollars)}$$

Find the present value of this income stream for the next 10 years, if money is worth 6%, compounded continuously.

11. The income from an established chain of laundromats is a continuous stream with its annual rate of flow at time t given by $f(t) = 63{,}000$ (dollars). If money is worth 7%, compounded continuously, find the present value and future value of this chain over the next 5 years.
12. The profit from an insurance agency can be considered as a continuous income stream with an annual rate of flow at time t given by $f(t) = 84{,}000$ (dollars). Find the present value and future value of this agency over the next 12 years, if money is worth 8%, compounded continuously.
13. Suppose that a printing firm considers the production of its presses as a continuous income stream. If the annual rate of flow at time t is given by

$$f(t) = 97.5e^{-0.2(t+3)}$$

in thousands of dollars, and if money is worth 6%, compounded continuously, find the present value and future value of the presses over the next 10 years.

14. Suppose that a vending machine company is considering selling some of its machines. Suppose further that the income from these particular machines is a continuous stream with an annual rate of flow at time t given by

$$f(t) = 12e^{-0.4(t+3)}$$

in thousands of dollars. Find the present value and future value of the machines over the next 5 years if money is worth 10%, compounded continuously.

15. A 58-year-old couple is considering opening a business of their own. They will either purchase an established Gift and Card Shoppe or open a new Video Rental Palace. The Gift Shoppe has a continuous income stream with an annual rate of flow at time t given by

$$G(t) = 30{,}000 \quad \text{(dollars)}$$

and the Video Palace has a continuous income stream with a projected annual rate of flow at time t given by

$$V(t) = 21{,}600e^{0.08t} \quad \text{(dollars)}$$

The initial investment is the same for both businesses, and money is worth 10%, compounded continuously. Find the present value of each business over the next 7 years (until the couple reaches age 65) to see which is the better buy.

16. If the couple in Problem 15 plans to keep the business until age 70 (for the next 12 years), find each present value to see which business is the better buy in this case.

Consumer's Surplus

17. The demand function for a product is $p = 34 - x^2$. If the equilibrium price is \$9, what is the consumer's surplus?
18. The demand function for a product is $p = 100 - 4x$. If the equilibrium price is \$40, what is the consumer's surplus?
19. The demand function for a product is $p = 200/(x + 2)$. If the equilibrium quantity is 8 units, what is the consumer's surplus?
20. The demand function for a certain product is $p = 100/(1 + 2x)$. If the equilibrium quantity is 12 units, what is the consumer's surplus?
21. The demand function for a certain product is $p = 81 - x^2$ and the supply function is $p = x^2 + 4x + 11$. Find the equilibrium point and the consumer's surplus there.
22. The demand function for a product is $p = 49 - x^2$ and the supply function is $p = 4x + 4$. Find the equilibrium point and the consumer's surplus there.
23. If the demand function for a product is $p = 12/(x + 1)$ and the supply function is $p = 1 + 0.2x$, find the consumer's surplus under pure competition.
24. If the demand function for a good is $p = 110 - x^2$ and the supply function for it is $p = 2 - \frac{6}{5}x + \frac{1}{5}x^2$, find the consumer's surplus under pure competition.
25. A monopoly has a total cost function $C = 1000 + 120x + 6x^2$ for its product, which has demand function $p = 360 - 3x - 2x^2$. Find the consumer's surplus at the point where the monopoly has maximum profit.
26. A monopoly has a total cost function $C = 500 + 2x^2 + 10x$ for its product, which has demand function $p = -\frac{1}{3}x^2 - 2x + 30$. Find the consumer's surplus at the point where the monopoly has maximum profit.

Producer's Surplus

27. Suppose that the supply function for a good is $p = 4x^2 + 2x + 2$. If the equilibrium price is \$422, what is the producer's surplus there?
28. Suppose that the supply function for a good is $p = 0.1x^2 + 3x + 20$. If the equilibrium price is \$36, what is the producer's surplus there?
29. If the supply function for a commodity is $p = 10e^{x/3}$, what is the producer's surplus when 15 units are sold?
30. If the supply function for a commodity is $p = 40 + 100(x + 1)^2$, what is the producer's surplus at $x = 20$?
31. Find the producer's surplus for a product if its demand function is $p = 81 - x^2$ and its supply function is $p = x^2 + 4x + 11$.
32. Find the producer's surplus for a product if its demand function is $p = 49 - x^2$ and its supply function is $p = 4x + 4$.
33. Find the producer's surplus for a product with demand function $p = 12/(x + 1)$ and supply function $p = 1 + 0.2x$.
34. Find the producer's surplus for a product with demand function $p = 110 - x^2$ and supply function $p = 2 - \frac{6}{5}x + \frac{1}{5}x^2$.
35. The demand function for a certain product is $p = 144 - 2x^2$ and the supply function is $p = x^2 + 33x + 48$. Find the producer's surplus at the equilibrium point.
36. The demand function for a product is $p = 280 - 4x - x^2$ and the supply function for it is $p = 160 + 4x + x^2$. Find the producer's surplus at the equilibrium point.

13.5 Using Tables of Integrals

OBJECTIVE

- To use tables of integrals to evaluate certain integrals

APPLICATION PREVIEW

In 1982, the rate of increase of new AIDS cases was predicted to be

$$\frac{dN}{dt} = 2^t(700)$$

where t is the number of years after 1982. The number of AIDS cases was 1012 in 1982 ($t = 0$). Finding the predicted number of cases for 1986 ($t = 4$) requires evaluating the integral

$$\int 2^t(700)\, dt$$

Evaluating this integral is made easier by the existence of a formula such as those given in Table 13.2. The formulas in this table, and others listed in other resources, such as the Chemical Rubber Company's *Standard Mathematical Tables*, extend the number of integrals that can be evaluated. Using the formulas is not quite as easy as it may sound because finding the correct formula and using it properly may present problems. The examples in this section illustrate how some of these formulas are used.

TABLE 13.2 Integration Formulas

1. $\displaystyle\int u^n\, du = \frac{u^{n+1}}{n+1} + C, \quad \text{for } n \neq -1$
2. $\displaystyle\int \frac{du}{u} = \int u^{-1}\, du = \ln|u| + C$
3. $\displaystyle\int a^u\, du = a^u \log_a e + C = \frac{a^u}{\ln a} + C$
4. $\displaystyle\int e^u\, du = e^u + C$
5. $\displaystyle\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C$
6. $\displaystyle\int \sqrt{u^2 + a^2}\, du = \frac{1}{2}\left(u\sqrt{u^2 + a^2} + a^2 \ln\left|u + \sqrt{u^2 + a^2}\right|\right) + C$
7. $\displaystyle\int \sqrt{u^2 - a^2}\, du = \frac{1}{2}\left(u\sqrt{u^2 - a^2} - a^2 \ln\left|u + \sqrt{u^2 - a^2}\right|\right) + C$
8. $\displaystyle\int \frac{du}{\sqrt{u^2 + a^2}} = \ln\left|u + \sqrt{u^2 + a^2}\right| + C$
9. $\displaystyle\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$
10. $\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$
11. $\displaystyle\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 + u^2}}{u}\right| + C$

12. $\displaystyle\int \frac{u\,du}{au+b} = \frac{u}{a} - \frac{b}{a^2}\ln|au+b| + C$

13. $\displaystyle\int \frac{du}{u(au+b)} = \frac{1}{b}\ln\left|\frac{u}{au+b}\right| + C$

14. $\displaystyle\int \ln u\,du = u(\ln u - 1) + C$

15. $\displaystyle\int \frac{u\,du}{(au+b)^2} = \frac{1}{a^2}\left(\ln|au+b| + \frac{b}{au+b}\right) + C$

16. $\displaystyle\int u\sqrt{au+b}\,du = \frac{2(3au-2b)(au+b)^{3/2}}{15a^2} + C$

17. $\displaystyle\int u\,dv = uv - \int v\,du$

EXAMPLE 1

Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2+4}}$.

Solution

We must find a formula in Table 13.2 that is of the same form as this integral. We see that formula 8 has the desired form, *if* we let $u = x$ and $a = 2$. Thus

$$\int \frac{dx}{\sqrt{x^2+4}} = \ln|x + \sqrt{x^2+4}| + C$$

EXAMPLE 2

Evaluate $\displaystyle\int_1^2 \frac{dx}{x^2+2x}$.

Solution

There does not appear to be any formula having exactly the same form as our integral. But if we rewrite our integral as

$$\int_1^2 \frac{dx}{x(x+2)}$$

we see that formula 13 will work. Letting $u = x$, $a = 1$, and $b = 2$, we get

$$\begin{aligned}\int_1^2 \frac{dx}{x(x+2)} = \frac{1}{2}\ln\left|\frac{x}{x+2}\right|\Bigg|_1^2 &= \frac{1}{2}\ln\left|\frac{2}{4}\right| - \frac{1}{2}\ln\left|\frac{1}{3}\right| \\ &= \frac{1}{2}\left(\ln\frac{1}{2} - \ln\frac{1}{3}\right) \\ &= \frac{1}{2}\ln\frac{3}{2} \\ &= \frac{1}{2}\ln 1.5\end{aligned}$$

Although the formulas in Table 13.2 are given in terms of the variable u, they may be used with any variable.

EXAMPLE 3

Evaluate $\int \frac{dq}{9-q^2}$.

Solution

The formula that applies in this case is formula 5, with $a = 3$ and $u = q$. Then

$$\int \frac{dq}{9-q^2} = \frac{1}{2 \cdot 3} \ln \left|\frac{3+q}{3-q}\right| + C = \frac{1}{6} \ln \left|\frac{3+q}{3-q}\right| + C$$

EXAMPLE 4

Evaluate $\int \ln (2x + 1)\, dx$.

Solution

This integral has the form of formula 14, with $u = 2x + 1$. But if $u = 2x + 1$, du must be represented by the differential of $2x + 1$ (that is, $2\,dx$). Thus

$$\int \ln (2x+1)\, dx = \frac{1}{2} \int \ln (2x+1)(2\, dx)$$

$$= \frac{1}{2}(2x+1)[\ln (2x+1) - 1] + C$$

CHECKPOINT

1. Can both $\int \frac{dx}{\sqrt{x^2-4}}$ and $-\int \frac{dx}{\sqrt{4-x^2}}$ be evaluated with formula 10 in Table 13.2?
2. Determine the formula used to evaluate $\int \frac{3x}{4x-5}\, dx$, and show how the formula would be applied.
3. True or false: In order for us to use a formula, the given integral must correspond exactly to the formula, including du.
4. True or false: $\int \frac{dx}{x^2(3x^2-7)}$ can be evaluated with formula 13.
5. True or false: $\int \frac{dx}{(6x+1)^2}$ can be evaluated with either formula 1 or formula 15.
6. True or false: $\int \sqrt{x^2+4}\, dx$ can be evaluated with formula 1, formula 6, or formula 16.

EXAMPLE 5

Evaluate $\int_1^2 \frac{dx}{x\sqrt{81-9x^2}}$.

Solution

This integral is similar to that of formula 9 in Table 13.2. Letting $a = 9$, letting $u = 3x$, and multiplying the numerator and denominator by 3 give the proper form.

$$\begin{aligned}\int_1^2 \frac{dx}{x\sqrt{81-9x^2}} &= \int_1^2 \frac{3\,dx}{3x\sqrt{81-9x^2}}\\ &= -\frac{1}{9}\ln\left|\frac{9+\sqrt{81-9x^2}}{3x}\right|\Bigg|_1^2\\ &= \left[-\frac{1}{9}\ln\left(\frac{9+\sqrt{45}}{6}\right)\right]-\left[-\frac{1}{9}\ln\left(\frac{9+\sqrt{72}}{3}\right)\right]\\ &\approx 0.0889\end{aligned}$$

Remember that the formulas given in Table 13.2 represent only a very small sample of all possible integration formulas. Additional formulas may be found in books of mathematical tables.

EXAMPLE 6

In 1982, the rate of increase of new AIDS cases was predicted to be

$$\frac{dN}{dt} = 2^t(700)$$

where t is the number of years since 1982. If the number of AIDS cases was 1012 in 1982 ($t = 0$), what would be the predicted number of cases for 1986 ($t = 4$), before added attention was given to prevention?

Solution

Finding the predicted number of cases N in 1986 requires evaluating the integral

$$N = \int 2^t(700)\,dt = 700\int 2^t\,dt$$

and using the fact that $N = 1012$ when $t = 0$ to find the constant C. The integral has the form of formula 3 with $a = 2$ and $u = t$.

$$N = 700\int 2^t\,dt = 700\cdot\frac{2^t}{\ln 2} + C$$

Using $N = 1012$ when $t = 0$ gives

$$\begin{aligned}1012 &= 700\cdot\frac{2^0}{\ln 2} + C\\ 1012 &= 1010 + C\\ C &= 2,\end{aligned}$$

so

$$N = 700\cdot\frac{2^t}{\ln 2} + 2$$

The number of cases in 1986 (when $t = 4$) is predicted by this model to be

$$N = 700\cdot\frac{2^4}{\ln 2} + 2 = 16{,}160$$

The actual number of cases in 1986 was 13,898. Paying more attention to prevention has reduced the rate of spread of the disease somewhat, changing the function that models the number of cases. Models using more recent data are discussed in other sections of the text.

Graphing Utilities Numerical integration with a graphing calculator or computer software is especially useful in evaluating definite integrals when the formulas for the integral are difficult to use or are not available. For example, evaluating the definite integral in Example 5 above with the numerical integration feature of a graphing utility gives 0.08892484. The decimal approximation of the answer in Example 5 is 0.088924836, so the numerical approximation of the answer agrees for the first eight decimal places.

CHECKPOINT SOLUTIONS

1. No. Although $\int \frac{dx}{\sqrt{x^2-4}}$ can be evaluated with formula 10 from Table 13.2, $\int \frac{dx}{\sqrt{4-x^2}}$ cannot, because $\sqrt{4-x^2}$ cannot be rewritten in the form $\sqrt{u^2-a^2}$ as is needed for this formula to be used.
2. Use formula 12 with $u = x$, $a = 4$, $b = -5$, and $du = dx$.

$$\int \frac{3x}{4x-5}\,dx = 3\int \frac{x\,dx}{4x-5} = 3\int \frac{u\,du}{au+b} = 3\left(\frac{u}{a} - \frac{b}{a^2}\ln|au+b| + C\right)$$
$$= \frac{3x}{4} + \frac{15}{16}\ln|4x-5| + C$$

3. True. An exact correspondence with the formula and du is necessary.
4. False. With $u = x^2$, we must have $du = 2x\,dx$. In this problem there is no x with dx, so the problem cannot correspond to formula 13.
5. False. The integral can be evaluated only with formula 1, not with formula 15. The correspondence is $u = 6x + 1$, $du = 6\,dx$, and $n = -2$.
6. False. The integral can be evaluated only with formula 6, with $u = x$, $du = dx$, and $a = 2$.

EXERCISE 13.5

Evaluate the integrals in Problems 1–32.

1. $\int \frac{dx}{16-x^2}$
2. $\int \frac{dx}{x(3x+5)}$
3. $\int_1^4 \frac{dx}{x\sqrt{9+x^2}}$
4. $\int \frac{dx}{x\sqrt{9-x^2}}$
5. $\int \ln w\,dw$
6. $\int \frac{dv}{v(3v+8)}$
7. $\int_0^2 \frac{q\,dq}{6q+9}$
8. $\int_1^5 \frac{dq}{q\sqrt{25+q^2}}$
9. $\int 3^x\,dx$
10. $\int_0^3 \sqrt{x^2+16}\,dx$
11. $\int_5^7 \sqrt{x^2-25}\,dx$
12. $\int \frac{x\,dx}{(3x+2)^2}$
13. $\int w\sqrt{4w+5}\,dw$
14. $\int \frac{dy}{\sqrt{9+y^2}}$
15. $\int x\,5^{x^2}\,dx$
16. $\int \sqrt{9x^2+4}\,dx$
17. $\int_0^3 x\sqrt{x^2+4}\,dx$
18. $\int x\sqrt{x^4-36}\,dx$
19. $\int \frac{5\,dx}{x\sqrt{4-9x^2}}$
20. $\int x e^{x^2}\,dx$
21. $\int \frac{dx}{\sqrt{9x^2-4}}$
22. $\int \frac{dx}{25-4x^2}$
23. $\int_5^6 \frac{dx}{x^2-16}$
24. $\int_0^1 \frac{x\,dx}{6-5x}$

25. $\displaystyle\int \frac{dx}{\sqrt{(3x+1)^2+1}}$ 26. $\displaystyle\int \frac{dx}{9-(2x+3)^2}$

27. $\displaystyle\int_0^3 x\sqrt{(x^2+1)^2+9}\,dx$ 28. $\displaystyle\int_1^e x \ln x^2\,dx$

29. $\displaystyle\int \frac{x\,dx}{7-3x^2}$ 30. $\displaystyle\int_0^1 \frac{e^x}{1+e^x}\,dx$

31. $\displaystyle\int \frac{dx}{\sqrt{4x^2+7}}$ 32. $\displaystyle\int e^{2x}\sqrt{3e^x+1}\,dx$

Use formulas or numerical integration with a graphing calculator or computer to evaluate the definite integral in Problems 33–36.

33. $\displaystyle\int_2^3 \frac{e^{\sqrt{x-1}}}{\sqrt{x-1}}\,dx$ 34. $\displaystyle\int_2^4 \frac{3x}{\sqrt{x^4-9}}\,dx$

35. $\displaystyle\int_0^1 \frac{x^3\,dx}{(4x^2+5)^2}$ 36. $\displaystyle\int_0^1 (e^x+1)^3 e^x\,dx$

Applications

37. ***Producer's surplus*** If the supply function for a commodity is $p = 40 + 100 \ln (x+1)^2$, what is the producer's surplus at $x = 20$?

38. ***Consumer's surplus*** If the demand function for a good is $p = 5000e^{-x} + 4$, where x is the number of hundreds of bushels of wheat, what is the consumer's surplus at $x = 7, p = 9.10$?

39. ***Cost*** (a) If the marginal cost for a good is $\overline{MC} = \sqrt{x^2+9}$ and if the fixed cost is \$300, what is the total cost function of the good?
    (b) What is the total cost of producing 4 units of this good?

40. ***Consumer's surplus*** Suppose that the demand function for an appliance is

$$p = \frac{400q + 400}{(q+2)^2}$$

What is the consumer's surplus if the equilibrium price is \$19 and the equilibrium quantity is 18?

41. ***Income streams*** Suppose that when a new oil well is opened, its production is viewed as a continuous income stream with monthly rate of flow

$$f(t) = 10 \ln (t+1) - 0.1t$$

where t is time in months and $f(t)$ is in thousands of dollars. Find the total income over the next 10 years (120 months).

42. ***Spread of disease*** An isolated community of 1000 people susceptible to a certain disease is exposed when one member returns carrying the disease. If x represents the number infected with the disease at time t (in days), then the rate of change of x is proportional to the product of the number infected, x, and the number still susceptible, $1000 - x$. That is,

$$\frac{dx}{dt} = kx(1000-x) \quad \text{or} \quad \frac{dx}{x(1000-x)} = k\,dt$$

(a) If $k = 0.001$, integrate both sides to solve this differential equation.
(b) Find how long it will be before half the population of the community is affected.
(c) Find the rate of new cases, dx/dt, every other day for the first 13 days.

13.6 Integration by Parts

OBJECTIVE

- *To evaluate integrals using the method of integration by parts*

APPLICATION PREVIEW

If the value of oil produced by a piece of oil extraction equipment is considered a continuous income stream with an annual rate of flow at time t given by

$$f(t) = 300{,}000 - 2500t, \qquad 0 \le t \le 10$$

and if money can be invested at 8%, compounded continuously, then the present value of the piece of equipment is

$$\int_0^{10} (300{,}000 - 2500t)e^{-0.08t}\,dt$$
$$= 300{,}000 \int_0^{10} e^{-0.08t}\,dt - 2500 \int_0^{10} te^{-0.08t}\,dt$$

The first integral can be evaluated with the formula for the integral of $e^u\,du$. Evaluating the second integral can be done by using **integration by parts,** which is a special technique that uses formula 17 from Table 13.2. This technique involves rewriting an integral in a form that can be evaluated.

Formula 17 in Table 13.2 follows from the Product Rule for derivatives (actually differentials) as follows.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{so} \quad d(uv) = u\,dv + v\,du$$

Rearranging the differential form and integrating both sides give the following.

$$u\,dv = d(uv) - v\,du$$
$$\int u\,dv = \int d(uv) - \int v\,du$$
$$\int u\,dv = uv - \int v\,du$$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Integration by parts is very useful if the integral we seek to evaluate can be treated as the product of one function, u, and the differential dv of a second function, so that the two integrals $\int dv$ and $\int v \cdot du$ can be found. Let us consider an example using this method.

EXAMPLE 1

Evaluate $\int xe^x\,dx$.

Solution

We cannot evaluate this integral using methods we have learned. But we can "split" the integrand into two parts, setting one part equal to u and the other part equal to dv. This "split" must be done in such a way that $\int dv$ and $\int v\,du$ can be evaluated. Letting $u = x$ and letting $dv = e^x\,dx$ are possible choices. If we make these choices, we have

$$u = x \qquad dv = e^x\,dx$$
$$du = 1\,dx \qquad v = \int e^x\,dx = e^x$$

Then

$$\begin{aligned}\int xe^x\,dx &= u \cdot v - \int v\,du \\ &= x \cdot e^x - \int e^x\,dx \\ &= xe^x - e^x + C\end{aligned}$$

We see that choosing $u = x$ and $dv = e^x\,dx$ worked in evaluating $\int xe^x\,dx$ in Example 1. If we had chosen $u = e^x$ and $dv = x\,dx$, the results would not have been so successful.

How can we select u and dv to make integration by parts work? There are no general rules for separating the integrand into u and dv, but the goal is to select a dv that is integrable and will result in an $\int v\,du$ that is also integrable. There are usually just two reasonable choices, and it may be necessary to try both. Practice will enhance your insight and lead to increasingly successful choices. Consider the following examples.

EXAMPLE 2

Evaluate $\int x \ln x\,dx$.

Solution

Let $u = \ln x$ and $dv = x\,dx$. Then

$$du = \frac{1}{x}\,dx \quad \text{and} \quad v = \frac{x^2}{2}$$

so

$$\begin{aligned}\int x \ln x\,dx &= u \cdot v - \int v\,du \\ &= (\ln x)\frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx \\ &= \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx \\ &= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C\end{aligned}$$

Note that letting $dv = \ln x\,dx$ would lead to great difficulty in evaluating $\int dv$ and $\int v\,du$, so it would not be wise choice.

EXAMPLE 3

Evaluate $\int \ln x^2\,dx$.

Solution

It is frequently good practice to let expressions involving logarithms be part of u in integrating by parts, because the derivatives of logarithmic expressions are usually simple.

In this problem, we can let $u = \ln x^2$. Thus

$$\begin{aligned} u &= \ln x^2 & dv &= dx \\ du &= \frac{2x}{x^2}\,dx = \frac{2}{x}\,dx & v &= x \end{aligned}$$

Then

$$\begin{aligned}\int \ln x^2\,dx &= x \ln x^2 - \int x \cdot \frac{2}{x}\,dx \\ &= x \ln x^2 - 2x + C\end{aligned}$$

Note that if we write $\ln x^2$ as $2 \ln x$, we can also evaluate this integral using formula 14 in Table 13.2, so integration by parts would not be needed.

CHECKPOINT

1. True or false: In evaluating $\int u\, dv$ by parts,
   (a) the parts u and dv are selected and the parts du and v are calculated.
   (b) the differential (often dx) is always chosen as part of dv.
   (c) the parts du and v are found from u and dv as follows:
   $$du = u'\, dx \qquad \text{and} \qquad v = \int dv$$
   (d) For $\displaystyle\int \frac{3x}{e^{2x}}\, dx$, we could choose $u = 3x$ and $dv = e^{2x}\, dx$.
2. For $\displaystyle\int \frac{\ln x}{x^4}\, dx$,
   (a) identify u and dv.
   (b) find du and v.
   (c) complete the evaluation of the integral.

Sometimes it is necessary to repeat the integration by parts to complete the evaluation. As before, the goal is to produce a new integral that is simpler.

EXAMPLE 4

Evaluate $\int x^2 e^{2x}\, dx$.

Solution

Let $u = x^2$ and $dv = e^{2x}\, dx$, so $du = 2x\, dx$ and $v = \frac{1}{2}e^{2x}$. Then

$$\int x^2 e^{2x}\, dx = \frac{1}{2}x^2 e^{2x} - \int x e^{2x}\, dx$$

We cannot evaluate $\int xe^{2x}\, dx$ directly, but this new integral is simpler than the original, and a second integration by parts will be successful. Letting $u = x$ and $dv = e^{2x}\, dx$ gives $du = dx$ and $v = \frac{1}{2}e^{2x}$. Thus

$$\begin{aligned}\int x^2 e^{2x}\, dx &= \frac{1}{2}x^2 e^{2x} - \left(\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\, dx\right)\\ &= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C\\ &= \frac{1}{4}e^{2x}(2x^2 - 2x + 1) + C\end{aligned}$$

The most obvious choices for u and dv are not always the correct ones, as the following example shows. Integration by parts still requires some trial and error.

EXAMPLE 5

Evaluate $\int x^3\sqrt{x^2 + 1}\, dx$.

Solution

Because x^3 can be integrated easily, it may appear that the new integral would be simplified if we let $u = \sqrt{x^2+1}$ and $dv = x^3$. But then du would be $\frac{1}{2}(x^2+1)^{-1/2}2x\,dx$, making $\int v\,du$ more complicated than the original integral. However, we can use $\sqrt{x^2+1}$ as part of dv, and we can evaluate $\int dv$ if we let $dv = x\sqrt{x^2+1}\,dx$. Then

$$u = x^2 \qquad dv = (x^2+1)^{1/2}x\,dx$$

$$du = 2x\,dx \qquad v = \int (x^2+1)^{1/2}(x\,dx) = \frac{1}{2}\int (x^2+1)^{1/2}(2x\,dx)$$

$$v = \frac{1}{2}\frac{(x^2+1)^{3/2}}{3/2} = \frac{1}{3}(x^2+1)^{3/2}$$

Then $$\int x^3\sqrt{x^2+1}\,dx = \frac{x^2}{3}(x^2+1)^{3/2} - \int \frac{1}{3}(x^2+1)^{3/2}(2x\,dx)$$

$$= \frac{x^2}{3}(x^2+1)^{3/2} - \frac{1}{3}\frac{(x^2+1)^{5/2}}{5/2} + C$$

$$= \frac{x^2}{3}(x^2+1)^{3/2} - \frac{2}{15}(x^2+1)^{5/2} + C$$

We now consider the problem in the Application Preview.

EXAMPLE 6

Suppose that the value of oil produced by a piece of oil extraction equipment is considered a continuous income stream with an annual rate of flow at time t given by

$$f(t) = 300{,}000 - 2500t, \qquad 0 \le t \le 10$$

and that money is worth 8%, compounded continuously. Find the present value of the piece of equipment.

Solution

The present value of the piece of equipment is given by

$$\int_0^{10} (300{,}000 - 2500t)e^{-0.08t}\,dt$$

$$= 300{,}000\int_0^{10} e^{-0.08t}\,dt - 2500\int_0^{10} te^{-0.08t}\,dt$$

$$= \frac{300{,}000}{-0.08}e^{-0.08t}\Big|_0^{10} - 2500\int_0^{10} te^{-0.08t}\,dt$$

The value of the first integral is

$$= \frac{300{,}000}{-0.08}e^{-0.08t}\Big|_0^{10} = \frac{300{,}000}{-0.08}e^{-0.8} - \frac{300{,}000}{-0.08}$$

$$\approx -1{,}684{,}983.615 + 3{,}750{,}000$$

$$= 2{,}065{,}016.385$$

The second of these integrals can be evaluated by using integration by parts, with $u = t$ and $dv = e^{-0.08t}\,dt$. Then $du = 1\,dt$ and $v = \dfrac{e^{-0.08t}}{-0.08}$, and this integral is

$$
\begin{aligned}
-2500\int_0^{10} te^{-0.08t}\,dt &= -2500\frac{te^{-0.08t}}{-0.08}\bigg|_0^{10} + 2500\int_0^{10}\frac{e^{-0.08t}}{-0.08}dt\\
&= \frac{2500}{0.08}te^{-0.08t}\bigg|_0^{10} + \frac{2500}{0.0064}e^{-0.08t}\bigg|_0^{10}\\
&= \frac{2500}{0.08}10e^{-0.8} + \frac{2500}{0.0064}e^{-0.8} - \frac{2500}{0.0064}\\
&\approx -74{,}690.572
\end{aligned}
$$

Thus the sum of the integrals is

$$2{,}065{,}016.385 + (-74{,}690.572) = 1{,}990{,}325.823$$

and the present value of this piece of equipment is \$1,990,325.82.

One further note about integration by parts. It can be very useful on certain types of problems, but not on all types. Don't attempt to use integration by parts when easier methods are available.

Graphing Utilities Using numerical integration with a graphing calculator or computer software to evaluate the integral in Example 6 gives the present value of \$1,990,325.80, so this answer is the same as that found in Example 6, to the nearest dollar.

CHECKPOINT SOLUTIONS

1. (a) True (b) True (c) True
   (d) False. The product of u and dv must equal the original integrand. Rewrite as
   $$\int \frac{3x}{e^{2x}}\,dx = \int 3xe^{-2x}\,dx$$
   and then choose $u = 3x$ and $dv = e^{-2x}\,dx$.
2. (a) $u = \ln x$ and $dv = x^{-4}\,dx$
   (b) $du = \dfrac{1}{x}\,dx$ and $v = \displaystyle\int x^{-4}\,dx = \frac{x^{-3}}{-3}$
   (c)
   $$
   \begin{aligned}
   \int \frac{\ln x}{x^4}\,dx &= uv - \int v\,du\\
   &= (\ln x)\left(\frac{x^{-3}}{-3}\right) - \int \frac{x^{-3}}{-3}\cdot\frac{1}{x}\,dx\\
   &= \frac{-\ln x}{3x^3} + \frac{1}{3}\int x^{-4}\,dx\\
   &= -\frac{\ln x}{3x^3} + \frac{1}{3}\left(\frac{x^{-3}}{-3}\right) + C'\\
   &= -\frac{\ln x}{3x^3} - \frac{1}{9x^3} + C
   \end{aligned}
   $$

EXERCISE 13.6

In Problems 1–16, use integration by parts to evaluate the integral.

1. $\int xe^{2x}\,dx$
2. $\int xe^{-x}\,dx$
3. $\int x^2 \ln x\,dx$
4. $\int x^3 \ln x\,dx$
5. $\int_4^6 q\sqrt{q-4}\,dq$
6. $\int_0^1 y(1-y)^{3/2}\,dy$
7. $\int \frac{\ln x}{x^2}\,dx$
8. $\int \frac{\ln (x-1)}{\sqrt{x-1}}\,dx$
9. $\int_1^e \ln x\,dx$
10. $\int \frac{x}{\sqrt{x-3}}\,dx$
11. $\int x \ln (2x-3)\,dx$
12. $\int x \ln (4x)\,dx$
13. $\int q^3\sqrt{q^2-3}\,dq$
14. $\int \frac{x^3}{\sqrt{9-x^2}}\,dx$
15. $\int_0^4 x^3\sqrt{x^2+9}\,dx$
16. $\int \sqrt{x}\ln x\,dx$

In Problems 17–24, use integration by parts to evaluate the integral. Note that evaluation will require integration by parts more than once.

17. $\int x^2e^{-x}\,dx$
18. $\int_0^1 x^2e^x\,dx$
19. $\int_0^2 x^3e^{x^2}\,dx$
20. $\int x^3e^x\,dx$
21. $\int x^3 \ln^2 x\,dx$
22. $\int \frac{x^2}{\sqrt{x-3}}\,dx$
23. $\int e^{2x}\sqrt{e^x+1}\,dx$
24. $\int_1^2 (\ln x)^2\,dx$

In Problems 25–30, match each of the integrals with the formula or method (I–IV) that should be used to evaluate it. Then evaluate the integral.

I. Integration by parts II. $\int e^u\,du$ III. $\int \frac{du}{u}$ IV. $\int u^n\,du$

25. $\int xe^{x^2}\,dx$
26. $\int \frac{x}{\sqrt{9-x^2}}\,dx$
27. $\int e^x\sqrt{e^x+1}\,dx$
28. $\int 4x^2\,e^{x^3}\,dx$
29. $\int_0^4 \frac{t}{e^t}\,dt$
30. $\int x^2\sqrt{x-1}\,dx$

Applications

31. ***Producer's surplus*** If the supply function for a commodity is $p = 30 + 50 \ln (2x+1)^2$, what is the producer's surplus at $x = 30$?

32. ***Cost*** If the marginal cost function for a product is $\overline{MC} = 1 + 3 \ln (x+1)$, and if the fixed cost is \$100, find the total cost function.

33. ***Present value*** Suppose that a machine's production can be considered as a continuous income stream with annual rate of flow at time t given by

$$f(t) = 10{,}000 - 500t \quad \text{(dollars)}$$

If money is worth 10%, compounded continuously, find the present value of the machine over the next 5 years.

34. ***Present value*** Suppose that the production of a machine used to mine coal is considered as a continuous income stream with annual rate of flow at time t given by

$$f(t) = 280{,}000 - 14{,}000t \quad \text{(dollars)}$$

If money is worth 7%, compounded continuously, find the present value of this machine over the next 8 years.

35. ***Income distribution*** Suppose the Lorenz curve for the distribution of income of a certain country is given by

$$y = xe^{x-1}$$

Find the Gini coefficient of income.

36. ***Income streams*** Suppose the income from an Internet access business is a continuous income stream with annual rate of flow given by

$$f(t) = 100te^{-0.1t}$$

in thousands of dollars. Find the total income over the next 10 years.

13.7 Improper Integrals and Their Applications

OBJECTIVES

- *To evaluate improper integrals*
- *To apply improper integrals to continuous income streams and to probability density functions*

APPLICATION PREVIEW

We saw in Section 13.4, "Applications of Definite Integrals in Business and Economics," that the present value of a continuous income stream over a fixed number of years can be found by using a definite integral. When this notion is extended to an infinite time interval, the result is called the **capital value** of the income stream and is given by

$$\textbf{Capital value} = \int_0^{\infty} f(t)e^{-rt}\,dt$$

where $f(t)$ is the annual rate of flow at time t, and r is the annual interest rate, compounded continuously. This is called an **improper integral.**

Other applications of calculus to business and to statistics use improper integrals to find the area of a region that extends infinitely to the left or right along the x-axis (see Figure 13.24).

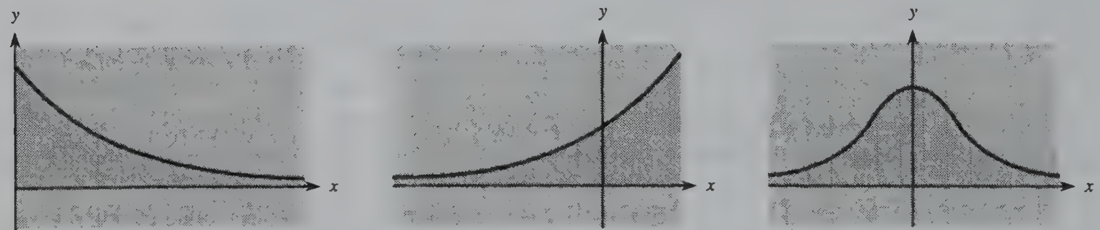


Figure 13.24

Let us consider how to find the area between the curve $y = 1/x^2$ and x-axis to the right of $x = 1$.

To find the area under this curve from $x = 1$ to $x = b$, where b is any number greater than 1 (see Figure 13.25), we evaluate

$$A = \int_1^b \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right|_1^b = \frac{-1}{b} - \left(\frac{-1}{1}\right) = 1 - \frac{1}{b}$$

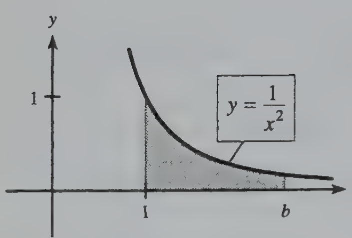


Figure 13.25

Note that the larger b is, the closer the area is to 1. If $b = 100$, $A = 0.99$; if $b = 1000$, $A = 0.999$; and if $b = 1{,}000{,}000$, $A = 0.999999$.

We can represent the area of the region under $1/x^2$ to the right of 1 using the notation

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left(1 - \frac{1}{b}\right)$$

where $\lim\limits_{b\to\infty}$ represents the limit as b gets larger without bound. Clearly,

$$\lim_{b\to\infty} \frac{1}{b} = 0$$

so

$$\lim_{b\to\infty} \left(1 - \frac{1}{b}\right) = 1$$

Thus the area under the curve $y = 1/x^2$ to the right of $x = 1$ is 1.

In general, we define the area under a curve $y = f(x)$ to the right of $x = a$, with $f(x) \geq 0$, to be

$$\text{Area} = \lim_{b\to\infty} (\text{area from } a \text{ to } b) = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

This motivates the definition that follows.

Improper Integral

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

If the limit defining the improper integral is a unique finite number, we say that the integral *converges;* otherwise, we say that the integral *diverges.*

EXAMPLE 1

Evaluate the following improper integrals, if they converge.

(a) $\displaystyle\int_1^{\infty} \frac{1}{x^3}\,dx$ (b) $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$

Solution

(a) $$\int_1^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^3}\,dx = \lim_{b\to\infty} \left[\frac{x^{-2}}{-2}\right]_1^b$$

$$= \lim_{b\to\infty} \left[\frac{-1}{2b^2} - \left(\frac{-1}{2(1)^2}\right)\right]$$

$$= \lim_{b\to\infty} \left(\frac{-1}{2b^2} + \frac{1}{2}\right)$$

Now as $b \to \infty$, $\dfrac{-1}{2b^2} \to 0$, so the limit, and the integral, converge to $\frac{1}{2}$. That is,

$$\int_1^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2}$$

(b) $$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \Big[\ln|x|\Big]_1^b$$

$$= \lim_{b\to\infty} (\ln b - \ln 1)$$

Now $\ln b$ increases without bound as $b \to \infty$, so the limit, and the integral, diverge. We write this as

$$\int_1^{\infty} \frac{1}{x}\,dx = \infty$$

From Example 1 we can conclude that the area under the curve $y = 1/x^3$ to the right of $x = 1$ is $\frac{1}{2}$, whereas the corresponding area under the curve $y = 1/x$ is infinite. (We have already seen that the corresponding area under $y = 1/x^2$ is 1.)

As Figure 13.26 shows, the graphs of the curves look similar, but the graph of $1/x^2$ gets "close" to the x-axis much more rapidly than the graph of $1//x$. The area under $y = 1/x$ does not converge to a finite number because as $x \to \infty$ the graph of $1/x$ does not approach the x-axis rapidly enough.

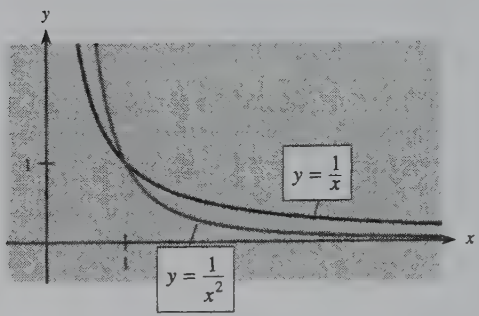


Figure 13.26

EXAMPLE 2

Suppose that an organization wants to establish a trust fund that will provide a continuous income stream with an annual rate of flow at time t given by $f(t) = 10{,}000$. If the interest rate remains at 10%, compounded continuously, find the capital value of the fund.

Solution

The capital value of the fund is given by

$$\int_0^\infty f(t)\, e^{-rt}\, dt$$

where $f(t)$ is the annual rate of flow at time t, and r is the annual interest rate, compounded continuously.

$$\begin{aligned}\int_0^\infty 10{,}000e^{-0.10t}\, dt &= \lim_{b\to\infty} \int_0^b 10{,}000e^{-0.10t}\, dt \\ &= \lim_{b\to\infty} \Big[-100{,}000e^{-0.10t}\Big]_0^b \\ &= \lim_{b\to\infty} \left(\frac{-100{,}000}{e^{0.10b}} + 100{,}000\right) \\ &= 100{,}000\end{aligned}$$

Thus the capital value of the fund is \$100,000.

Another term for a fund such as the one in Example 2 is a **perpetuity.** Usually the rate of flow of a perpetuity is a constant. If the rate of flow is a constant A, it can be shown that the capital value is given by A/r (see Problem 37 in the exercise set).

CHECKPOINT

1. True or false:
   (a) $\lim_{b\to+\infty} \frac{1}{b^p} = 0$ if $p > 0$ (b) $\lim_{b\to+\infty} b^p = +\infty$ if $p > 0$
   (c) $\lim_{b\to+\infty} e^{-pb} = 0$ if $p > 0$
2. Evaluate the following (if they exist).
   (a) $\int_1^{\infty} \frac{1}{x^{4/3}}\,dx = \lim_{b\to\infty} \int_1^b x^{-4/3}\,dx$ (b) $\int_0^{\infty} \frac{dx}{\sqrt{x+1}}$

A second improper integral has the form

$$\int_{-\infty}^{b} f(x)\,dx$$

and is defined by

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to\infty} \int_{-a}^{b} f(x)\,dx$$

The integral converges if the limit is finite. In addition, the improper integral

$$\int_{-\infty}^{\infty} f(x)\,dx$$

is defined by

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to\infty} \int_{-a}^{c} f(x)\,dx + \lim_{b\to\infty} \int_{c}^{b} f(x)\,dx$$

for any finite constant c. (0 is often used for c.) If both limits are finite, the improper integral converges; otherwise, it diverges.

EXAMPLE 3

Evaluate the following integrals.

(a) $\int_{-\infty}^{4} e^{3x}\,dx$ (b) $\int_{-\infty}^{\infty} \frac{x^3}{(x^4+3)^2}\,dx$

Solution

(a)
$$\int_{-\infty}^{4} e^{3x}\,dx = \lim_{a\to\infty} \int_{-a}^{4} e^{3x}\,dx$$
$$= \lim_{a\to\infty} \left[\left(\frac{1}{3}\right)e^{3x}\right]_{-a}^{4}$$
$$= \lim_{a\to\infty} \left[\left(\frac{1}{3}\right)e^{12} - \left(\frac{1}{3}\right)e^{-3a}\right]$$

$$= \lim_{a\to\infty}\left[\left(\frac{1}{3}\right)e^{12} - \left(\frac{1}{3}\right)\left(\frac{1}{e^{3a}}\right)\right]$$

$$= \frac{1}{3}e^{12} \qquad \text{(because } 1/e^{3a} \to 0 \text{ as } a \to \infty\text{)}$$

(b) $$\int_{-\infty}^{\infty} \frac{x^3}{(x^4+3)^2}\,dx = \lim_{a\to\infty}\int_{-a}^{0}\frac{x^3}{(x^4+3)^2}\,dx + \lim_{b\to\infty}\int_{0}^{b}\frac{x^3}{(x^4+3)^2}\,dx$$

$$= \lim_{a\to\infty}\left[\frac{1}{4}\,\frac{(x^4+3)^{-1}}{-1}\right]_{-a}^{0} + \lim_{b\to\infty}\left[\frac{1}{4}\,\frac{(x^4+3)^{-1}}{-1}\right]_{0}^{b}$$

$$= \lim_{a\to\infty}\left[-\frac{1}{4}\left(\frac{1}{3}-\frac{1}{a^4+3}\right)\right] + \lim_{b\to\infty}\left[-\frac{1}{4}\left(\frac{1}{b^4+3}-\frac{1}{3}\right)\right]$$

$$= -\frac{1}{12} + 0 + 0 + \frac{1}{12} = 0$$

$$\left(\text{since } \lim_{a\to\infty}\frac{1}{a^4+3} = 0 \quad \text{and} \quad \lim_{b\to\infty}\frac{1}{b^4+3} = 0\right)$$

In Section 13.2, "The Definite Integral," we calculated the probability that a computer component will last between 3 and 5 years when the probability density function for the life span is $f(x) = 0.10e^{-0.10x}$, where x is the number of years, $x \geq 0$. The probability that the component will last more than 3 years is given by the improper integral

$$\int_{3}^{\infty} 0.10e^{-0.10x}\,dx$$

which gives

$$\lim_{b\to\infty}\int_{3}^{b} 0.10e^{-0.10x}\,dx = \lim_{b\to\infty}\left[-e^{-0.10x}\right]_3^b$$

$$= \lim_{b\to\infty}\left(-e^{-0.10b} + e^{-0.3}\right)$$

$$= e^{-0.3} = 0.7408$$

We noted in Chapter 8, "Further Topics in Probability" that the sum of the probabilities for a probability distribution (a probability density function) equals 1. In particular, we stated that the area under the normal probability curve is 1.

Probability Density Function

In general, if $f(x) \geq 0$ for all x, then f is a probability density function for a continuous random variable if and only if

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

The complete probability density function for the life span of the computer component mentioned above is

$$f(x) = \begin{cases} 0.10e^{-0.10x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

We can verify that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

for this function.

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} 0.10e^{-0.10x}\,dx \\
&= 0 + \lim_{b\to\infty}\int_0^b 0.1e^{-0.10x}\,dx \\
&= \lim_{b\to\infty}\left[-e^{-0.10x}\right]_0^b \\
&= \lim_{b\to\infty}\left[-e^{-0.10b}+1\right] \\
&= 1
\end{aligned}$$

In Section 8.2, "Discrete Probability Distributions," we found the expected value (mean) of a discrete probability distribution using the formula

$$E(x) = \sum x\Pr(x)$$

For continuous probability distributions, such as the normal probability distribution, the expected value, or mean, can be found by evaluating the improper integral

$$\int_{-\infty}^{\infty} xf(x)\,dx$$

Mean (Expected Value) If x is a continuous random variable with probability density function f, then the mean of the probability distribution is

$$\mu = \int_{-\infty}^{\infty} xf(x)\,dx$$

The normal distribution density function, in standard form, is

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

so the mean of the normal probability distribution is given by

$$\begin{aligned}
\mu &= \int_{-\infty}^{\infty} x\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)dx \\
&= \lim_{a\to\infty}\int_{-a}^{0}\frac{1}{\sqrt{2\pi}}xe^{-x^2/2}\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{\sqrt{2\pi}}xe^{-x^2/2}\,dx \\
&= \lim_{a\to\infty}\frac{1}{\sqrt{2\pi}}\left[-e^{-x^2/2}\right]_{-a}^{0} + \lim_{b\to\infty}\frac{1}{\sqrt{2\pi}}\left[-e^{-x^2/2}\right]_0^b \\
&= \frac{1}{\sqrt{2\pi}}(-1+0) + \frac{1}{\sqrt{2\pi}}(0+1) = 0
\end{aligned}$$

This verifies the statement in Chapter 8 that the mean of the standard normal distribution is 0.

EXAMPLE 4

Find the area of the region below the graph of $f(x) = \dfrac{\ln x}{x^2}$ and above the x-axis.

This can be done by evaluating the integral over the interval where $f(x)$ is above the x-axis. It is not possible to find two points where $f(x) = 0$, so we will use graphs to find the interval and to evaluate the integral.

(a) Graph $y = f(x)$ to find the interval over which to integrate.
(b) Evaluate the integral of $f(x)$ over this interval.

Solution

(a) The graph of $y = f(x)$ is shown in Figure 13.27(a). Using TRACE and ZOOM shows that $f(1) = 0$ and $f(x)$ approaches 0 as x approaches $+\infty$. Thus the area is found by evaluating

$$\int_1^{+\infty} f(x)\,dx = \int_1^{+\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to+\infty} \int_1^b \frac{\ln x}{x^2}\,dx$$

(b) Evaluating $\displaystyle\int_1^b \frac{\ln x}{x^2}\,dx$ requires the use of integration by parts, with

$$u = \ln x \quad \text{and} \quad dv = x^{-2}$$

Thus

$$du = \frac{1}{x}\,dx \quad \text{and} \quad v = \frac{x^{-1}}{-1}$$

$$\int_1^b \frac{\ln x}{x^2}\,dx = (\ln x)\left(\frac{x^{-1}}{-1}\right)\Big|_1^b - \int_1^b (-x^{-1})\frac{1}{x}\,dx = -\frac{\ln x}{x}\Big|_1^b + \frac{x^{-1}}{-1}\Big|_1^b$$

$$= \left(-\frac{\ln b}{b} + \frac{\ln 1}{1}\right) - \left(\frac{1}{b} - 1\right) = 1 - \frac{1}{b} - \frac{\ln b}{b}$$

$$\int_1^\infty f(x)\,dt = \lim_{b\to\infty}\left(1 - \frac{1}{b} - \frac{\ln b}{b}\right) = \lim_{b\to\infty} 1 - \lim_{b\to\infty}\frac{1}{b} - \lim_{b\to\infty}\frac{\ln b}{b}$$

We have not developed a method to evaluate $\displaystyle\lim_{b\to\infty}\left(\frac{\ln b}{b}\right)$ or, equivalently, $\displaystyle\lim_{x\to\infty}\left(\frac{\ln x}{x}\right)$. But we can use ZOOM and TRACE to see that the graph of $y = 1 - \dfrac{1}{x} - \dfrac{\ln x}{x}$ approaches 1 as x approaches $+\infty$ (see Figure 13.27b).

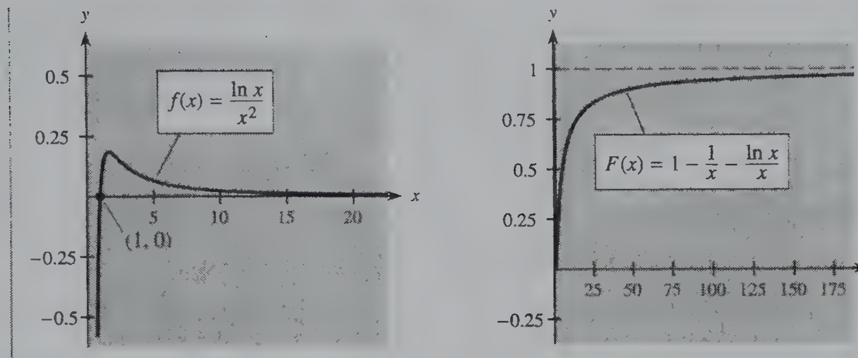


Figure 13.27

Thus $\lim_{b\to\infty}\left(1 - \frac{1}{b} - \frac{\ln b}{b}\right) = 1$, and the area under $y = f(x)$ is $\int_1^\infty f(x)\,dx = \lim_{b\to\infty}\left(1 - \frac{1}{b} - \frac{\ln b}{b}\right) = 1$.

CHECKPOINT SOLUTIONS

1. (a) True (b) True (c) True

2. (a) $\lim_{b\to\infty}\int_1^b x^{-4/3}\,dx = \lim_{b\to\infty}\left.\frac{x^{-1/3}}{-1/3}\right|_1^b = \lim_{b\to\infty}\left(\frac{-3}{b^{1/3}} - \frac{-3}{1}\right) = 0 + 3 = 3$

(b) $\lim_{b\to\infty}\int_0^b (x+1)^{-1/2}\,dx = \lim_{b\to\infty}\left.\frac{(x+1)^{1/2}}{1/2}\right|_0^b = \lim_{b\to\infty} 2\sqrt{x+1}\Big|_0^b$

$= \lim_{b\to\infty}(2\sqrt{b+1} - 2) = \infty$

(Integral diverges)

EXERCISE 13.7

In Problems 1–20, evaluate the improper integrals that converge.

1. $\int_1^\infty \frac{dx}{x^6}$
2. $\int_1^\infty \frac{1}{x^4}\,dx$
3. $\int_1^\infty \frac{dt}{t^{3/2}}$
4. $\int_5^\infty \frac{dx}{(x-1)^3}$
5. $\int_1^\infty e^{-x}\,dx$
6. $\int_0^\infty x^2 e^{-x^3}\,dx$
7. $\int_1^\infty \frac{dt}{t^{1/3}}$
8. $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$
9. $\int_0^\infty e^{3x}\,dx$
10. $\int_1^\infty x e^{x^2}\,dx$
11. $\int_{-\infty}^{-1} \frac{10}{x^2}\,dx$
12. $\int_{-\infty}^{-2} \frac{x}{\sqrt{x^2-1}}\,dx$
13. $\int_{-\infty}^{0} x^2 e^{-x^3}\,dx$
14. $\int_{-\infty}^{0} \frac{x}{(x^2+1)^2}\,dx$
15. $\int_{-\infty}^{-1} \frac{6}{x}\,dx$
16. $\int_{-\infty}^{-2} \frac{3x}{x^2+1}\,dx$
17. $\int_{-\infty}^{\infty} \frac{2x}{x^2+1}\,dx$
18. $\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2}\,dx$
19. $\int_{-\infty}^{\infty} x^3 e^{-x^4}\,dx$
20. $\int_{-\infty}^{\infty} x^4 e^{-x^5}\,dx$
21. For what value of c does
$$\int_0^\infty \frac{c}{e^{0.5t}}\,dt = 1?$$
22. For what value of c does
$$\int_{10}^\infty \frac{c}{x^3}\,dx = 1?$$

In Problems 23–26, find the area, if it exists, of the region under the graph of $y = f(x)$ and to the right of $x = 1$.

23. $f(x) = \dfrac{x}{e^{x^2}}$

24. $f(x) = \dfrac{1}{\sqrt[5]{x^3}}$

25. $f(x) = \dfrac{1}{\sqrt[3]{x^5}}$

26. $f(x) = \dfrac{1}{x\sqrt{x}}$

27. Show that the function

$$f(x) = \begin{cases} \dfrac{200}{x^3} & \text{if } x \geq 10 \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function.

28. Show that

$$f(x) = \begin{cases} 3e^{-3t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

is a probability density function.

29. For what value of c is the function

$$f(x) = \begin{cases} c/x^2 & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

a probability density function?

30. For what value of c is the function

$$f(x) = \begin{cases} c/x^3 & \text{if } x \geq 100 \\ 0 & \text{otherwise} \end{cases}$$

a probability density function?

31. If

$$f(x) = \begin{cases} ce^{-x/4} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

is a probability density function, what must be the value of c?

32. If

$$f(x) = \begin{cases} ce^{-kx} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

is a probability density function, what must be the value of c?

33. Find the mean of the probability distribution if the probability density function is

$$f(x) = \begin{cases} \dfrac{200}{x^3} & \text{if } x \geq 10 \\ 0 & \text{otherwise} \end{cases}$$

34. Find the mean of the probability distribution if the probability density function is

$$f(x) = \begin{cases} c/x^3 & \text{if } x \geq 100 \\ 0 & \text{otherwise} \end{cases}$$

35. Find the area below the graph of $y = f(x)$ and above the x-axis for $f(x) = 24xe^{-3x}$. Use the graph of $y = f(x)$ to find the interval for which $f(x) \geq 0$ and the graph of the integral of $f(x)$ over this interval to find the area.

36. Find the area below the graph of $y = f(x)$ and above the x-axis for $f(x) = x^2e^{-x}$ and $x \geq 0$. Use the graph of the integral of $f(x)$ over this interval to find the area.

Applications

37. ***Capital value*** Suppose that a continuous income stream has an annual rate of flow at time t given by $f(t) = A$, where A is a constant. If the interest rate is r (as a decimal, $r > 0$), compounded continuously, show that the capital value of the stream is A/r.

38. ***Capital value*** Suppose that a donor wishes to provide a cash gift to a hospital that will generate a continuous income stream with an annual rate of flow at time t given by $f(t) = \$20{,}000$. If the annual interest rate is 12%, compounded continuously, find the capital value of this perpetuity.

39. ***Capital value*** Suppose that a business provides a continuous income stream with an annual rate of flow at time t given by $f(t) = 120e^{0.04t}$ in thousands of dollars. If the interest rate is 9%, compounded continuously, find the capital value of the business.

40. ***Capital value*** Suppose that the output of the machinery in a factory can be considered as a continuous income stream with annual rate of flow at time t given by $f(t) = 450e^{-0.09t}$ in thousands of dollars. If the annual interest rate is 6%, compounded continuously, find the capital value of the machinery.

41. ***Capital value*** A business has a continuous income stream with an annual rate of flow at time t given by $f(t) = 56{,}000e^{0.02t}$ (dollars). If the interest rate is 10%, compounded continuously, find the capital value of the business.

42. ***Capital value*** Suppose that a business provides a continuous income stream with an annual rate of flow at time t given by $f(t) = 10{,}800e^{0.06t}$ (dollars). If money is worth 12%, compounded continuously, find the capital value of the business.

43. ***Radioactive waste*** Suppose that the rate at which a nuclear power plant produces radioactive waste is proportional to the number of years it has been operating, according to $f(t) = 500t$ in pounds per year. Suppose also that the waste decays exponentially at a

rate of 3% per year. Then the amount of radioactive waste that will accumulate in b years is given by

$$\int_0^b 500te^{-0.03(b-t)}\,dt$$

(a) Evaluate this integral.
(b) How much waste will accumulate in the long run? Take the limit as $b \to \infty$ in (a).

44. ***Field intensity*** The field intensity around an infinitely long, straight electrical wire is

$$F = \frac{\mu IM}{10}\int_{-\infty}^{\infty}\frac{dx}{(x^2+r^2)^{3/2}}$$

where μ, r, I, and M are constants. Evaluate this integral.

45. ***Quality control*** The probability density function for the life span of an electronics part is $f(t) = 0.08e^{-0.08t}$, where t is the number of months in service. Find the probability that any given part of this type lasts longer than 24 months.

46. ***Warranties*** A transmission repair firm that wants to offer a lifetime warranty on its repairs has determined that the probability density function for transmission failure after repair is $f(t) = 0.3e^{-0.3t}$, where t is the number of months after repair. What is the probability that a transmission chosen at random will last
(a) 3 months or less?
(b) more than 3 months?

KEY TERMS AND FORMULAS

Section	Key Terms	Formula
13.1	Sigma notation	$\sum_{i=1}^{n} 1 = n;\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
		$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
	Area	
	Right-hand endpoints	$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\frac{b-a}{n}$
	Left-hand endpoints	$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i-1})\frac{b-a}{n}$
13.2	Riemann sum	$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$
	Definite integral	$\int_a^b f(x)\,dx = \lim_{\substack{\max \Delta x_i \to 0\\(n\to\infty)}} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$
	Fundamental Theorem of Calculus	$\int_a^b f(x)\,dx = F(b) - F(a)$, where $F'(x) = f(x)$
	Definite Integral Properties	$\int_a^a f(x)\,dx = 0$
		$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
		$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
		$\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$

Section	Key Terms	Formula
		$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$
	Area under $f(x)$, where $f(x) \geq 0$	$A = \int_a^b f(x)\,dx$
13.3	Area between $f(x)$ and $g(x)$, where $f(x) \geq g(x)$	$A = \int_a^b [f(x) - g(x)]\,dx$
	Average value over $[a, b]$	$\frac{1}{b-a}\int_a^b f(x)\,dx$
	Lorenz curve	
	Gini coefficient	$2\int_0^1 [x - L(x)]\,dx$
13.4	Continuous income streams	
	Total income	$\int_0^k f(t)\,dt$ (for k years)
	Present value	$\int_0^k f(t)e^{-rt}\,dt$, where r is the interest rate
	Future value	$e^{rk}\int_0^k f(t)e^{-rt}\,dt$
	Consumer's surplus [demand is $f(x)$]	$CS = \int_0^{x_1} f(x)\,dx - p_1x_1$
	Producer's surplus [supply is $g(x)$]	$PS = p_1x_1 - \int_0^{x_1} g(x)\,dx$
13.5	Integration from tables	See Table 13.2.
13.6	Integration by parts	$\int u\,dv = uv - \int v\,du$
13.7	Improper integrals	$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$
		$\int_{-\infty}^b f(x)\,dx = \lim_{a\to\infty}\int_{-a}^b f(x)\,dx$
		$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$
	Capital value of a continuous income stream	$\int_0^\infty f(t)\,e^{-rt}\,dt$
	Probability distribution	$\int_{-\infty}^\infty f(x)\,dx = 1$
	Mean	$\mu = \int_{-\infty}^\infty xf(x)\,dx$

REVIEW EXERCISES

Section 13.1

1. Calculate $\sum_{k=1}^{8} (k^2 + 1)$.
2. Use formulas to simplify
$$\sum_{i=1}^{n} \frac{3i}{n^3}$$
3. Use 6 subintervals of the same size to approximate the area under the graph of $y = 3x^2$ from $x = 0$ to $x = 1$. Use the right-hand endpoints of the subintervals to find the heights of the rectangles.
4. Use rectangles to find the area under the graph of $y = 3x^2$ from $x = 0$ to $x = 1$. Use n equal subintervals.

Section 13.2

5. Use a definite integral to find the area under the graph of $y = 3x^2$ from $x = 0$ to $x = 1$.
6. Find the area between the graph of $y = x^3 - 4x + 5$ and the x-axis from $x = 1$ to $x = 3$.

Evaluate the integrals in Problems 7–18.

7. $\int_1^4 4\sqrt{x^3}\, dx$
8. $\int_{-3}^{2} (x^3 - 3x^2 + 4x + 2)\, dx$
9. $\int_0^5 (x^3 + 4x)\, dx$
10. $\int_{-2}^{3} (x + 2)^2\, dx$
11. $\int_{-3}^{-1} (x + 1)\, dx$
12. $\int_2^3 \frac{x^2}{2x^3 - 7}\, dx$
13. $\int_{-1}^{2} (x^2 + x)\, dx$
14. $\int_1^4 \left(\frac{1}{x} + \sqrt{x}\right) dx$
15. $\int_0^4 (2x + 1)^{1/2}\, dx$
16. $\int_0^1 \frac{x}{x^2 + 1}\, dx$
17. $\int_0^1 e^{-2x}\, dx$
18. $\int_0^1 xe^{x^2}\, dx$

Section 13.3

Find the area between the curves in Problems 19–22.

19. $y = x^2 - 3x + 2$ and $y = x^2 + 4$ from $x = 0$ to $x = 5$
20. $y = x^2$ and $y = 4x + 5$
21. $y = x^3$ and $y = x$ from $x = -1$ to $x = 0$
22. $y = x^3 - 1$ and $y = x - 1$

Section 13.5

Evaluate the integrals in Problems 23–26, using Table 13.2.

23. $\int \sqrt{x^2 - 4}\, dx$
24. $\int_0^1 3^x\, dx$
25. $\int x \ln x^2\, dx$
26. $\int \frac{dx}{x(3x + 2)}$

Section 13.6

In Problems 27–30, use integration by parts to evaluate.

27. $\int x^5 \ln x\, dx$
28. $\int xe^{-2x}\, dx$
29. $\int \frac{x\, dx}{\sqrt{x + 5}}$
30. $\int_1^e \ln x\, dx$

Section 13.7

Evaluate the improper integrals in Problems 31–34.

31. $\int_1^{\infty} \frac{1}{x}\, dx$
32. $\int_{-\infty}^{-1} \frac{200}{x^3}\, dx$
33. $\int_0^{\infty} 5e^{-3x}\, dx$
34. $\int_{-\infty}^{0} \frac{x}{(x^2 + 1)^2}\, dx$

Applications

Section 13.2

35. ***Maintenance*** Maintenance costs for buildings increase as the buildings age. If the rate of increase in maintenance costs for a building is
$$M'(t) = \frac{14{,}000}{\sqrt{t + 16}}$$
where M is in dollars and t is time in years, $0 \le t \le 15$, find the total maintenance cost for the first 9 years ($t = 0$ to $t = 9$).
36. ***Quality control*** Suppose the probability density function for the life expectancy of a "disposable" telephone is
$$f(x) = \begin{cases} 1.4e^{-1.4x} & x \ge 0 \\ 0 & x < 0 \end{cases}$$
Find the probability that the telephone lasts 2 years.

Section 13.3

37. ***Savings*** The future value of \$1000 invested in a savings account at 10%, compounded continuously, is $S = 1000e^{0.1t}$, where t is in years. Find the average amount in the savings account during the first 5 years.
38. ***Income streams*** Suppose the total income from a video machine is given by
$$I = 50e^{0.2t}, \quad 0 \le t \le 4,\ t \text{ in hours}$$
Find the average income over this 4-hour period.

Section 13.4

39. ***Consumer's surplus*** The demand function for a product under pure competition is $p = \sqrt{64 - 4x}$, and the supply function is $p = x - 1$.
    (a) Find the market equilibrium.
    (b) Find the consumer's surplus at market equilibrium.

40. ***Producer's surplus*** Find the producer's surplus at market equilibrium for Problem 39.

41. ***Income streams*** Find the total income over the next 10 years from a continuous income stream that has an annual flow rate at time t given by
$$f(t) = 125e^{0.05t}$$
in thousands of dollars.

42. ***Income streams*** Suppose that a machine's production is considered a continuous income stream with an annual rate of flow at time t given by
$$f(t) = 150e^{-0.2t}$$
in thousands of dollars. If money is worth 8%, compounded continuously, find (a) the present value of the machine's production over the next 5 years, and (b) the future value of the production 5 years from now.

Section 13.5

43. ***Average cost*** Suppose the cost function for a product is given by $C(x) = \sqrt{40{,}000 + x^2}$. Find the average cost over the first 150 units.

Section 13.6

44. ***Income streams*** Suppose the present value of a continuous income stream over the next 5 years is given by
$$P = 9000 \int_0^5 te^{-0.08t}\, dt, \quad P \text{ in dollars, } t \text{ in years}$$
Find the present value.

45. ***Cost*** If the marginal cost for a product is $\overline{MC} = 3 + x\ln(x + 1)$ and if the fixed cost is \$2000, find the total cost function.

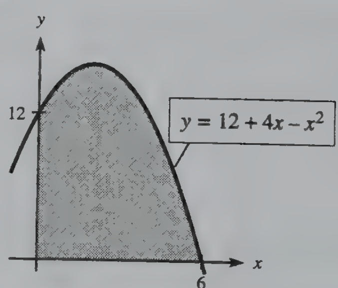
Section 13.7

46. ***Quality control*** Find the probability that a telephone lasts more than 1 year if the probability density function for its life expectancy is given by
$$f(x) = \begin{cases} 1.4e^{-1.4x} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

47. ***Capital value*** Find the capital value of a business if its income is considered a continuous income stream with annual rate of flow given by
$$f(t) = 120e^{0.03t}$$
in thousands of dollars, and the current interest rate is 6%, compounded continuously.

CHAPTER TEST

1. Use left-hand endpoints and $n = 4$ subdivisions to approximate the area under $f(x) = \sqrt{4 - x^2}$ on the interval $[0, 2]$.
2. Consider $f(x) = 5 - 2x$ from $x = 0$ to $x = 1$ with n equal subdivisions.
    (a) If $f(x)$ is evaluated at right-hand endpoints, find a formula for the sum, S, of the areas of the n rectangles.
    (b) Find $\lim_{n \to \infty} S$.
3. Express the area in quadrant I under $y = 12 + 4x - x^2$ (shaded in the figure) as an integral. Then evaluate the integral to find the area.



4. Evaluate the following integrals with the Fundamental Theorem.
    (a) $\int_0^4 (9 - 4x)\, dx$ (b) $\int_0^3 x(8x^2 + 9)^{-1/2} dx$

(c) $\int_1^4 \frac{5}{4x-1}\,dx$ (d) $\int_1^\infty \frac{7}{x^2}\,dx$

5. Use integration by parts to evaluate the following.

(a) $\int 3xe^x\,dx$ (b) $\int x\ln(2x)\,dx$

6. If $\int_1^4 f(x)\,dx = 3$ and $\int_3^4 f(x)\,dx = 7$, find $\int_1^3 2f(x)\,dx$.

7. Use Table 13.2 on page 966 to evaluate each of the following.

(a) $\int \ln(2x)\,dx$ (b) $\int x\sqrt{3x-7}\,dx$

8. Use the numerical integration feature of a graphing utility to approximate $\int_1^4 \sqrt{x^3+10}\,dx$.

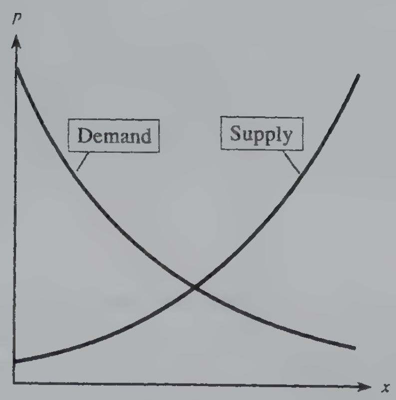
9. Suppose the supply function for a product is $p = 40 + 0.001x^2$ and the demand function is $p = 120 - 0.2x$, where x is the number of units and p is the price in dollars. If the market equilibrium price is \$80, find (a) the consumer's surplus and (b) the producer's surplus.

10. Suppose a continuous income stream has an annual rate of flow $f(t) = 85e^{-0.01t}$, in thousands of dollars, and the current interest rate is 7%, compounded continuously.

(a) Find the total income over the next 12 years.
(b) Find the present value over the next 12 years.
(c) Find the capital value of the stream.

11. Find the area between $y = 2x + 4$ and $y = x^2 - x$.

12. The figure shows typical supply and demand curves. On the figure, sketch and shade the region whose area represents the consumer's surplus.



13. In an effort to make the distribution of income more nearly equal, the government of a country passes a tax law that changes the Lorenz curve from $y = 0.998x^{2.6}$ for one year to $y = 0.57x^2 + 0.43x$ for the next year. Find the Gini coefficient of income for both years and determine whether the distribution of income is more or less equitable after the tax law is passed. Interpret the result.

14. With data from the U.S. Department of Energy, the number of millions of barrels of oil imported from the Persian Gulf each year from 1985 to 1996 can be modeled by the function

$$f(t) = 1.107t^3 - 46.288t^2 + 614.913t - 1952.342$$

where t is the number of years past 1980 (*Monthly Energy Review*, June 1997).

(a) Find the average number of barrels imported per year from 1985 to 1996 (that is, from $t = 5$ to $t = 16$).
(b) Find the average number of barrels imported per year during the 1990s (from $t = 10$ to $t = 16$).

Extended Applications / Group Projects

I. Retirement Planning

A 52-year-old client asks an accountant how to plan for his future retirement at age 62. He expects income from Social Security in the amount of $16,000 per year and a retirement pension of $30,000 per year from his employer. He wants to make monthly contributions to an investment plan that pays 8%, compounded monthly, for 10 years so that he will have a total income of $62,000 per year for 30 years. What will the size of the monthly contributions have to be to accomplish this goal, if it is assumed that money will be worth 8%, compounded continuously throughout the period after he is 62?

To help you answer this question, complete the following.

1. How much money must the client withdraw annually from his investment plan during his retirement so that his total income goal is met?
2. How much money S must the client's account contain when he is 62 so that it will generate this annual amount for 30 years? *Hint:* S can be considered the present value over 30 years of a continuous income stream with the amount you found in question 1 as its annual rate of flow.
3. The monthly contribution R that would, after 10 years, amount to the present value S found in question 2 can be obtained from the formula

$$R = S\left[\frac{i}{(1+i)^n - 1}\right]$$

where i represents the monthly interest rate and n the number of months. Find the client's monthly contribution, R.

M II. Purchasing Electrical Power

In order to plan its purchases of electrical power from suppliers over the next 5 years, the PAC Electric Company needs to model its load data (demand for power by its customers) and use this model to predict future loads. The company pays for the electrical power each month on the basis of the peak load (demand) at any point during the month. The table gives, for the years 1980–1998, the load in megawatts (million watts) for the month when the maximum load occurred and the load in megawatts for the month when the minimum load occurred. The maximum loads occurred in summer, and the minimum loads occurred in spring or fall.

Year	*Maximum Monthly Load*	*Minimum Monthly Load*
1980	40.9367	19.4689
1981	45.7127	22.1504
1982	48.0460	25.3670
1983	56.1712	28.7254
1984	55.5793	31.0460
1985	62.4285	31.3838
1986	76.6536	34.8426
1987	73.8214	38.4544
1988	74.8844	40.6080
1989	83.0590	47.3621
1990	88.3914	45.8393
1991	88.7704	48.7956
1992	94.2620	48.3313
1993	105.1596	52.7710
1994	95.8301	54.4757
1995	97.8854	55.2210
1996	102.8912	55.1360
1997	109.5541	57.2162
1998	111.2516	58.3216

The company wishes to predict the average monthly load over the next 5 years so that it can plan its future monthly purchases. To assist the company, proceed as follows.

1. (a) Using the years and the maximum monthly load given for each year, graph the data, with x representing the number of years from 1980 and y representing the load in megawatts.
   (b) Find the equation that best fits the data, using both a quadratic model and a cubic model.
   (c) Graph the data and both of these models from 1980 to 2000 (that is, from $x = 0$ to $x = 20$).
2. Do the two models appear to fit the data equally well in the interval 1980–2000? Which model appears to be a better predictor for the next decade?
3. Use the quadratic model to predict the maximum monthly load in the year 2003. How can this value be used by the company? Should this number be used to plan monthly power purchases for each month in 2003?
4. To create a "typical" monthly load function:
   (a) Create a table with the year as the independent variable and the average of the maximum and minimum monthly loads as the dependent variable.
   (b) Find the quadratic model that best fits these data points, using $x = 0$ in 1980.
5. Use a definite integral with the typical monthly load function to predict the average monthly load over the years 2000–2005.
6. What factors in addition to the average monthly load should be considered when the company plans future purchases of power?

Warm-up

Prerequisite Problem Type	For Section	Answer	Section for Review
If $y = f(x)$, x is the independent variable and y is the ________ variable.	14.1	Dependent	1.2 Functions
What is the domain of $f(x) = \frac{3x}{x-1}$?	14.1	All reals except $x = 1$	1.2 Domains
If $C(x) = 5 + 5x$, what is $C(0.20)$?	14.1	6	1.2 Functional notation
(a) Solve for x and y: $\begin{cases} 0 = 50 - 2x - 2y \\ 0 = 60 - 2x - 4y \end{cases}$ (b) Solve for x and y: $\begin{cases} x = 2y \\ x + y - 9 = 0 \end{cases}$	14.4 14.5	(a) $x = 20$, $y = 5$ (b) $x = 6$, $y = 3$	1.5 Systems of equations
If $z = 4x^2 + 5x^3 - 7$, what is $\frac{dz}{dx}$?	14.2 14.3 14.4 14.5	$\frac{dz}{dx} = 8x + 15x^2$	9.4 Derivatives
If $f(x) = (x^2 - 1)^2$, what is $f'(x)$?	14.2	$f'(x) = 4x(x^2 - 1)$	9.6 Derivatives
If $z = 10y - \ln y$, what is $\frac{dz}{dy}$?	14.2	$\frac{dz}{dy} = 10 - \frac{1}{y}$	11.1 Derivatives of logarithmic functions
If $z = 5x^2 + e^x$, what is $\frac{dz}{dx}$?	14.2	$\frac{dz}{dx} = 10x + e^x$	11.2 Derivatives of exponential functions
Find the slope of the tangent to $y = 4x^3 - 4e^x$ at $(0, -4)$.	14.2	-4	9.3, 9.4, 11.2 Derivatives

Chapter 14

Functions of Two or More Variables

Although we have been dealing primarily with functions of one variable, many real-life situations involve quantities that are functions of two or more variables. For example, the grade you receive in a course is a function of several test grades. The cost of manufacturing a product may involve the cost of labor, the cost of materials, and overhead expenses. The concentration of a substance at any point in a vein after an injection is a function of time since the injection t, the velocity of the blood v, and the distance the point is from the point of injection.

In this chapter we will extend our study to functions of two or more variables. We will extend the derivative concept to functions of several variables by taking partial derivatives, and we will learn how to maximize functions of two variables. We will use these concepts to solve problems in the management, social, and life sciences. In particular, we will discuss joint cost functions, marginal cost, marginal productivity, and marginal demand functions. We will use Lagrange multipliers to optimize functions of two variables subject to a condition that constrains the variables.

14.1 Functions of Two or More Variables

OBJECTIVES

- *To find the domain and range of a function of two or more variables*
- *To evaluate a function of two or more variables given values for the independent variables*

APPLICATION PREVIEW

The relations we have studied up to this point have been limited to two variables, with one of the variables assumed to be a function of the other. But there are many instances where one variable may depend on two or more other variables. For example, the output or production Q (for quantity) of a company can be modeled according to the equation

$$Q = AK^{\alpha}L^{1-\alpha}$$

where A is a constant, K is the company's capital investment, L is the size of the labor force (in work-hours), and α is a constant with $0 < \alpha < 1$. Functions of this type are called **Cobb-Douglas production functions,** and they are frequently used in economics. For example, suppose the Cobb-Douglas production function for a company is given by

$$Q = 4\,K^{0.4}L^{0.6}$$

where Q is dollars of production value. We could use this function to determine the production value for a given amount of capital investment and available work-hours of labor. We could also find how production is affected by changes in capital investment or available work-hours.

In addition, the demand function for a commodity frequently depends on the price of the commodity, available income, and prices of competing goods. Other examples from economics will be presented later in this chapter.

We write $z = f(x, y)$ to state that z is a function of both x and y. The variables x and y are called the **independent variables** and z is called the **dependent variable.** Thus the function f associates with each pair of possible values for the independent variables (x and y) exactly one value of the dependent variable (z).

The equation $z = x^2 - xy$ defines z as a function of x and y. We can denote this by writing $z = f(x, y) = x^2 - xy$. The domain of the function is the set of all ordered pairs (of real numbers), and the range is the set of all real numbers.

EXAMPLE 1

Give the domain of the function

$$g(x, y) = \frac{x^2 - 3y}{x - y}$$

Solution

The domain of the function is the set of ordered pairs that do not give a 0 denominator. That is, the domain is the set of all ordered pairs where the first and second elements are not equal (that is, where $x \neq y$).

CHECKPOINT

1. Find the domain of the function

$$f(x, y) = \frac{2}{\sqrt{x^2 - y^2}}$$

We graph the function $z = f(x, y)$ by using three dimensions. We can construct a three-dimensional coordinate space by drawing three mutually perpendicular axes, as in Figure 14.1. By setting up a scale of measurement along the three axes from the origin 0, we can determine the three coordinates (x, y, z) for any point P. The point shown in Figure 14.1 is +2 units in the x-direction, +3 units in the y-direction, and +4 units in the z-direction, so the coordinates of the point are (2, 3, 4).

The pairs of axes determine the three **coordinate planes;** the xy-plane, the yz-plane, and the xz-plane. The planes divide the space into eight **octants.** The point $P(2, 3, 4)$ is in the first octant.

If we are given a function $z = f(x, y)$, we can find the z-value corresponding to $x = a$ and $y = b$ by evaluating $f(a, b)$.

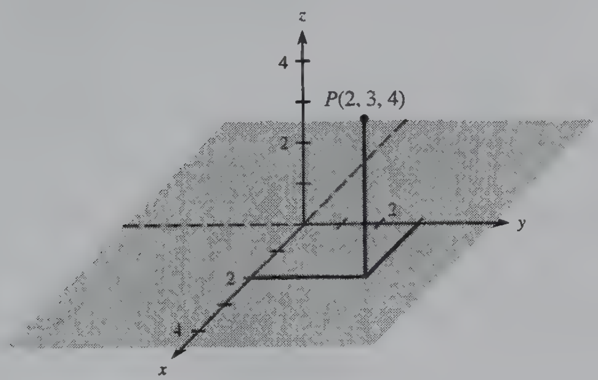


Figure 14.1

EXAMPLE 2

If $z = f(x, y) = x^2 - 4xy + xy^3$, find the following.

(a) $f(1, 2)$ (b) $f(2, 5)$ (c) $f(-1, 3)$

Solution

(a) $f(1, 2) = 1^2 - 4(1)(2) + (1)(2)^3 = 1$
(b) $f(2, 5) = 2^2 - 4(2)(5) + (2)(5)^3 = 214$
(c) $f(-1, 3) = (-1)^2 - 4(-1)(3) + (-1)(3)^3 = -14$

CHECKPOINT

2. If $f(x, y, z) = x^2 + 2y - z$, find $f(2, 3, 4)$.

EXAMPLE 3

A small furniture company's cost (in dollars) to manufacture 1 unit of several different all-wood items is given by

$$C(x, y) = 5 + 5x + 22y$$

where x represents the number of board feet of material used and y represents the number of work-hours of labor required for assembly and finishing. A certain bookcase uses 20 board feet of material and requires 2.5 work-hours for assembly and finishing. Find the cost of manufacturing this bookcase.

Solution

The desired cost is

$$C(20, 2.5) = 5 + 5(20) + 22(2.5) = 160 \quad \text{(dollars)}$$

For a given function $z = f(x, y)$, we can construct a table of values by assigning values to x and y and finding the corresponding values of z. To each pair of values for x and y there corresponds a unique value of z, and thus a unique point in a three-dimensional coordinate system. From a table of values such as this, a finite number of points can be plotted. All points that satisfy the equation form a "surface" in space. Because z is a function of x and y, lines parallel to the z-axis will intersect such a surface in at most one point. The graph of the equation $z = 4 - x^2 - y^2$ is a surface like that shown in Figure 14.2. The portion of the surface above the xy-plane resembles a bullet and is called a **paraboloid.**

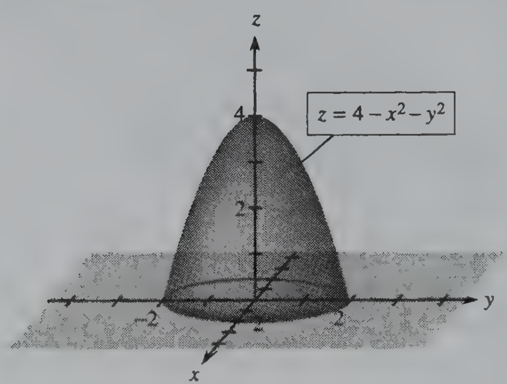


Figure 14.2

In practical applications of functions of two variables, we will have little need to construct the graphs of the surfaces. For this reason, we will not discuss methods of sketching the graphs. Although you will not be asked to sketch graphs of these surfaces, the fact that the graphs do *exist* will be used in studying relative maxima and minima of functions of two variables.

The properties of functions of one variable can be extended to functions of two variables. The precise definition of continuity for functions of two variables is technical and may be found in more advanced books. We will limit our study to functions that are continuous and have continuous derivatives in the domain of interest to us. We may think of continuous functions as functions whose graphs consist of surfaces without "holes" or "breaks" in them.

Let the function $U = f(x, y)$ represent the **utility** (that is, satisfaction) derived by a consumer from the consumption of two goods, X and Y, where x and y represent the amounts of X and Y, respectively. Because we will assume that the utility function is continuous, a given level of utility can be derived from an infinite number of combinations of x and y. The graph of all points (x, y) that give the same utility is called an **indifference curve.** A set of indifference curves corresponding to different levels of utility is called an **indifference map** (see Figure 14.3).

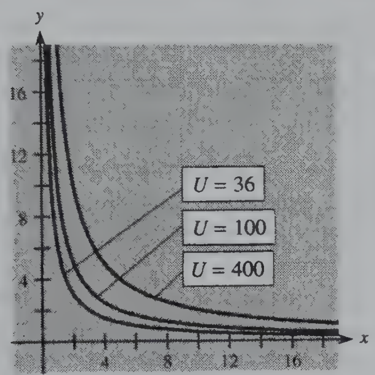


Figure 14.3

EXAMPLE 4

Suppose that the utility function for two goods, X and Y, is $U = x^2y^2$ and a consumer purchases 10 units of X and 2 units of Y.

(a) If the consumer purchases 5 units of X, how many units of Y must be purchased to retain the same level of utility?
(b) Graph the indifference curve for this level of utility.
(c) Graph the indifference curves for this utility function if $U = 100$ and if $U = 36$.

Solution

(a) If $x = 10$ and $y = 2$ satisfy the utility function, then $U = 10^2 \cdot 2^2 = 400$. Thus if x is 5, y must satisfy $400 = 5^2y^2$, so $y = 4$.
(b) The indifference curve for $U = 400$ is $400 = x^2y^2$. The graph for positive x and y is shown in Figure 14.3.
(c) The indifference map in Figure 14.3 contains these indifference curves.

Graphing Utilities A graphing utility can be used to graph each indifference curve in the indifference map shown in Figure 14.3. To graph the indifference curve for a given value of U, we must recognize that y will be positive and solve for y to express it as a function of x. Try it for $U = 100$. Does your graph agree with the one shown in Figure 14.3?

Sometimes functions of two variables are studied by fixing a value for one variable and graphing the resulting function of a single variable. We'll do this in Section 14.3 with production functions.

EXAMPLE 5

Suppose a company has the Cobb-Douglas production function introduced in the Application Preview,

$$Q = 4K^{0.4}L^{0.6}$$

where Q is thousands of dollars of production value, K is hundreds of dollars of capital investment per week, and L is work-hours of labor per week.

(a) If current capital investment is \$72,900 per week and work-hours are 3072 per week, find the current weekly production value.
(b) If weekly capital investment is increased to \$97,200 and new employees are hired so that there are 4096 total weekly work-hours, find the percentage increase in the production value.

Solution

(a) Capital investment of \$72,900 means that $K = 729$. We use this value and $L = 3072$ in the production function.

$$Q = 4(729)^{0.4}(3072)^{0.6} = 6912$$

Thus the weekly production value is \$6,912,000.

(b) In this case we use $K = 972$ and $L = 4096$.

$$Q = 4(972)^{0.4}(4096)^{0.6} = 9216$$

This is an increase in production value of $9216 - 6912 = 2304$, which is equivalent to a weekly increase of

$$\frac{2304}{6912} = 0.33\tfrac{1}{3} = 33\tfrac{1}{3}\%$$

CHECKPOINT SOLUTIONS

1. The domain is the set of ordered pairs of real numbers where $x^2 - y^2 > 0$ or $x^2 > y^2$—that is, where $|x| > |y|$.
2. $f(2, 3, 4) = 4 + 6 - 4 = 6$

EXERCISE 14.1

Give the domain of each function in Problems 1–8.

1. $z = x^2 + y^2$
2. $z = 4x - 3y$
3. $z = \dfrac{4x - 3}{y}$
4. $z = \dfrac{x + y^2}{\sqrt{x}}$
5. $z = \dfrac{4x^3y - x}{2x - y}$
6. $z = \sqrt{x - y}$
7. $q = \sqrt{p_1} + 3p_2$
8. $q = \dfrac{p_1 + p_2}{\sqrt{p_1}}$

In Problems 9–22, evaluate the following functions at the given values of the independent variables.

9. $z = x^3 + 4xy + y^2$; $x = 1, y = -1$
10. $z = 4x^2 - 3xy^3$; $x = 2, y = 2$
11. $z = \dfrac{x - y}{x + y}$; $x = 4, y = -1$
12. $z = \dfrac{x^2 + xy}{x - y}$; $x = 3, y = 2$
13. $C(x_1, x_2) = 600 + 4x_1 + 6x_2$; $x_1 = 400, x_2 = 50$
14. $C(x_1, x_2) = 500 + 5x_1 + 7x_2$; find $C(200, 300)$.
15. $q_1(p_1, p_2) = \dfrac{p_1 + 4p_2}{p_1 - p_2}$; find $q_1(40, 35)$.
16. $q_1(p_1, p_2) = \dfrac{5p_1 - p_2}{p_1 + 3p_2}$; find $q_1(50, 10)$.
17. $z(x, y) = xe^{x+y}$; find $z(3, -3)$.

18. $f(x, y) = ye^{2x} + y^2$; find $f(0, 7)$.
19. $f(x, y) = \dfrac{\ln(xy)}{x^2 + y^2}$; find $f(-3, -4)$.
20. $z(x, y) = x \ln y - y \ln x$; find $z(1, 1)$.
21. $w = \dfrac{x^2 + 4yz}{xyz}$ at $(1, 3, 1)$
22. $u = f(w, x, y, z) = \dfrac{wx - yz^2}{xy - wz}$ at $(2, 3, 1, -1)$

Applications

23. ***Investment*** The future value S of an investment earning 6%, compounded continuously, is a function of the principal P and the length of time t that the principal has been invested. It is given by

$$S = f(P, t) = Pe^{0.06t}$$

Find $f(2000, 20)$, and interpret your answer.

24. ***Amortization*** If \$100,000 is borrowed to purchase a home, then the monthly payment R is a function of the interest rate i (expressed as a percent) and the number of years n before the mortgage is paid. It is given by

$$R = f(i, n) = 100{,}000\left[\frac{0.01(i/12)}{1 - (1 + 0.01(i/12))^{-12n}}\right]$$

Find $f(7.25, 30)$ and interpret your answer.

25. ***Wilson's lot size formula*** In economics, the most economical quantity Q of goods (TVs, dresses, gallons of paint, etc.) for a store to order is given by Wilson's lot size formula

$$Q = f(K, M, h) = \sqrt{2KM/h}$$

where K is the cost of placing the order, M is the number of items sold per week, and h is the weekly holding cost for each item (the cost of storage space, utilities, taxes, security, etc.). Find $f(200, 625, 1)$ and interpret your answer.

26. ***Gas law*** Suppose that a gas satisfies the universal gas law, $V = nRT/P$, with n equal to 10 moles of the gas and R, the universal gas constant, equal to 0.082054. What is V if $T = 10$ K (kelvins, the units in which temperature is measured on the Kelvin scale) and $P = 1$ atmosphere?

Temperature-humidity models There are different models for measuring the effects of high temperature and humidity. Two of these are the Summer Simmer Index (S) and the Apparent Temperature (A),* and they are given by

$$S = 1.98T - 1.09(1 - H)(T - 58) - 56.9$$
$$A = 2.70 + 0.885T - 78.7H + 1.20TH$$

where T is the air temperature (in degrees Fahrenheit) and H is the relative humidity (expressed as a decimal). Use these models in Problems 27 and 28.

27. At the Dallas–Fort Worth Airport, the average daily temperatures and humidities for July are

Maximum: 97.8°F with 44% humidity
Minimum: 74.7°F with 80% humidity**

Calculate the Summer Simmer Index S and the Apparent Temperature A for both the average daily maximum and the average daily minimum temperature.

28. In Orlando, Florida, the following represent the average daily temperatures and humidities for August.

Maximum: 91.6°F with 60% humidity
Minimum: 73.4°F with 92% humidity**

Calculate the Summer Simmer Index S and the Apparent Temperature A for both the average daily maximum and the average daily minimum temperature.

29. The tables below and on the next page show that a monthly mortgage payment, R, is a function of the amount financed, A, in thousands of dollars; the duration of the loan, n, in years; and the annual interest rate, r, as a percent. If $R = f(A, n, r)$, use the tables to find the following, and then write a sentence of explanation for each.
(a) $f(90, 20, 8)$ (b) $f(160, 15, 9)$

8% Annual Percentage Rate
Monthly Payments (Principal and Interest)

Amount Financed	*10 Years*	*15 Years*	*20 Years*	*25 Years*	*30 Years*
\$50,000	\$606.64	\$477.83	\$418.22	\$385.91	\$366.88
60,000	727.97	573.39	501.86	463.09	440.26
70,000	849.29	668.96	585.51	540.27	513.64
80,000	970.62	764.52	669.15	617.45	587.01
90,000	1091.95	860.09	752.80	694.63	660.39
100,000	1213.28	955.65	836.44	771.82	733.76
120,000	1455.94	1146.78	1003.72	926.18	880.52
140,000	1698.58	1337.92	1171.02	1080.54	1027.28
160,000	1941.24	1529.04	1338.30	1234.90	1174.02
180,000	2183.90	1720.18	1505.60	1389.26	1320.78
200,000	2426.56	1911.30	1672.88	1543.64	1467.52

*Bosch, W., and L. G. Cobb, "Temperature-Humidity Indices," UMAP Unit 691, *The UMAP Journal*, 10(3), Fall 1989, 237–256.

**Ruffner, James, and Frank Bair (eds.), *Weather of U.S. Cities*, Gale Research Co., Detroit, MI, 1987.

9% Annual Percentage Rate
Monthly Payments (Principal and Interest)

Amount Financed	*10 Years*	*15 Years*	*20 Years*	*25 Years*	*30 Years*
$50,000	$633.38	$507.13	$449.86	$419.60	$402.31
60,000	760.05	608.56	539.84	503.52	482.77
70,000	886.73	709.99	629.81	587.44	563.24
80,000	1013.41	811.41	719.78	671.36	643.70
90,000	1140.08	912.84	809.75	755.28	724.16
100,000	1266.76	1014.27	899.73	839.20	804.62
120,000	1520.10	1217.12	1079.68	1007.04	965.54
140,000	1773.46	1419.96	1259.62	1174.88	1126.48
160,000	2026.82	1622.82	1439.56	1342.72	1287.40
180,000	2280.16	1825.68	1619.50	1510.56	1448.32
200,000	2533.52	2028.54	1799.46	1678.40	1609.24

Source: *The Mortgage Money Guide*, Federal Trade Commission

30. Wind and cold temperatures combine to make the air temperature feel colder than it actually is. This combination is reported as wind chill. The table below shows that wind chill temperatures, WC, are a function of wind speed, s, and air temperature, t. If $WC = f(s, t)$, use the table to find the following, and then write a sentence of explanation for each.
    (a) $f(25, 5)$ (b) $f(15, -15)$

Wind Speed (mph) \ *Air Temperature (°F)*	*35*	*25*	*15*	*5*	*−5*	*−15*	*−25*
5	33	21	12	0	−10	−21	−31
15	16	2	−11	−25	−38	−51	−65
25	8	−7	−22	−36	−51	−66	−81
35	4	−12	−27	−43	−58	−74	−89
45	2	−14	−30	−46	−62	−78	−93

Source: *World Almanac*, 1998

31. ***Utility*** Suppose that the utility function for two goods X and Y is given by $U = xy^2$, and a consumer purchases 9 units of X and 6 units of Y.
    (a) If the consumer purchases 9 units of Y, how many units of X must be purchased to retain the same level of utility?
    (b) If the consumer purchases 81 units of X, how many units of Y must be purchased to retain the same level of utility?
    (c) Graph the indifference curve for the utility level found in (a) and (b). Use the graph to confirm your answers to (a) and (b).

32. ***Utility*** Suppose that an indifference curve for two goods, X and Y, has the equation $xy = 400$.
    (a) If 25 units of X are purchased, how many units of Y must be purchased to remain on this indifference curve?
    (b) Graph this indifference curve and confirm your results in (a).

33. ***Production*** Suppose that a company's production is given by the Cobb-Douglas production function

$$Q = 30K^{1/4}L^{3/4}$$

where K is dollars of capital investment and L is labor hours.
    (a) Find Q if $K = \$10{,}000$ and $L = 625$ hours.
    (b) Show that if *both* K and L are doubled, then the output is doubled.
    (c) If capital investment is held at $10,000, graph Q as a function of L.

34. ***Production*** Suppose that a company's production is given by the Cobb-Douglas production function

$$Q = 70K^{2/3}L^{1/3}$$

where K is dollars of capital investment and L is labor hours.
    (a) Find Q if $K = \$64{,}000$ and $L = 512$ hours.
    (b) Show that if both K and L are halved, then Q is also halved.
    (c) If capital investment is held at $64,000, graph Q as a function of L.

35. ***Production*** Suppose that the number of units of a good produced, z, is given by $z = 20xy$, where x is the number of machines working properly and y is the average number of work-hours per machine. Find the production for a week in which
    (a) 12 machines are working properly and the average number of work-hours per machine is 30.
    (b) 10 machines are working properly and the average number of work-hours per machine is 25.

36. ***Profit*** The Kirk Kelly Kandy Company makes two kinds of candy, Kisses and Kreams. The profit function for the company is

$$P(x, y) = 100x + 64y - 0.01x^2 - 0.25y^2$$

where x is the number of pounds of Kisses sold per week and y is the number of pounds of Kreams. What is the company's profit if it sells
    (a) 20 pounds of Kisses and 10 pounds of Kreams?
    (b) 100 pounds of Kisses and 16 pounds of Kreams?
    (c) 10,000 pounds of Kisses and 256 pounds of Kreams?

37. ***Epidemic*** The cost per day to society of an epidemic is

$$C(x, y) = 20x + 200y$$

where C is in dollars, x is the number of people infected on a given day, and y is the number of people who die on a given day. If 14,000 people are infected and 20 people die on a given day, what is the cost to society?

38. ***Pesticide*** An area of land is to be sprayed with two brands of pesticide: x liters of brand 1 and y liters of brand 2. If the number of insects killed is given by

$$f(x, y) = 10{,}000 - 6500e^{-0.01x} - 3500e^{-0.02y}$$

how many insects would be killed if 80 liters of brand 1 and 120 liters of brand 2 were used?

14.2 Partial Differentiation

OBJECTIVES

- *To find partial derivatives of functions of two or more variables*
- *To evaluate partial derivatives of functions of two or more variables at given points*
- *To use partial derivatives to find slopes of tangents to surfaces*
- *To find and evaluate second- and higher-order partial derivatives of functions of two variables*

APPLICATION PREVIEW

We used derivatives to find the rate of change of cost with respect to the quantity produced. If cost is given as a function of two variables (such as material costs x and labor costs y), then we can find the rate of change of cost with respect to *one* of these independent variables. This is done by finding the **partial derivative** of the function with respect to one variable, while holding the other one constant.

For example, suppose that the cost of manufacturing a good is given by

$$C(x, y) = 5 + 5x + 2y$$

where x represents the cost of 1 ounce of material used and y represents the labor cost in dollars per hour. To find the rate at which the cost changes with respect to material used x, we treat the y-variable as though it were a constant and take the derivative of C with respect to x. This derivative is

$$\frac{\partial C}{\partial x} = 0 + 5 + 0, \quad \text{or} \quad \frac{\partial C}{\partial x} = 5$$

Note that the partial derivative of $2y$ with respect to x is 0 because y is treated as a constant. The partial derivative is a new type of derivative, so we use a new symbol to denote it, $\partial C/\partial x$.

The partial derivative $\partial C/\partial x = 5$ tells us that a change of \$1 in the cost of materials will cause an increase of \$5 in total costs, *if* labor costs remain constant. To see the rate at which total cost changes with respect to labor costs, we find $\partial C/\partial y = 0 + 0 + 2$, or $\partial C/\partial y = 2$. Thus if the cost of materials is held constant, an increase of \$1 in labor costs will cause an increase of \$2 in the total cost of the good.

First-Order Partial Derivatives

In general, if $z = f(x, y)$ we denote the partial derivative of z with respect to x as $\partial z/\partial x$ and the partial derivative of z with respect to y as $\partial z/\partial y$. Note that dz/dx represents the derivative of a function of one variable, x, and that $\partial z/\partial x$ represents the partial derivative of a function of two or more variables.

As mentioned in the Application Preview, $\partial z/\partial x$ is found by treating y as a constant and taking the derivative of $z = f(x, y)$ with respect to x. Other notations used to represent the partial derivative of $z = f(x, y)$ with respect to x are

$$\frac{\partial z}{\partial x}, \quad \frac{\partial f}{\partial x}, \quad \frac{\partial}{\partial x}f(x, y), \quad f_x(x, y), \quad f_x, \quad \text{and} \quad z_x$$

If x is held constant in the function $z = f(x, y)$ and the derivative is taken with respect to y, we have the partial derivative of z with respect to y, denoted by

$$\frac{\partial z}{\partial y}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial}{\partial y}f(x, y), \quad f_y(x, y), \quad f_y, \quad \text{or} \quad z_y$$

EXAMPLE 1

If $z = 4x^2 + 5x^2y^2 + 6y^3 - 7$, find $\partial z/\partial x$ and $\partial z/\partial y$.

Solution

$$\frac{\partial z}{\partial x} = 8x + 10y^2x$$

$$\frac{\partial z}{\partial y} = 10x^2y + 18y^2$$

EXAMPLE 2

If $z = x^2y + e^x - \ln y$, find z_x and z_y.

Solution

$$z_x = \frac{\partial z}{\partial x} = 2yx + e^x$$

$$z_y = \frac{\partial z}{\partial y} = x^2 - \frac{1}{y}$$

EXAMPLE 3

If $f(x, y) = (x^2 - y^2)^2$, find the following.

(a) f_x (b) f_y

Solution

(a) $f_x = 2(x^2 - y^2)2x = 4x^3 - 4xy^2$
(b) $f_y = 2(x^2 - y^2)(-2y) = -4x^2y + 4y^3$

CHECKPOINT

1. If $z = 100x + 10xy - y^2$, find the following.
   (a) z_x
   (b) $\dfrac{\partial z}{\partial y}$

EXAMPLE 4

If $q = \dfrac{p_1p_2 + 2p_1}{p_1p_2 - 2p_2}$, find $\partial q/\partial p_1$.

Solution

$$\begin{aligned}\frac{\partial q}{\partial p_1} &= \frac{(p_1p_2 - 2p_2)(p_2 + 2) - (p_1p_2 + 2p_1)p_2}{(p_1p_2 - 2p_2)^2}\\ &= \frac{p_1p_2^2 + 2p_1p_2 - 2p_2^2 - 4p_2 - p_1p_2^2 - 2p_1p_2}{(p_1p_2 - 2p_2)^2}\\ &= \frac{-2p_2^2 - 4p_2}{p_2^2(p_1 - 2)^2}\\ &= \frac{-2p_2(p_2 + 2)}{p_2^2(p_1 - 2)^2}\\ &= \frac{-2(p_2 + 2)}{p_2(p_1 - 2)^2}\end{aligned}$$

We may evaluate partial derivatives by substituting values for x *and* y in the same way we did with derivatives of functions of one variable. For example, if $\partial z/\partial x = 2x - xy$, the value of the partial derivative with respect to x at $x = 2$, $y = 3$ is

$$\left.\frac{\partial z}{\partial x}\right|_{(2,\,3)} = 2(2) - 2 \cdot 3 = -2$$

Other notations used to denote evaluation of partial derivatives with respect to x at (a, b) are

$$\frac{\partial}{\partial x} f\,(a, b) \quad \text{and} \quad f_x(a, b)$$

We denote the evaluation of partial derivatives with respect to y at (a, b) by

$$\left.\frac{\partial z}{\partial y}\right|_{(a,\,b)}, \quad \frac{\partial}{\partial y} f(a, b), \quad \text{or} \quad f_y(a, b)$$

EXAMPLE 5

Find the partial derivative of $f(x, y) = x^2 + 3xy + 4$ with respect to x at the point $(1, 2, 11)$.

Solution

$$f_x(x, y) = 2x + 3y$$
$$f_x(1, 2) = 2(1) + 3(2) = 8$$

CHECKPOINT

2. If $g(x, y) = 4x^2 - 3xy + 10y^2$, find the following.
   (a) $\dfrac{\partial g}{\partial x}(1, 3)$
   (b) $g_y(4, 2)$

EXAMPLE 6

Suppose that a company's sales are related to its television advertising by

$$s = 20{,}000 + 10nt + 20n^2$$

where n is the number of commercials per day and t is the length of the commercials in seconds. Find the partial derivative of s with respect to n, and use the result to find the instantaneous rate of change of sales with respect to the number of commercials per day, if the company is currently running ten 30-second commercials.

Solution

The partial derivative of s with respect to n is $\partial s/\partial n = 10t + 40n$. At $n = 10$ and $t = 30$, the rate of change in sales is approximately

$$\left.\frac{\partial s}{\partial n}\right|_{\substack{n=10\\t=30}} = 10(30) + 40(10) = 700$$

Thus increasing the number of commercials by 1 would result in approximately 700 additional sales.

We have seen that the partial derivative $\partial z/\partial x$ is found by holding y constant and taking the derivative of z with respect to x and that the partial derivative $\partial z/\partial y$ is found by holding x constant and taking the derivative of z with respect to y. We now give formal definitions of these partial derivatives.

Partial Derivatives The partial derivative of $z = f(x, y)$ with respect to x at the point (x, y) is

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

provided this limit exists.

The partial derivative of $z = f(x, y)$ with respect to y at the point (x, y) is

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

provided this limit exists.

We have already stated that the graph of $z = f(x, y)$ is a surface in three dimensions. The partial derivative with respect to x of such a function may be thought of as the slope of the tangent to the surface at a point (x, y, z) on the surface in the *positive direction of the x-axis.* That is, if a plane parallel to the xz-plane cuts the surface, passing through the point (x_0, y_0, z_0), the line in the plane that is tangent to the surface will have a slope equal to $\partial z/\partial x$ evaluated at the point. Thus

$$\left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)}$$

represents the slope of the tangent to the surface in the positive direction of the x-axis (see Figure 14.4).

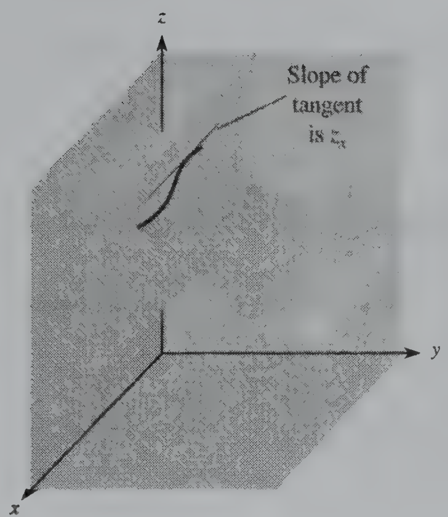


Figure 14.4

Similarly,

$$\left.\frac{\partial z}{\partial y}\right|_{(x_0, y_0)} = \frac{\partial}{\partial y} f(x_0, y_0)$$

represents the slope of the tangent to the surface at (x_0, y_0, z_0) in the positive direction of the y-axis (see Figure 14.5).

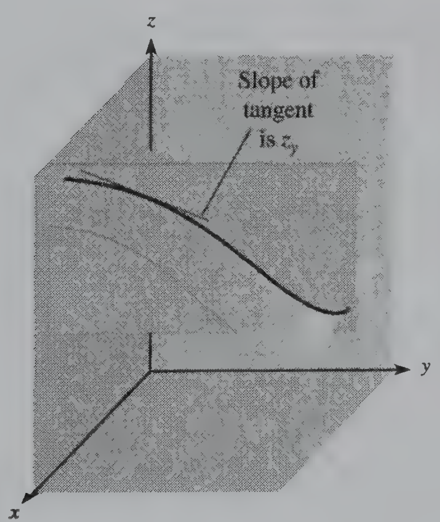


Figure 14.5

EXAMPLE 7

Let $z = 4x^3 - 4e^x + 4y^2$ and let P be the point (0, 2, 12). Find the slope of the tangent to z at the point P in the positive direction of (a) the x-axis and (b) the y-axis.

Solution

(a) The slope of z at P in the positive x-direction is given by $\frac{\partial z}{\partial x}$, evaluated at P.

$$\frac{\partial z}{\partial x} = 12x^2 - 4e^x \quad \text{and} \quad \left.\frac{\partial z}{\partial x}\right|_{(0,\,2)} = 12(0)^2 - 4e^0 = -4$$

This tells us that z *decreases* approximately 4 units for an increase of 1 unit in x at this point.

(b) The slope of z at P in the positive y-direction is given by $\frac{\partial z}{\partial y}$, evaluated at P.

$$\frac{\partial z}{\partial y} = 8y \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(0,\,2)} = 8(2) = 16$$

Thus, at the point P, in the positive y-direction the function *increases* approximately 16 units in the z-value for a unit increase in y.

Up to this point, we have considered derivatives of functions of two variables. We can easily extend the concept to functions of three or more variables. We can find the partial derivative with respect to any one independent variable by taking the derivative of the function with respect to that variable while holding all other independent variables constant.

EXAMPLE 8

If $u = f(w, x, y, z) = 3x^2y + w^3 - 4xyz$, find the following.

(a) $\frac{\partial u}{\partial w}$ (b) $\frac{\partial u}{\partial x}$ (c) $\frac{\partial u}{\partial y}$ (d) $\frac{\partial u}{\partial z}$

Solution

(a) $\frac{\partial u}{\partial w} = 3w^2$ (b) $\frac{\partial u}{\partial x} = 6xy - 4yz$

(c) $\frac{\partial u}{\partial y} = 3x^2 - 4xz$ (d) $\frac{\partial u}{\partial z} = -4xy$

EXAMPLE 9

If $C = 4x_1 + 2x_1^2 + 3x_2 - x_1x_2 + x_3^2$, find the following.

(a) $\frac{\partial C}{\partial x_1}$ (b) $\frac{\partial C}{\partial x_2}$ (c) $\frac{\partial C}{\partial x_3}$

Solution

(a) $\frac{\partial C}{\partial x_1} = 4 + 4x_1 - x_2$ (b) $\frac{\partial C}{\partial x_2} = 3 - x_1$ (c) $\frac{\partial C}{\partial x_3} = 2x_3$

CHECKPOINT

3. If $f(w, x, y, z) = 8xy^2 + 4yz - xw^2$, find
   (a) $\frac{\partial f}{\partial x}$.   (b) $\frac{\partial f}{\partial w}$.   (c) $\frac{\partial f}{\partial y}(1, 2, 1, 3)$.   (d) $\frac{\partial f}{\partial z}(0, 2, 1, 3)$.

Higher-Order Partial Derivatives

Just as we have taken derivatives of derivatives to obtain higher-order derivatives of functions of one variable, we may also take partial derivatives of partial derivatives to obtain higher-order partial derivatives of a function of more than one variable. If $z = f(x, y)$, then the partial derivative functions z_x and z_y are called *first partials.* Partial derivatives of z_x and z_y are called *second partials,* so $z = f(x, y)$ has *four* **second partial derivatives.** The notations for these second partial derivatives follow.

Second Partial Derivatives

$z_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$: both derivatives taken with respect to x.

$z_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$: both derivatives taken with respect to y.

$z_{xy} = \frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$: first derivative taken with respect to x, second with respect to y.

$z_{yx} = \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$: first derivative taken with respect to y, second with respect to x.

EXAMPLE 10

If $z = x^3y - 3xy^2 + 4$, find each of the second partial derivatives of the function.

Solution

Because

$$z_x = 3x^2y - 3y^2 \quad \text{and} \quad z_y = x^3 - 6xy,$$

$$z_{xx} = \frac{\partial}{\partial x}(3x^2y - 3y^2) = 6xy$$

$$z_{xy} = \frac{\partial}{\partial y}(3x^2y - 3y^2) = 3x^2 - 6y$$

$$z_{yy} = \frac{\partial}{\partial y}(x^3 - 6xy) = -6x$$

$$z_{yx} = \frac{\partial}{\partial x}(x^3 - 6xy) = 3x^2 - 6y$$

Note that z_{xy} and z_{yx} are equal for the function in Example 10. This will always occur if the derivatives of this function are continuous.

$z_{xy} = z_{yx}$ If the second partial derivatives z_{xy} and z_{yx} of a function $z = f(x, y)$ are continuous at a point, they are equal there.

EXAMPLE 11

Find each of the second partial derivatives of $z = x^2y + e^{xy}$.

Solution

Because $z_x = 2xy + e^{xy} \cdot y = 2xy + ye^{xy}$,

$$\begin{aligned} z_{xx} &= 2y + e^{xy} \cdot y^2 = 2y + y^2e^{xy} \\ z_{xy} &= 2x + (e^{xy} \cdot 1 + ye^{xy} \cdot x) \\ &= 2x + e^{xy} + xye^{xy} \end{aligned}$$

Because $z_y = x^2 + e^{xy} \cdot x = x^2 + xe^{xy}$,

$$\begin{aligned} z_{yx} &= 2x + (e^{xy} \cdot 1 + xe^{xy} \cdot y) \\ &= 2x + e^{xy} + xye^{xy} \\ z_{yy} &= 0 + xe^{xy} \cdot x = x^2e^{xy} \end{aligned}$$

CHECKPOINT

4. If $z = 4x^3y^4 + 4xy$, find the following.
   (a) z_{xx} (b) z_{yy} (c) z_{xy} (d) z_{yx}
5. If $z = x^2 + 4e^{xy}$, find z_{xy}.

We can find partial derivatives of order higher than the second. For example, we can find the third-order partial derivatives z_{xyx} and z_{xyy} for the function in Example 10 from the second derivative $z_{xy} = 3x^2 - 6y$.

$$\begin{aligned} z_{xyx} &= 6x \\ z_{xyy} &= -6 \end{aligned}$$

EXAMPLE 12

If $z = x^3y^2 + 4 \ln x$, find z_{xyy}.

Solution

$$\begin{aligned} z_x &= 3x^2y^2 + 4 \cdot \frac{1}{x} \\ z_{xy} &= 3x^2(2y) + 0 = 6x^2y \\ z_{xyy} &= 6x^2 \end{aligned}$$

CHECKPOINT SOLUTIONS

1. (a) $z_x = 100 + 10y$
   (b) $\dfrac{\partial z}{\partial y} = 10x - 2y$
2. (a) $\dfrac{\partial g}{\partial x} = 8x - 3y$ and $\dfrac{\partial g}{\partial x}(1, 3) = 8(1) - 3(3) = -1$
   (b) $g_y = -3x + 20y$ and $g_y(4, 2) = -3(4) + 20(2) = 28$
3. (a) $\dfrac{\partial f}{\partial x} = 8y^2 - w^2$ (b) $\dfrac{\partial f}{\partial w} = -2xw$
   (c) $\dfrac{\partial f}{\partial y} = 16xy + 4z$ and $\dfrac{\partial f}{\partial y}(1, 2, 1, 3) = 16(2)(1) + 4(3) = 44$
   (d) $\dfrac{\partial f}{\partial z} = 4y$ and $\dfrac{\partial f}{\partial z}(0, 2, 1, 3) = 4(1) = 4$
4. $z_x = 12x^2y^4 + 4y$ and $z_y = 16x^3y^3 + 4x$
   (a) $z_{xx} = 24xy^4$ (b) $z_{yy} = 48x^3y^2$
   (c) $z_{xy} = 48x^2y^3 + 4$ (d) $z_{yx} = 48x^2y^3 + 4$
5. $z_x = 2x + 4e^{xy}(y) = 2x + 4ye^{xy}$
   Calculation of z_{xy} requires the Product Rule.
   $$z_{xy} = 0 + (4y)(e^{xy}x) + (e^{xy})(4) = 4xye^{xy} + 4e^{xy}$$

EXERCISE 14.2

1. If $z = x^4 - 5x^2 + 6x + 3y^3 - 5y + 7$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.
2. If $z = x^5 - 6x + 4y^4 - y^2$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.
3. If $z = x^3 + 4x^2y + 6y^2$, find z_x and z_y.
4. If $z = 3xy + y^2$, find z_x and z_y.
5. If $f(x, y) = (x^3 + 2y^2)^3$, find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.
6. If $f(x, y) = (xy^3 + y)^2$, find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.
7. If $f(x, y) = \sqrt{2x^2 - 5y^2}$, find f_x and f_y.
8. If $g(x, y) = x\sqrt{y - x}$, find g_x and g_y.
9. If $C(x, y) = 600 - 4xy + 10x^2y$, find $\dfrac{\partial C}{\partial x}$ and $\dfrac{\partial C}{\partial y}$.
10. If $C(x, y) = 1000 - 4x + xy^2$, find $\dfrac{\partial C}{\partial x}$ and $\dfrac{\partial C}{\partial y}$.
11. If $Q(s, t) = \dfrac{2s - 3t}{s^2 + t^2}$, find $\dfrac{\partial Q}{\partial s}$ and $\dfrac{\partial Q}{\partial t}$.
12. If $q = \dfrac{5p_1 + 4p_2}{p_1 + p_2}$, find $\dfrac{\partial q}{\partial p_1}$ and $\dfrac{\partial q}{\partial p_2}$.
13. If $z = e^{2x} + y \ln x$, find z_x and z_y.
14. If $z = \ln(1 + x^2y) - ye^{-x}$, find z_x and z_y.
15. If $f(x, y) = 100e^{xy}$, find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.
16. If $f(x, y) = \ln(xy + 1)$, find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.
17. Find the partial derivative of
    $$f(x, y) = 4x^3 - 5xy + y^2$$
    with respect to x at the point $(1, 2, -2)$.
18. Find the partial derivative of
    $$f(x, y) = 3x^2 + 4x + 6xy$$
    with respect to y at $x = 2$, $y = -1$.
19. Find the slope of the tangent in the positive x-direction to the surface $z = 5x^3 - 4xy$ at the point $(1, 2, -3)$.
20. Find the slope of the tangent in the positive y-direction to the surface $z = x^3 - 5xy$ at $(2, 1, -2)$.
21. Find the slope of the tangent in the positive y-direction to the surface $z = e^{xy}$ at $(0, 1, 1)$.
22. Find the slope of the tangent in the positive x-direction to the surface $z = \ln(xy)$ at $(1, 1, 0)$.
23. If $u = f(w, x, y, z) = y^2 - x^2z + 4x$, find the following.
    (a) $\dfrac{\partial u}{\partial w}$ (b) $\dfrac{\partial u}{\partial x}$ (c) $\dfrac{\partial u}{\partial y}$ (d) $\dfrac{\partial u}{\partial z}$
24. If $u = x^2 + 3xy + xz$, find the following.
    (a) u_x (b) u_y (c) u_z

25. If $C(x_1, x_2, x_3) = 4x_1^2 + 5x_1x_2 + 6x_2^2 + x_3$, find the following.
(a) $\dfrac{\partial C}{\partial x_1}$ (b) $\dfrac{\partial C}{\partial x_2}$ (c) $\dfrac{\partial C}{\partial x_3}$

26. If $f(x, y, z) = 2x\sqrt{yz - 1} + x^2z^3$, find the following.
(a) $\dfrac{\partial f}{\partial x}$ (b) $\dfrac{\partial f}{\partial y}$ (c) $\dfrac{\partial f}{\partial z}$

27. If $z = x^2 + 4x - 5y^3$, find the following.
(a) z_{xx} (b) z_{xy} (c) z_{yx} (d) z_{yy}

28. If $z = x^3 - 5y^2 + 4y + 1$, find the following.
(a) z_{xx} (b) z_{xy} (c) z_{yx} (d) z_{yy}

29. If $z = x^2y - 4xy^2$, find the following.
(a) z_{xx} (b) z_{xy} (c) z_{yx} (d) z_{yy}

30. If $z = xy^2 + 4xy - 5$, find the following.
(a) z_{xx} (b) z_{xy} (c) z_{yx} (d) z_{yy}

31. If $z = x^2 - xy + 4y^3$, find z_{xyx}.

32. If $z = x^3 - 4x^2y + 5y^3$, find z_{yyx}.

33. If $f(x, y) = x^3y + 4xy^4$, find $\left.\dfrac{\partial^2}{\partial x^2} f(x, y)\right|_{(1, -1)}$.

34. If $f(x, y) = x^4y^2 + 4xy$, find $\left.\dfrac{\partial^2}{\partial y^2} f(x, y)\right|_{(1, 2)}$.

35. If $f(x, y) = \dfrac{2x}{x^2 + y^2}$, find the following.
(a) $\left.\dfrac{\partial^2 f}{\partial x^2}\right|_{(-1, 4)}$ (b) $\left.\dfrac{\partial^2 f}{\partial y^2}\right|_{(-1, 4)}$

36. If $f(x, y) = \dfrac{2y^2}{3xy + 4}$, find the following.
(a) $\left.\dfrac{\partial^2 f}{\partial x^2}\right|_{(1, -2)}$ (b) $\left.\dfrac{\partial^2 f}{\partial y^2}\right|_{(1, -2)}$

37. If $z = x^2y + ye^{x^2}$, find $z_{yx}|_{(1, 2)}$.

38. If $z = xy^3 + x \ln y^2$, find $z_{xy}|_{(1, 2)}$.

39. If $f(x, y) = x^2 + e^{xy}$, find the following.
(a) $\dfrac{\partial^2 f}{\partial x^2}$ (b) $\dfrac{\partial^2 f}{\partial y\, \partial x}$ (c) $\dfrac{\partial^2 f}{\partial x\, \partial y}$ (d) $\dfrac{\partial^2 f}{\partial y^2}$

40. If $z = xe^{xy}$, find the following.
(a) z_{xx} (b) z_{yy} (c) z_{xy} (d) z_{yx}

41. If $f(x, y) = y^2 - \ln xy$, find the following.
(a) $\dfrac{\partial^2 f}{\partial x^2}$ (b) $\dfrac{\partial^2 f}{\partial y\, \partial x}$
(c) $\dfrac{\partial^2 f}{\partial x\, \partial y}$ (d) $\dfrac{\partial^2 f}{\partial y^2}$

42. If $f(x, y) = x^3 + \ln(xy - 1)$, find the following.
(a) $\dfrac{\partial^2 f}{\partial x^2}$ (b) $\dfrac{\partial^2 f}{\partial y^2}$
(c) $\dfrac{\partial^2 f}{\partial x\, \partial y}$ (d) $\dfrac{\partial^2 f}{\partial y\, \partial x}$

43. If $w = 4x^3y + y^2z + z^3$, find the following.
(a) w_{xxy} (b) w_{xyx} (c) w_{xyz}

44. If $w = 4xyz + x^3y^2z + x^3$, find the following.
(a) w_{xyz} (b) w_{xzz} (c) w_{yyz}

Applications

45. ***Mortgage*** When a homeowner has a 25-year variable-rate mortgage loan, the monthly payment R is a function of the amount of the loan A and the current interest rate i (as a percent); that is, $R = f(A, i)$. Interpret each of the following.
(a) $f(100{,}000, 8) = 1289$
(b) $\dfrac{\partial f}{\partial i}(100{,}000, 8) = 62.51$

46. ***Mass transportation ridership*** Suppose that in a certain city, the number of people N using the mass transportation system is a function of the fare f and the daily cost of downtown parking p, so that $N = N(f, p)$. Interpret each of the following.
(a) $N(5, 10) = 6500$
(b) $\dfrac{\partial N}{\partial f}(5, 10) = -400$
(c) $\dfrac{\partial N}{\partial p}(5, 10) = 250$

47. ***Wilson's lot size formula*** In economics, the most economical quantity Q of goods (TVs, dresses, gallons of paint, etc.) for a store to order is given by Wilson's lot size formula

$$Q = \sqrt{2KM/h}$$

where K is the cost of placing the order, M is the number of items sold per week, and h is the weekly holding costs for each item (the cost of storage space, utilities, taxes, security, etc.).
(a) Explain why $\dfrac{\partial Q}{\partial M}$ will be positive.
(b) Explain why $\dfrac{\partial Q}{\partial h}$ will be negative.

48. ***Cost*** Suppose that the total cost of producing a product is $C(x, y) = 25 + 2x^2 + 3y^2$, where x is the cost per pound for material and y is cost per hour for labor.
(a) If material costs are held constant, at what rate will the total cost increase for each \$1-per-hour increase in labor?
(b) If the labor costs are held constant, at what rate will the total cost increase for each increase of \$1 in material cost?

49. ***Pesticide*** Suppose that the number of insects killed by two brands of pesticide is given by

$$f(x, y) = 10{,}000 - 6500e^{-0.01x} - 3500e^{-0.02y}$$

where x is the number of liters of brand 1 and y is the number of liters of brand 2.
(a) What is the rate of change of insect deaths with respect to the number of liters of brand 1?
(b) What is the rate of change of insect deaths with respect to the number of liters of brand 2?

50. ***Profit*** Suppose that the profit from the sale of Kisses and Kreams is given by

$$P(x, y) = 100x + 64y - 0.01x^2 - 0.25y^2$$

where x is the number of pounds of Kisses and y is the number of pounds of Kreams.
(a) Find $\partial P/\partial x$, and give the approximate rate of change of profit with respect to the number of pounds of Kisses if present sales are 20 pounds of Kisses and 10 pounds of Kreams.
(b) Find $\partial P/\partial y$, and give the approximate rate of change of profit with respect to the number of pounds of Kreams that are sold if 100 pounds of Kisses and 16 pounds of Kreams are currently being sold.

51. ***Utility*** If $U = f(x, y)$ is the utility function for goods X and Y, the *marginal utility* of X is $\partial U/\partial x$ and the *marginal utility* of Y is $\partial U/\partial y$. If $U = x^2y^2$, find the marginal utility of
(a) X. (b) Y.

52. ***Utility*** If the utility function for goods X and Y is $U = xy + y^2$, find the marginal utility of
(a) X. (b) Y.

53. ***Production*** Suppose that the output Q (in thousands of units) of a certain company is $Q = 75K^{1/3}L^{2/3}$, where K is the capital expenditures in thousands of dollars and L is the number of labor hours. Find $\partial Q/\partial K$ and $\partial Q/\partial L$ when capital expenditures are \$729,000 and the labor hours total 1728. Interpret each answer.

54. ***Production*** Suppose that the production Q (in hundreds of gallons of paint) of a paint manufacturer can be modeled by $Q = 140K^{1/2}L^{1/2}$, where K is the company's capital expenditures in thousands of dollars and L is the size of the labor force (in hours worked). Find $\partial Q/\partial K$ and $\partial Q/\partial L$ when capital expenditures are \$250,000 and the labor hours are 1225. Interpret each answer.

Wind chill factor Dr. Paul Siple conducted studies testing the effect of wind on the formation of ice at various temperatures and developed the concept of the wind chill factor, which we hear reported during winter weather reports. The wind chill temperatures for selected air temperatures and wind speeds are shown in the table below. For example, the table shows that an air temperature of 15°F together with a wind speed of 35 mph feels the same as an air temperature of −27°F when there is no wind.

Wind Speed (mph)	*Air Temperature (°F)*						
	35	*25*	*15*	*5*	*−5*	*−15*	*−25*
5	33	21	12	0	−10	−21	−31
15	16	2	−11	−25	−38	−51	−65
25	8	−7	−22	−36	−51	−66	−81
35	4	−12	−27	−43	−58	−74	−89
45	2	−14	−30	−46	−62	−78	−93

Source: *World Almanac*, 1991

One form of the formula that meteorologists use to calculate wind chill temperatures is

$$\begin{aligned} WC &= 48.064 + 0.474t - 0.020ts - 1.85s \\ &\quad + 0.304t\sqrt{s} - 27.74\sqrt{s} \end{aligned}$$

where s is wind speed and t is the actual air temperature. Use this equation to answer Problems 55 and 56.

55. (a) To see how the wind chill temperature changes with wind speed, find $\partial WC/\partial s$.
(b) Find $\partial WC/\partial s$ when the temperature is 10°F and the wind speed is 25 mph. What does this mean?

56. (a) To see how wind chill temperature changes with temperature, find $\partial WC/\partial t$.
(b) Find $\partial WC/\partial t$ when the temperature is 10°F and the wind speed is 25 mph. What does this mean?

14.3 Applications of Functions of Two Variables in Business and Economics

OBJECTIVES

- To evaluate cost functions at given levels of production
- To find marginal costs from total cost and joint cost functions
- To find marginal productivity for given production functions
- To find marginal demand functions from demand functions for a pair of related products

APPLICATION PREVIEW

In this section, we consider three classes of applications of functions of two variables and their partial derivatives. We begin with joint cost functions and their marginals. Next, we consider production functions and revisit Cobb-Douglas production functions. Marginal productivity is introduced and its meaning is explained. Finally, we consider demand functions for two products in a competitive market. Partial derivatives are used to define marginal demands, and these marginals are used to classify the products as competitive or complementary.

Joint Cost and Marginal Cost

Suppose that a firm produces two commodities using the same inputs in different proportions. In such a case the **joint cost function** is of the form $C = Q(x, y)$, where x and y represent the quantities of each commodity and C represents the total cost for the two commodities. Then $\partial C/\partial x$ is the **marginal cost** with respect to product x, and $\partial C//\partial y$ is the **marginal cost** with respect to product y.

EXAMPLE 1

If the joint cost function for two products is

$$C = Q(x, y) = 50 + x^2 + 8xy + y^3$$

find the marginal cost with respect to the following.

(a) x (b) y (c) x at (5, 3) (d) y at (5, 3)

Solution

(a) The marginal cost with respect to x is $\partial C/\partial x = 2x + 8y$.

(b) The marginal cost with respect to y is $\dfrac{\partial C}{\partial y} = 8x + 3y^2$.

(c) $\left.\dfrac{\partial C}{\partial x}\right|_{(5,\,3)} = 2(5) + 8(3) = 34$

Thus if 5 units of product x and 3 units of product y are produced, the total cost will increase approximately \$34 for each unit increase in product x if y is held constant.

(d) $\left.\dfrac{\partial C}{\partial y}\right|_{(5,\,3)} = 8(5) + 3(3)^2 = 67$

Thus if 5 units of product x and 3 units of product y are produced, the total cost will increase approximately \$67 for each unit increase in product y if x is held constant.

Production Functions

An important problem in economics concerns how the factors necessary for production determine the output of a product. For example, the output of a product depends on available labor, land, capital, material, and machines. If the amount of output z of a product depends on the amounts of two inputs x and y, then the quantity z is given by the **production function** $z = f(x, y)$.

EXAMPLE 2

Suppose that it is known that z bushels of a crop can be harvested according to the function

$$z = (21)\frac{6xy - 4x^2 - 3y}{2x + 0.01y}$$

when $100x$ work-hours of labor are employed on y acres of land. What would be the output (in bushels) if 200 work-hours were used on 300 acres?

Solution

Because $z = f(x, y)$,

$$\begin{aligned} f(2, 300) &= (21)\frac{6(2)(300) + 4(2)^2 - 3(300)}{2(2) + 3} \\ &= (21)\frac{3600 + 16 - 900}{7} = 8148 \quad \text{(bushels)} \end{aligned}$$

If $z = f(x, y)$ is a production function, $\partial z/\partial x$ represents the change in the output z with respect to input x while input y remains constant. This partial derivative is called the **marginal productivity of x.** The partial derivative $\partial z/\partial y$ is the **marginal productivity of y** and measures the rate of change of z with respect to input y.

Marginal productivity (for either input) will be positive over a wide range of inputs, but it increases at a decreasing rate, and it may eventually reach a point where it no longer increases and begins to decrease.

EXAMPLE 3

If a production function is given by $z = 5x^{1/2}y^{1/4}$, find the marginal productivity of

(a) x. (b) y.

Solution

(a) $\dfrac{\partial z}{\partial x} = \dfrac{5}{2}x^{-1/2}y^{1/4}$ (b) $\dfrac{\partial z}{\partial y} = \dfrac{5}{4}x^{1/2}y^{-3/4}$

Note that the marginal productivity of x is positive for all values of x but that it decreases as x gets larger (because of the negative exponent). The same is true for the marginal productivity of y.

Graphing Utilities If we have a production function and fix a value for one variable, then we can use a graphing utility to analyze the marginal productivity with respect to the other variable.

EXAMPLE 4
Suppose the Cobb-Douglas production function for a company is given by

$$z = 100x^{1/4}y^{3/4}$$

where x is the company's capital investment and y is the size of the labor force (in work-hours).

(a) Find the marginal productivity of x.
(b) If the current labor force is 625 work-hours, substitute $y = 625$ in your answer to (a) and graph the result.
(c) From the graph in (b), what can be said about the effect on production of additional capital investment when the work-hours remain at 625?
(d) Find the marginal productivity of y.
(e) If current capital investment is \$10,000, substitute $x = 10{,}000$ in your answer to (d) and graph the result.
(f) From the graph in (e), what can be said about the effect on production of additional work-hours when capital investment remains at \$10,000?

Solution

(a) $z_x = 25x^{-3/4}y^{3/4}$
(b) If $y = 625$, then z_x becomes

$$z_x = 25x^{-3/4}(625)^{3/4} = 25\left(\frac{1}{x^{3/4}}\right)(125) = \frac{3125}{x^{3/4}}$$

The graph of z_x can be limited to quadrant I because the capital investment is $x > 0$, and hence $z_x > 0$. Knowledge of asymptotes can help us determine range values for x and z_x that give an accurate graph. See Figure 14.6.

(c) Figure 14.6 shows that $z_x > 0$ for $x > 0$. This means that any increases in capital investment will result in increases in productivity. However, Figure 14.6 also shows that z_x is decreasing for $x > 0$, which means that increases in capital investment have a diminishing impact on productivity.

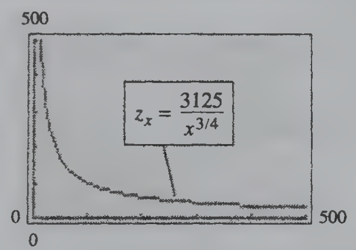


Figure 14.6

(d) $z_y = 75x^{1/4}y^{-1/4}$
(e) If $x = 10{,}000$, then z_y becomes

$$z_y = 75(10{,}000)^{1/4}\left(\frac{1}{y^{1/4}}\right) = \frac{750}{y^{1/4}}$$

The graph is shown in Figure 14.7.

(f) Figure 14.7 also shows that $z_y > 0$ when $y > 0$, so increasing work-hours increases productivity. Note that z_y is decreasing for $y > 0$ but that it does so more slowly than z_x. This indicates that increases in work-hours have a diminishing impact on productivity, but still a more significant one than increases in capital expenditures (as seen in Figure 14.6).

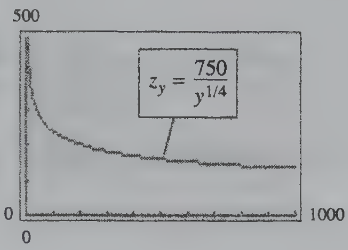


Figure 14.7

Demand Functions

Suppose that two products are sold at prices p_1 and p_2, respectively, in a competitive market consisting of a fixed number of consumers with given tastes and incomes. Then the amount of each *one* of the products demanded by the consumers is dependent on the prices of *both* products on the market. If q_1 represents the demand for the first product, then $q_1 = f(p_1, p_2)$ is the **demand function** for that product. The graph of such a function is called a **demand surface.** An example of a demand function in two variables is $q_1 = 400 - 2p_1 - 4p_2$. Here q_1 is a function of two variables p_1 and p_2. If $p_1 = \$10$ and $p_2 = \$20$, the demand would equal $400 - 2(10) - 4(20) = 300$.

EXAMPLE 5

The demand functions for two products are

$$q_1 = 50 - 5p_1 - 2p_2$$
$$q_2 = 100 - 3p_1 - 8p_2$$

(a) What is the demand for each of the products if the price of the first is $p_1 = \$5$ and the price of the second is $p_2 = \$8$?

(b) Find a pair of prices p_1 and p_2 such that the demands for product 1 and product 2 are equal.

Solution

(a)
$$q_1 = 50 - 5(5) - 2(8) = 9$$
$$q_2 = 100 - 3(5) - 8(8) = 21$$

Thus if these are the prices, the demand for product 2 is higher than the demand for product 1.

(b) We want q_1 to equal q_2. Setting $q_1 = q_2$, we see that

$$50 - 5p_1 - 2p_2 = 100 - 3p_1 - 8p_2$$
$$6p_2 - 50 = 2p_1$$
$$p_1 = 3p_2 - 25$$

Now, any pair of positive values that satisfies this equation will make the demands equal. Letting $p_2 = 10$, we see that $p_1 = 5$ will satisfy the equation. Thus the prices $p_1 = 5$ and $p_2 = 10$ will make the demands equal. The prices $p_1 = 2$ and $p_2 = 9$ will also make the demands equal. Many pairs of values (that is, all those satisfying $p_1 = 3p_2 - 25$) will equalize the demands.

If the demand functions for a pair of related products, product 1 and product 2, are $q_1 = f(p_1, p_2)$ and $q_2 = g(p_1, p_2)$, respectively, then the partial derivatives of q_1 and q_2 are called **marginal demand functions.**

$\dfrac{\partial q_1}{\partial p_1}$ is the marginal demand of q_1 with respect to p_1.

$\dfrac{\partial q_1}{\partial p_2}$ is the marginal demand of q_1 with respect to p_2.

$\dfrac{\partial q_2}{\partial p_1}$ is the marginal demand of q_2 with respect to p_1.

$\dfrac{\partial q_2}{\partial p_2}$ is the marginal demand of q_2 with respect to p_2.

For typical demand functions, if the price of product 2 is fixed, the demand for product 1 will decrease as its price p_1 increases. In this case the marginal demand of q_1 with respect to p_1 will be negative; that is, $\partial q_1/\partial p_1 < 0$. Similarly, $\partial q_2/\partial p_2 < 0$.

But what about $\partial q_2/\partial p_1$ and $\partial q_1/\partial p_2$? If $\partial q_2/\partial p_1$ and $\partial q_1/\partial p_2$ are both positive, the two products are **competitive** because an increase in price p_1 will result in an increase in demand for product 2 (q_2) if the price p_2 is held constant, and an increase in price p_2 will increase the demand for product 1 (q_1) if p_1 is held constant. Stated more simply, an increase in the price of one of the two products will result in an increased demand for the other, so the products are in competition. For example, an increase in the price of a Japanese automobile will result in an increase in demand for an American automobile if the price of the American automobile is held constant.

If $\partial q_2/\partial p_1$ and $\partial q_1/\partial p_2$ are both negative, the products are **complementary** because an increase in the price of one product will cause a decrease in demand for the other product if the price of the second product doesn't change. Under these conditions, a *decrease* in the price of product 1 will result in an *increase* in the demand for product 2, and a decrease in the price of product 2 will result in an increase in the demand for product 1. For example, a decrease in the price of gasoline will result in an increase in the demand for large automobiles.

If the signs of $\partial q_2/\partial p_1$ and $\partial q_1/\partial p_2$ are different, the products are neither competitive nor complementary. This situation rarely occurs but is possible.

EXAMPLE 6

The demand functions for two related products, product 1 and product 2, are given by

$$q_1 = 400 - 5p_1 + 6p_2 \qquad q_2 = 250 + 4p_1 - 5p_2$$

(a) Determine the four marginal demands.
(b) Are product 1 and product 2 complementary or competitive?

Solution

(a) $\frac{\partial q_1}{\partial p_1} = -5 \qquad \frac{\partial q_2}{\partial p_2} = -5 \qquad \frac{\partial q_1}{\partial p_2} = 6 \qquad \frac{\partial q_2}{\partial p_1} = 4$

(b) Because $\partial q_1/\partial p_2$ and $\partial q_2/\partial p_1$ are positive, products 1 and 2 are competitive.

CHECKPOINT

1. If the joint cost function for two products is

$$C = 100 + 3x + 10xy + y^2$$

find the marginal cost with respect to

(a) x. (b) y at (7, 3).

2. If the production function for a product is

$$P = 10x^{1/4}y^{1/2}$$

find the marginal productivity of x.

3. If the demand functions for two products are

$$q_1 = 200 - 3p_1 - 4p_2 \quad \text{and} \quad q_2 = 50 - 6p_1 - 5p_2$$

find the marginal demand of

(a) q_1 with respect to p_1. (b) q_2 with respect to p_2.

CHECKPOINT SOLUTIONS

1. (a) $\frac{\partial C}{\partial x} = 3 + 10y$ (b) $\frac{\partial C}{\partial y} = 10x + 2y$

$\frac{\partial C}{\partial y}(7, 3) = 10(7) + 2(3) = 76$

2. $\frac{\partial P}{\partial x} = \frac{2.5y^{1/2}}{x^{3/4}}$

3. (a) $\frac{\partial q_1}{\partial p_1} = -3$ (b) $\frac{\partial q_2}{\partial p_2} = -5$

EXERCISE 14.3

Joint Cost and Marginal Cost

1. The cost of manufacturing one item is given by

$$C(x, y) = 30 + 3x + 5y$$

where x is the cost of 1 hour of labor and y is the cost of 1 pound of material. If the hourly cost of labor is \$4, and the material costs \$3 per pound, what is the cost of manufacturing one of these items?

2. The manufacture of 1 unit of a product has a cost given by

$$C(x, y, z) = 10 + 8x + 3y + z$$

where x is the cost of 1 pound of one raw material, y is the cost of 1 pound of a second material, and z is the cost of 1 work-hour of labor. If the cost of the first raw material is \$16 per pound, the cost of the second raw material is \$8 per pound, and labor costs \$8 per work-hour, what will it cost to produce 1 unit of the product?

3. The total cost of producing 1 unit of a product is

$$C(x, y) = 30 + 2x + 4y + \frac{xy}{50}$$

where x is the cost per pound of raw materials and y is the cost per hour of labor.
   (a) If labor costs are held constant, at what rate will the total cost increase for each increase of \$1 per pound in material cost?
   (b) If material costs are held constant, at what rate will the total cost increase for each \$1 per hour increase in labor costs?
4. The total cost of producing an item is

$$C(x, y) = 40 + 4x + 6y + \frac{x^2y}{100}$$

where x is the cost per pound of raw materials and y is the cost per hour for labor. How will an increase of
   (a) \$1 per pound of raw materials affect the total cost?
   (b) \$1 per hour in labor costs affect the total cost?
5. The total cost of producing 1 unit of a product is given by

$$C(x, y) = 20x + 70y + \frac{x^2}{1000} + \frac{xy^2}{100}$$

where x represents the cost per pound of raw materials and y represents the hourly rate for labor. The present cost for raw materials is \$10 per pound and the present hourly rate for labor is \$4. How will an increase of
   (a) \$1 per pound for raw materials affect the total cost?
   (b) \$1 per hour in labor costs affect the total cost?
6. The total cost of producing 1 unit of a product is given by

$$C(x, y) = 30 + 10x^2 + 20y - xy$$

where x is the hourly labor rate and y is the cost per pound of raw materials. The current hourly rate is \$5, and the raw materials cost \$6 per pound. How will an increase of
   (a) \$1 per pound for the raw materials affect the total cost?
   (b) \$1 in the hourly labor rate affect the total cost?
7. The joint cost function for two products is

$$C(x, y) = 30 + x^2 + 3y + 2xy$$

where x represents the quantity of product X produced and y represents the quantity of product Y produced.
   (a) Find the marginal cost with respect to x if 8 units of product X and 10 units of product Y are produced.
   (b) Find the marginal cost with respect to y if 8 units of product X and 10 units of product Y are produced.
8. The joint cost function for products X and Y is

$$C(x, y) = 40 + 3x^2 + y^2 + xy$$

where x represents the quantity of X and y represents the quantity of Y.
   (a) Find the marginal cost with respect to x if 20 units of product X and 15 units of product Y are produced.
   (b) Find the marginal cost with respect to y if 20 units of X and 15 units of Y are produced.
9. If the joint cost function for two products is

$$C(x, y) = x\sqrt{y^2 + 1}$$

   (a) Find the marginal cost (function) with respect to x.
   (b) Find the marginal cost with respect to y.
10. Suppose the joint cost function for x units of product X and y units of product Y is given by

$$C(x, y) = 2500\sqrt{xy + 1}$$

Find the marginal cost with respect to
   (a) x. (b) y.
11. Suppose that the joint cost function for two products is

$$C(x, y) = 1200 \ln(xy + 1) + 10{,}000$$

Find the marginal cost with respect to
   (a) x. (b) y.
12. Suppose that the joint cost function for two products is

$$C(x, y) = y \ln(x + 1)$$

Find the marginal cost with respect to
   (a) x. (b) y.

Production Functions

13. Suppose that the production function for a product is $z = \sqrt{4xy}$, where x represents the number of work-hours per month and y is the number of available machines. Determine the marginal productivity of
   (a) x. (b) y.

14. Suppose the production function for a product is

$$z = 60x^{2/5}y^{3/5}$$

where x is the capital expenditures and y is the number of work-hours. Find the marginal productivity of
(a) x. (b) y.

15. Suppose that the production function for a product is $z = \sqrt{x}\ln(y + 1)$, where x represents the number of work-hours and y represents the available capital (per week). Find the marginal productivity of
(a) x. (b) y.

16. Suppose that a company's production function for a certain product is

$$z = (x + 1)^{1/2}\ln(y^2 + 1)$$

where x is the number of work-hours of unskilled labor and y is the number of work-hours of skilled labor. Find the marginal productivity of
(a) x. (b) y.

For Problems 17–19, suppose that the production function for an agricultural product is given by

$$z = \frac{11xy - 0.0002x^2 - 5y}{0.03x + 3y}$$

where x is the number of hours of labor and y is the number of acres of the crop.

17. Find the output when $x = 300$ and $y = 500$.
18. Find the marginal productivity of the number of acres of the crop (y) when $x = 300$ and $y = 500$.
19. Find the marginal productivity of the number of hours of labor (x) when $x = 300$ and $y = 500$.
20. If a production function is given by $z = 12x^{3/4}y^{1/3}$, find the marginal productivity of
(a) x. (b) y.

21. Suppose the Cobb-Douglas production function for a company is given by

$$z = 400x^{3/5}y^{2/5}$$

where x is the company's capital investment and y is the size of the labor force (in work-hours).
(a) Find the marginal productivity of x.
(b) If the current labor force is 1024 work-hours, substitute $y = 1024$ in your answer to (a) and graph the result.
(c) Find the marginal productivity of y.
(d) If the current capital investment is \$59,049, substitute $x = 59{,}049$ in your answer to (c) and graph the result.
(e) Interpret the graphs in (b) and (d) with regard to what they say about the effects on productivity of an increased capital investment (part b) and of an increased labor force (part d).

22. Suppose the Cobb-Douglas production function for a company is given by

$$z = 300x^{2/3}y^{1/3}$$

where x is the company's capital investment and y is the size of the labor force (in work-hours).
(a) Find the marginal productivity of x.
(b) If the current labor force is 729 work-hours, substitute $y = 729$ in your answer to (a) and graph the result.
(c) Find the marginal productivity of y.
(d) If the current capital investment is \$27,000, substitute $x = 27{,}000$ in your answer to (c) and graph the result.
(e) Interpret the graphs in (b) and (d) with regard to what they say about the effects on productivity of an increased capital investment (part b) and of an increased labor force (part d).

Demand Functions

23. The demand functions for two products are given by

$$q_1 = 300 - 8p_1 - 4p_2$$
$$q_2 = 400 - 5p_1 - 10p_2$$

Find the demand for each of the products if the price of the first is $p_1 = 10$ and the price of the second is $p_2 = 8$.

24. The demand functions for two products are given by

$$q_1 = 900 - 9p_1 + 2p_2$$
$$q_2 = 1200 + 6p_1 - 10p_2$$

Find the demands q_1 and q_2 if $p_1 = \$10$ and $p_2 = \$12$.

25. Find a pair of prices p_1 and p_2 such that the demands for the two products in Problem 23 will be equal.
26. Find a pair of prices p_1 and p_2 such that the demands for the two products in Problem 24 will be equal.

In Problems 27–30, the demand functions for two related products, A and B, are given. Complete (a)–(e) for each problem.
(a) Find the marginal demand of q_A with respect to p_A.
(b) Find the marginal demand of q_A with respect to p_B.
(c) Find the marginal demand of q_B with respect to p_B.
(d) Find the marginal demand of q_B with respect to p_A.
(e) Are the two goods competitive or complementary?

27. $\begin{cases} q_A = 400 - 3p_A - 2p_B \\ q_B = 250 - 5p_A - 6p_B \end{cases}$

28. $\begin{cases} q_A = 600 - 4p_A + 6p_B \\ q_B = 1200 + 8p_A - 4p_B \end{cases}$

29. $\begin{cases} q_A = 5000 - 50p_A - \dfrac{600}{p_B + 1} \\ q_B = 10{,}000 - \dfrac{400}{p_A + 4} + \dfrac{400}{p_B + 4} \end{cases}$

30. $\begin{cases} q_A = 2500 + \dfrac{600}{p_A + 2} - 40p_B \\ q_B = 3000 - 100p_A + \dfrac{400}{p_B + 5} \end{cases}$

14.4 Maxima and Minima

OBJECTIVES

- *To find relative maxima, minima, and saddle points of functions of two variables*
- *To apply linear regression formulas*

APPLICATION PREVIEW

Adele Lighting manufactures 20-inch lamps and 31-inch lamps. Suppose that x is the number of thousands of 20-inch lamps and that the demand for these is given by $p_1 = 50 - x$, where p_1 is in dollars. Similarly, suppose that y is the number of thousands of 31-inch lamps and that the demand for these is given by $p_2 = 60 - 2y$, where p_2 is also in dollars. Adele Lighting's joint cost function for these lamps is $C = 2xy$ (in thousands of dollars). Therefore, Adele Lighting's profit (in thousands of dollars) is a function of the two variables x and y. In order to determine Adele's maximum profit, we need to develop methods for finding maximum values for a function of two variables.

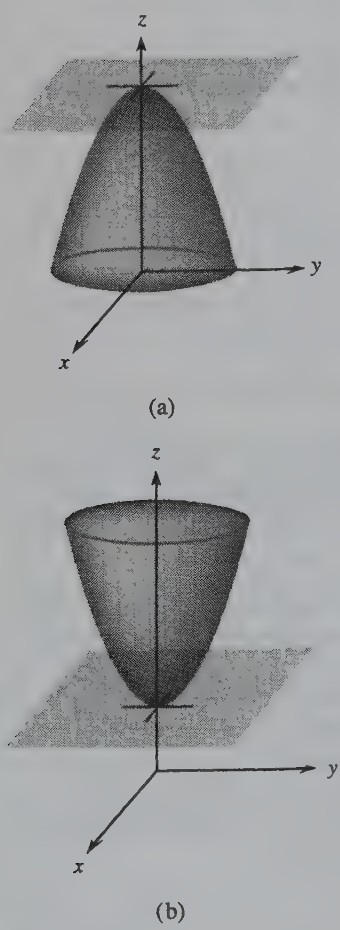


Figure 14.8

In our study of differentiable functions of one variable, we saw that for a relative maximum or minimum to occur at a point, the tangent line to the curve had to be horizontal at that point. The function $z = f(x, y)$ describes a surface in three dimensions. If all partial derivatives of $f(x, y)$ exist, then the surface described by $z = f(x, y)$ must have a horizontal plane tangent to the surface at a point in order to have a relative maximum or minimum at that point (see Figure 14.8). But if the plane tangent to the surface at the point is horizontal, then all the tangent lines to the surface at that point must also be horizontal, for they lie in the tangent plane. In particular, the tangent line in the direction of the x-axis will be horizontal, so $\partial z/\partial x = 0$ at the point; and the tangent line in the direction of the y-axis will be horizontal, so $\partial z/\partial y = 0$ at the point. Thus we can determine the *critical points* for a surface by finding those points where *both* $\partial z/\partial x = 0$ and $\partial z/\partial y = 0$.

How can we determine whether a critical point is a relative maximum, a relative minimum, or neither of these? Finding that $\partial^2 z/\partial x^2 < 0$ and $\partial^2 z/\partial y^2 < 0$ is not enough to tell us that we have a relative maximum. The "second derivative" test we must use involves the values of the second partial derivatives and the value of D at the critical point (a, b), where D is defined as follows:

$$D = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

We shall state, without proof, the result that determines whether there is a relative maximum, a relative minimum, or neither at the critical point (a, b).

Test for Maxima and Minima

Let $z = f(x, y)$ be a function for which

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0 \quad \text{at a point } (a, b)$$

and suppose that all second partial derivatives are continuous there. Evaluate

$$D = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

at the critical point (a, b), and conclude the following.

a. If $D > 0$ and $\partial^2 z/\partial x^2 > 0$ at (a, b), then a relative minimum occurs at (a, b). In this case, $\partial^2 z/\partial y^2 > 0$ at (a, b) also.
b. If $D > 0$ and $\partial^2 z/\partial x^2 < 0$ at (a, b), then a relative maximum occurs at (a, b). In this case, $\partial^2 z/\partial y^2 < 0$ at (a, b) also.
c. If $D < 0$ at (a, b), there is neither a relative maximum nor a relative minimum at (a, b).
d. If $D = 0$ at (a, b), the test fails; investigate the function near the point.

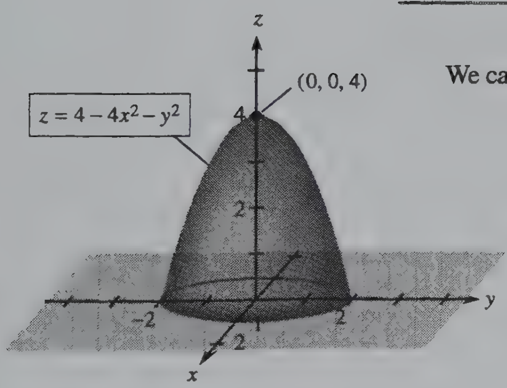


Figure 14.9

We can test for relative maxima and minima by using the following procedure.

Maxima and Minima of z = f(x, y)

Procedure	Example
To find relative maxima and minima of $z = f(x, y)$:	Test $z = 4 - 4x^2 - y^2$ for relative maxima and minima.
1. Find $\partial z/\partial x$ and $\partial z/\partial y$.	1. $\frac{\partial z}{\partial x} = -8x$; $\frac{\partial z}{\partial y} = -2y$
2. Find the point(s) that satisfy *both* $\partial z/\partial x = 0$ and $\partial z/\partial y = 0$. These are the critical points.	2. $\frac{\partial z}{\partial x} = 0$ if $x = 0$. $\frac{\partial z}{\partial y} = 0$ if $y = 0$. The critical point is $(0, 0, 4)$.
3. Find all second partial derivatives.	3. $\frac{\partial^2 z}{\partial x^2} = -8$; $\frac{\partial^2 z}{\partial y^2} = -2$; $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x} = 0$
4. Evaluate D at each critical point.	4. At $(0, 0)$, $D = (-8)(-2) - 0^2 = 16$.
5. Use the test for maxima and minima to determine whether relative maxima or minima occur.	5. $D > 0$, $\partial^2 z/\partial x^2 < 0$, and $\partial^2 z/\partial y^2 < 0$. A relative maximum occurs at $(0, 0)$. See Figure 14.9.

EXAMPLE 1

Test $z = x^2 + y^2 - 2x + 1$ for relative maxima and minima.

Solution

1. $\dfrac{\partial z}{\partial x} = 2x - 2; \quad \dfrac{\partial z}{\partial y} = 2y$
2. $\dfrac{\partial z}{\partial x} = 0$ if $x = 1$. $\quad \dfrac{\partial z}{\partial y} = 0$ if $y = 0$.
   Both are 0 if $x = 1$ *and* $y = 0$, so the critical point is $(1, 0, 0)$.
3. $\dfrac{\partial^2 z}{\partial x^2} = 2; \quad \dfrac{\partial^2 z}{\partial y^2} = 2; \quad \dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{\partial^2 z}{\partial y\, \partial x} = 0$
4. At $(1, 0)$, $D = 2 \cdot 2 - 0^2 = 4$.
5. $D > 0$, $\partial^2 z/\partial x^2 > 0$, and $\partial^2 z/\partial y^2 > 0$. A relative minimum occurs at $(1, 0)$. (See Figure 14.10.)

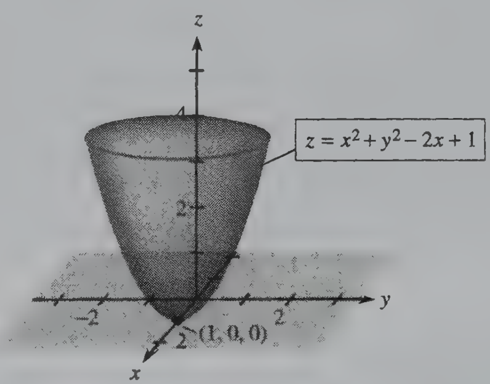


Figure 14.10

EXAMPLE 2

Test $z = y^2 - x^2$ for relative maxima and minima.

Solution

1. $\dfrac{\partial z}{\partial x} = -2x; \quad \dfrac{\partial z}{\partial y} = 2y$
2. $\dfrac{\partial z}{\partial x} = 0$ if $x = 0$; $\quad \dfrac{\partial z}{\partial y} = 0$ if $y = 0$.
   Thus both equal 0 if $x = 0$, $y = 0$. The critical point is $(0, 0, 0)$.
3. $\dfrac{\partial^2 z}{\partial x^2} = -2; \quad \dfrac{\partial^2 z}{\partial y^2} = 2; \quad \dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{\partial^2 z}{\partial y\, \partial x} = 0$
4. $D = (-2)(2) - 0 = -4$
5. $D < 0$, so the critical point is neither a relative maximum nor a relative minimum. As Figure 14.11 shows, the surface formed has the shape of a saddle. For this reason, critical points that are neither relative maxima nor relative minima are called **saddle points.**

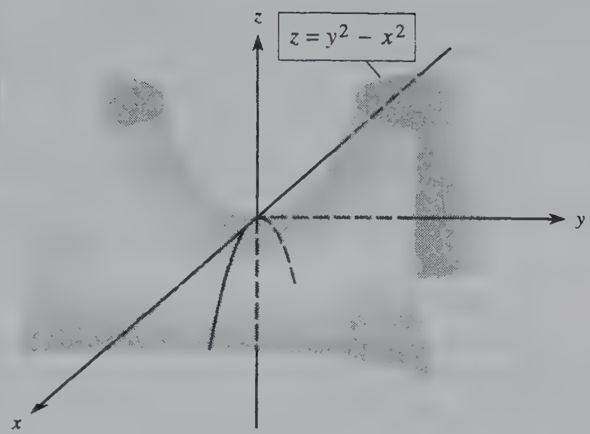


Figure 14.11

The following example involves a surface with two critical points.

EXAMPLE 3

Test $z = x^3 + y^3 + 6xy$ for relative maxima and minima.

Solution

1. $\dfrac{\partial z}{\partial x} = 3x^2 + 6y; \quad \dfrac{\partial z}{\partial y} = 3y^2 + 6x$
2. $\dfrac{\partial z}{\partial x} = 0$ if $0 = 3x^2 + 6y$—that is, if $y = -\frac{1}{2}x^2$.

   $\dfrac{\partial z}{\partial y} = 0$ if $0 = 3y^2 + 6x$—that is, if $x = -\frac{1}{2}y^2$.

   Because *both* conditions must be satisfied, we can substitute $-\frac{1}{2}y^2$ for x in $y = -\frac{1}{2}x^2$ and obtain

$$y = -\frac{1}{2}\left(-\frac{1}{2}y^2\right)^2$$
$$y = -\frac{1}{8}y^4$$
$$y + \frac{1}{8}y^4 = 0$$
$$y(8 + y^3) = 0$$

   Hence $y = 0$ or $y^3 = -8$; thus, $y = 0$ or $y = -2$. If $y = 0$, $x = -\frac{1}{2}(0)^2 = 0$, so one critical point is $(0, 0, 0)$. If $y = -2$, $x = -\frac{1}{2}(-2)^2 = -2$, so the second critical point is $(-2, -2, 8)$.
3. $\dfrac{\partial^2 z}{\partial x^2} = 6x; \qquad \dfrac{\partial^2 z}{\partial y^2} = 6y; \qquad \dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{\partial^2 z}{\partial y\, \partial x} = 6$

   Thus $D = (6x)(6y) - (6)^2$.
4. At $(0, 0)$, $D = 0 \cdot 0 - (6)^2 = -36 < 0$.

   At $(-2, -2)$, $D = (-12)(-12) - 36 = 108 > 0$.
5. At $(0, 0)$, $D < 0$, so a saddle point occurs at $(0, 0, 0)$.

   At $(-2, -2)$, $D > 0$, $\partial^2 z/\partial x^2 = 6(-2) = -12$, and $\partial^2 z/\partial y^2 = 6(-2) = -12$, so a relative maximum occurs at $(-2, -2)$. It is 8.

CHECKPOINT

Suppose that $z = 4 - x^2 - y^2 + 2x - 4y$.

1. Find z_x and z_y.
2. Solve $z_x = 0$ and $z_y = 0$ simultaneously to find the critical point(s) for the graph of this function.
3. Test the point(s) for relative maxima and minima.

Let us now solve the problem introduced in the Application Preview.

EXAMPLE 4

Maximize Adele Lighting's profit if the demand functions are $p_1 = 50 - x$ for 20-inch lamps and $p_2 = 60 - 2y$ for 31-inch lamps, and if the joint cost function is $C = 2xy$. Recall that x and y are in thousands of lamps, p_1 and p_2 are in dollars, and C is in thousands of dollars.

Solution

The profit function is $P(x, y) = p_1x + p_2y - C(x, y)$. Thus,

$$\begin{aligned} P(x, y) &= (50 - x)x + (60 - 2y)y - 2xy \\ &= 50x - x^2 + 60y - 2y^2 - 2xy \end{aligned}$$

gives the profit in thousands of dollars. To maximize the profit, we proceed as follows.

$$P_x = 50 - 2x - 2y \quad \text{and} \quad P_y = 60 - 4y - 2x$$

Solving simultaneously $P_x = 0$ and $P_y = 0$, we have

$$\begin{cases} 0 = 50 - 2x - 2y \\ 0 = 60 - 2x - 4y \end{cases}$$

Subtraction gives $-10 + 2y = 0$, so $y = 5$. Thus $0 = 40 - 2x$, so $x = 20$. Now

$$P_{xx} = -2, \quad P_{yy} = -4, \quad \text{and} \quad P_{xy} = -2, \quad \text{and}$$
$$D = (P_{xx})(P_{yy}) - (P_{xy})^2 = (-2)(-4) - (-2)^2 = +4$$

Because $P_{xx} < 0$, $P_{yy} < 0$, and $D > 0$, the values $x = 20$ and $y = 5$ yield maximum profit. Therefore, when $x = 20$ and $y = 5$, $p_1 = 30$, $p_2 = 50$, and the maximum profit is

$$P(20, 5) = 600 + 250 - 200 = 650$$

That is, Adele Lighting's maximum profit is \$650,000 when the company sells 20,000 of the 20-inch lamps at \$30 each and 5000 of the 31-inch lamps at \$50 each.

Linear Regression

We have used different types of functions to model cost, revenue, profit, demand, supply, and other real-world relationships. Sometimes we have used calculus to study the behavior of these functions, finding, for example, marginal cost, marginal revenue, producer's surplus, and so on. We now have the mathematical tools to understand and develop the formulas that graphing utilities and other technology use to find the equations for linear models.

The formulas used to find the equation of the straight line that is the best fit for a set of data are developed using max-min techniques for functions of two variables. This line is called the **regression line.** In Figure 14.12, we define line ℓ to be the best fit for the data points (that is, the regression line) if the sum of the squares of the differences between the actual y-values of the data points and the y-values of the points on the line is a minimum.

In general, to find the equation of the regression line, we assume that the relationship between x and y is approximately linear and that we can find a straight line with equation

$$\hat{y} = a + bx$$

where the values of $\hat{y}$ will approximate the y-values of the points we know. That is, for each given value of x, the point $(x, \hat{y})$ will be on the line. For any given x-value, x_i, we are interested in the deviation between the y-value of the data point (x_i, y_i) and the $\hat{y}$-value from the equation, $\hat{y}_i$, that results when x_i is substituted for x. These deviations are of the form

$$d_i = \hat{y}_i - y_i, \qquad \text{for } i = 1, 2, \ldots, n$$

(See Figure 14.12 for a general case with the deviations exaggerated.)

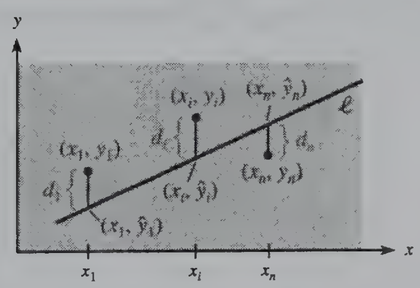


Figure 14.12

To measure the deviations in a way that accounts for the fact that some of the y-values will be above the line and some below, we will say that the line that is the best fit for the data is the one for which the sum of the squares of the deviations is a minimum. That is, we seek the a and b in the equation

$$\hat{y} = a + bx$$

such that the sum of the squares of the deviations,

$$\begin{aligned} S &= \sum_{i=1}^{n} (\hat{y}_i - y_i)^2 = \sum_{i=1}^{n} [(bx_i + a) - y_i]^2 \\ &= (bx_1 + a - y_1)^2 + (bx_2 + a - y_2)^2 + \cdots + (bx_n + a - y_n)^2 \end{aligned}$$

is a minimum. The procedure for determining a and b is called the **method of least squares.**

We seek the values of b and a that make S a minimum, so we find the values that make

$$\frac{\partial S}{\partial b} = 0 \quad \text{and} \quad \frac{\partial S}{\partial a} = 0$$

$$\frac{\partial S}{\partial b} = 2(bx_1 + a - y_1)x_1 + 2(bx_2 + a - y_2)x_2 + \cdots + 2(bx_n + a - y_n)x_n$$

$$\frac{\partial S}{\partial a} = 2(bx_1 + a - y_1) + 2(bx_2 + a - y_2) + \cdots + 2(bx_n + a - y_n)$$

Setting each equation equal to 0, dividing by 2, and using sigma notation give

$$0 = b \sum_{i=1}^{n} x_i^2 + a \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i y_i \qquad (1)$$

$$0 = b \sum_{i=1}^{n} x_i + a \sum_{i=1}^{n} 1 - \sum_{i=1}^{n} y_i \qquad (2)$$

We can write equations (1) and (2) as follows:

$$\sum_{i=1}^{n} x_i y_i = a \sum_{i=1}^{n} x_i + b \sum_{i=1}^{n} x_i^2 \qquad (3)$$

$$\sum_{i=1}^{n} y_i = an + b \sum_{i=1}^{n} x_i \qquad (4)$$

Multiplying equation (3) by n and equation (4) by $\sum_{i=1}^{n} x_i$ permits us to begin to solve for b.

$$n \sum_{i=1}^{n} x_i y_i = na \sum_{i=1}^{n} x_i + nb \sum_{i=1}^{n} x_i^2 \qquad (5)$$

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i = na \sum_{i=1}^{n} x_i + b\left(\sum_{i=1}^{n} x_i\right)^2 \qquad (6)$$

Subtracting equation (5) from equation (6) gives

$$\begin{aligned} \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i - n \sum_{i=1}^{n} x_i y_i &= b\left(\sum_{i=1}^{n} x_i\right)^2 - nb \sum_{i=1}^{n} x_i^2 \\ &= b\left[\left(\sum_{i=1}^{n} x_i\right)^2 - n \sum_{i=1}^{n} x_i^2\right] \end{aligned}$$

Thus

$$b = \frac{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i - n \sum_{i=1}^{n} x_i y_i}{\left(\sum_{i=1}^{n} x_i\right)^2 - n \sum_{i=1}^{n} x_i^2}$$

and, from equation (4),

$$a = \frac{\sum_{i=1}^{n} y_i - b \sum_{i=1}^{n} x_i}{n}$$

It can be shown that these values for b and a give a minimum value for S, so we have the following.

Linear Regression Equation

Given a set of data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, the equation of the line that is the best fit for these data is

$$\hat{y} = a + bx$$

where

$$b = \frac{\Sigma x \cdot \Sigma y - n \Sigma xy}{(\Sigma x)^2 - n \Sigma x^2}, \qquad a = \frac{\Sigma y - b \Sigma x}{n}$$

and each summation is taken over the entire data set (that is, from 1 to n).

EXAMPLE 5

The following data show the relation between the diameter of a partial roll of blue denim material at MacGregor Mills and the actual number of yards remaining on the roll. Use linear regression to find the linear equation that gives the number of yards as a function of the diameter.

Diameter (inches)	*Yards/Roll*	*Diameter (inches)*	*Yards/Roll*
14.0	120	22.5	325
15.0	145	24.0	360
16.5	170	24.5	380
17.75	200	25.25	405
18.5	220	26.0	435
19.8	255	26.75	460
20.5	270	27.0	470
22.0	305	28.0	500

Solution

Let x be the diameter of the partial rolls and y be the yards on a roll. Before finding the values for a and b, we evaluate some parts of the formulas:

$$\begin{aligned}
n &= 16\\
\Sigma x &= 348.05\\
\Sigma x^2 &= 7871.48\\
\Sigma y &= 5020\\
\Sigma xy &= 117{,}367.75\\
b &= \frac{\Sigma x \Sigma y - n\Sigma xy}{(\Sigma x)^2 - n\Sigma x^2}\\
&= \frac{(348.05)(5020) - 16(117{,}367.75)}{(348.05)^2 - 16(7871.48)} = 27.1959\\
a &= \frac{\Sigma y - b\Sigma x}{n}\\
&= \frac{(5020) - (27.1959)(348.05)}{16} = -277.8458
\end{aligned}$$

Thus the linear equation that can be used to estimate the number of yards of denim remaining on a roll is

$$\hat{y} = -277.85 + 27.20x$$

Note that if we use the linear regression capability of a graphing utility, we obtain exactly the same equation.

CHECKPOINT

4. Use linear regression to write the equation of the line that is the best fit for the following points.

x	50	25	10	5
y	2	4	10	20

Finally, we note that formulas for models other than linear ones, such as power models ($y = ax^b$), exponential models ($y = ab^x$), and logarithmic models [$y = a + b\ln(x)$], can also be developed with the least-squares method. That is, we apply max-min techniques for functions of two variables to minimize the sum of the squares of the deviations.

CHECKPOINT SOLUTIONS

1. $z_x = -2x + 2$, $z_y = -2y - 4$
2. $-2x + 2 = 0$ gives $x = 1$.
   $-2y - 4 = 0$ gives $y = -2$.
   Thus the critical point is $(1, -2, 9)$.
3. $z_{xx} = -2$, $z_{yy} = -2$, and $z_{xy} = 0$, so

$$D(x, y) = (-2)(-2) - (0)^2 = 4$$

Hence, at $(1, -2, 9)$ we have $D > 0$ and $z_{xx} < 0$, so $(1, -2, 9)$ is a relative minimum.

4.
$$\Sigma x = 50 + 25 + 10 + 5 = 90$$
$$\Sigma x^2 = 2500 + 625 + 100 + 25 = 3250$$
$$\Sigma y = 2 + 4 + 10 + 20 = 36$$
$$\Sigma xy = 100 + 100 + 100 + 100 = 400$$

Then

$$b = \frac{\Sigma x \cdot \Sigma y - n\Sigma xy}{(\Sigma x)^2 - n\Sigma x^2} = \frac{90 \cdot 36 - 4 \cdot 400}{90^2 - 4 \cdot 3250} = \frac{1640}{-4900} = -0.33,$$

and

$$a = \frac{\Sigma y - b\Sigma x}{n} = \frac{36 - (-0.33)(90)}{4} = \frac{65.7}{4} = 16.43$$

Thus the line that gives the best fit to these points is

$$\hat{y} = -0.33x + 16.43$$

EXERCISE 14.4

In Problems 1–16, test for relative maxima and minima.

1. $z = 9 - x^2 - y^2$
2. $z = 16 - 4x^2 - 9y^2$
3. $z = x^2 + y^2 + 4$
4. $z = x^2 + y^2 - 4$
5. $z = x^2 + y^2 - 2x + 4y + 5$
6. $z = 4x^2 + y^2 + 4x + 1$
7. $z = x^2 + 6xy + y^2 + 16x$
8. $z = x^2 - 4xy + y^2 - 6y$
9. $z = \dfrac{x^2 - y^2}{9}$
10. $z = \dfrac{y^2}{4} - \dfrac{x^2}{9}$
11. $z = 24 - x^2 + xy - y^2 + 36y$
12. $z = 46 - x^2 + 2xy - 4y^2$
13. $z = x^2 + xy + y^2 - 4y + 10x$
14. $z = x^2 + 5xy + 10y^2 + 8x - 40y$
15. $z = x^3 + y^3 - 6xy$
16. $z = x^3 + y^3 + 3xy$

In Problems 17 and 18, use the points given in the tables to write the equation of the line that is the best fit for the points.

17.

x	3	4	5	6
y	15	22	28	32

18.

x	10	20	30	40
y	2	6	5	6

Applications

19. ***Profit*** Suppose that the profit from the sale of Kisses and Kreams is given by

$$P(x, y) = 100x + 64y - 0.01x^2 - 0.25y^2$$

where x is the number of pounds of Kisses and y is the number of pounds of Kreams. Selling how many pounds of Kisses and Kreams will maximize profit?

20. ***Profit*** The profit from the sales of two products is given by

$$P(x, y) = 20x + 70y - x^2 - y^2$$

where x is the number of units of product 1 sold and y is the number of units of product 2. Selling how much of each product will maximize profit?

21. ***Nutrition*** A new food is designed to add weight to mature beef cattle. The increase in weight is given by $W = xy(20 - x - 2y)$, where x is the number of units of the first ingredient and y is the number of units of the second ingredient. How many units of each ingredient will maximize the weight gain?

22. ***Profit*** The profit for a grain crop is related to fertilizer and labor. The profit per acre is

$$P = 100x + 40y - 5x^2 - 2y^2$$

where x is the number of units of fertilizer and y is the number of work-hours. What values of x and y will maximize the profit?

23. ***Production*** Suppose that

$$P = 3.78x^2 + 1.5y^2 - 0.09x^3 - 0.01y^3$$

is the production function for a product with x units of one input and y units of a second input. Find the values of x and y that will maximize production.

24. ***Production*** Suppose that x units of one input and y units of a second input result in

$$P = 40x + 50y - x^2 - y^2 - xy$$

units of a product. Determine the inputs x and y that will maximize P.

25. ***Production*** Suppose that a manufacturer produces two brands of a product, brand 1 and brand 2. If the demand for brand 1 is $p_1 = 10 - x$, the demand for brand 2 is $p_2 = 40 - 2y$, and the joint cost function is $C = xy$, how many of each brand should be produced to maximize profit?

26. ***Production*** Suppose that a firm produces two products, A and B, that sell for \$$a$ and \$$b$, respectively, with the total cost of producing x units of A and y units of B equal to $C(x, y)$. Show that when the profit from these products is maximized,

$$\frac{\partial C}{\partial x}(x, y) = a \quad \text{and} \quad \frac{\partial C}{\partial y}(x, y) = b$$

27. ***Manufacturing*** Find the values for each of the dimensions of an open-top box of length x, width y, and height $500{,}000/(xy)$ such that the box requires the least amount of material to make.

28. ***Manufacturing*** Find the values for each of the dimensions of a closed-top box of length x, width y, and height z if the volume equals 27,000 cubic inches and the box requires the least amount of material to make. *Hint:* First write z in terms of x and y, as in Problem 27.

29. ***Profit*** A company manufactures two products, A and B. If x is the number of units of A and y is the number of units of B, then the cost and revenue functions are

$$C(x, y) = 2x^2 - 2xy + y^2 - 7x + 10y + 11$$
$$R(x, y) = 5x + 4y$$

Find the number of each type of product that should be manufactured to maximize profit.

30. ***Production*** Let x be the number of work-hours required and y be the amount of capital required to produce z units of a product. Show that the average production per work-hour, z/x, is maximized when

$$\frac{\partial z}{\partial x} = \frac{z}{x}$$

Use $z = f(x, y)$ and assume a maximum exists.

31. ***Retirement benefits*** The following table gives the approximate benefits for PepsiCo executives who earned an average of $250,000 per year during the last 5 years of service, based on the number of years of service, from 15 years to 45 years.
   (a) Use linear regression to find the linear equation that is the best fit for the data.
   (b) Use the equation to find the expected annual retirement benefits after 40 years of service.

Years of Service	*Annual Retirement*
25	$109,280
30	121,130
35	132,990
40	145,490
45	160,790

Source: TRICON Salaried Employees Retirement Plan

32. ***Health insurance*** The number of people enrolled in health insurance plans (in millions) is given in the table below for the years 1987 to 1995.
   (a) Use linear regression to find the equation of the line of best fit. Use $x = 0$ in 1987.
   (b) Use the equation to estimate the number enrolled in 1996.

Year	*Number of People Enrolled in Health Insurance Plans (millions)*
1987	241.2
1988	243.7
1989	246.2
1990	248.9
1991	251.4
1992	256.8
1993	259.8
1994	262.1
1995	264.3

Source: Bureau of the Census, 1996

33. ***Hourly earnings*** The following table shows the average hourly earnings for full-time workers in various industries for selected years.
   (a) Find the linear regression equation for hourly earnings as a function of time (with $x = 0$ in 1970).
   (b) What does this model predict for the average hourly earnings in 2005?
   (c) Write a sentence that interprets the slope of the linear regression equation.

Year	*Hourly Earnings*
1970	$3.23
1975	4.53
1980	6.66
1985	8.57
1986	8.76
1987	8.98
1988	9.28
1989	9.66
1990	10.01
1991	10.32
1992	10.57
1993	10.83
1994	11.12
1995	11.43
1996	11.81

Source: Bureau of Labor Statistics, U.S. Dept. of Labor

34. ***Tuition*** The following table gives the annual tuition for universities in Georgia from 1985 to 1994.

Year	*Tuition*	*Year*	*Tuition*
1985	$377	1990	528
1986	424	1991	552
1987	460	1992	574
1988	487	1993	597
1989	506	1994	615

Source: University System of Georgia

   (a) Let $t = 0$ in 1985 and use linear regression to write a linear equation representing the annual tuition at universities in Georgia as a function of the number of years past 1985.
   (b) Use the equation to predict tuition in 2003.

14.5 Maxima and Minima of Functions Subject to Constraints; Lagrange Multipliers

OBJECTIVE

- *To find the maximum or minimum value of a function of two or more variables subject to a condition that constrains the variables*

APPLICATION PREVIEW

Many practical problems require that a function of two or more variables be maximized or minimized subject to certain conditions, or constraints, that limit the variables involved. For example, a firm will want to maximize its profits within the limits (constraints) imposed by its production capacity. Similarly, a city planner may want to locate a new building to maximize access to public transportation yet may be constrained by the availability and cost of building sites.

Specifically, suppose that the utility function for commodities X and Y is given by $U = x^2y^2$, where x and y are the amounts of X and Y, respectively. If p_1 and p_2 represent the prices of X and Y, respectively, and I represents the consumer's income available to purchase these two commodities, the equation $p_1x + p_2y = I$ is called the *budget constraint*. If the price of X is \$2, the price of Y is \$4, and the income available is \$40, then the budget constraint is $2x + 4y = 40$. Thus we seek to maximize the consumer's utility $U = x^2y^2$ subject to the budget constraint $2x + 4y = 40$. In this section we develop methods to solve this type of constrained maximum or minimum.

We can obtain maxima and minima for a function $z = f(x, y)$ subject to the constraint $g(x, y) = 0$ by using the method of **Lagrange multipliers,** named for the famous eighteenth-century mathematician Joseph Louis Lagrange. Lagrange multipliers can be used with functions of two or more variables when the constraints are given by an equation.

In order to find the critical values of a function $f(x, y)$ subject to the constraint $g(x, y) = 0$, we will use the new variable λ to form the **objective function**

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

It can be shown that the critical values of $F(x, y, \lambda)$ will satisfy the constraint $g(x, y)$ and will also be critical points of $f(x, y)$. Thus we need only find the critical points of $F(x, y, \lambda)$ to find the required critical points.

To find the critical points of $F(x, y, \lambda)$, we must find the points that make all the partial derivatives equal to 0. That is, the points must satisfy

$$\partial F/\partial x = 0, \quad \partial F/\partial y = 0, \quad \text{and} \quad \partial F/\partial \lambda = 0$$

Because $F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$, these equations may be written as

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0$$
$$\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0$$
$$g(x, y) = 0$$

Finding the values of x and y that satisfy these three equations simultaneously gives the critical values.

This method will not tell us whether the critical points correspond to maxima or minima, but this can be determined either from the physical setting for the problem or by testing according to a procedure similar to that used for unconstrained maxima and minima. The following examples illustrate the use of Lagrange multipliers.

EXAMPLE 1

Find the maximum value of $z = x^2y$ subject to $x + y = 9$, $x \geq 0$, $y \geq 0$.

Solution

The function to be maximized is $f(x, y) = x^2y$.
The constraint is $g(x, y) = 0$, where $g(x, y) = x + y - 9$.
The objective function is

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

or

$$F(x, y, \lambda) = x^2y + \lambda(x + y - 9)$$

Thus

$$\frac{\partial F}{\partial x} = 2xy + \lambda(1) = 0, \quad \text{or} \quad 2xy + \lambda = 0$$

$$\frac{\partial F}{\partial y} = x^2 + \lambda(1) = 0, \quad \text{or} \quad x^2 + \lambda = 0$$

$$\frac{\partial F}{\partial \lambda} = 0 + 1(x + y - 9) = 0, \quad \text{or} \quad x + y - 9 = 0$$

Solving the first two equations for λ and substituting gives

$$\lambda = -2xy$$
$$\lambda = -x^2$$
$$2xy = x^2$$
$$2xy - x^2 = 0$$
$$x(2y - x) = 0$$

so

$$x = 0 \quad \text{or} \quad x = 2y$$

Because $x = 0$ could not make $z = x^2y$ a maximum, we substitute $x = 2y$ into $x + y - 9 = 0$.

$$2y + y = 9$$
$$y = 3$$
$$x = 6$$

Thus the function $z = x^2y$ is maximized at 108 when $x = 6$, $y = 3$, if the constraint is $x + y = 9$. Testing values near $x = 6$, $y = 3$, and satisfying the constraint shows that the function is maximized there. (Try $x = 5.5$, $y = 3.5$; $x = 7$, $y = 2$; and so on.)

EXAMPLE 2

Find the minimum value of the function $z = x^3 + y^3 + xy$ subject to the constraint $x + y - 4 = 0$.

Solution

The function to be minimized is $f(x, y) = x^3 + y^3 + xy$.
The constraint function is $g(x, y) = x + y - 4$.
The objective function is

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

or

$$F(x, y, \lambda) = x^3 + y^3 + xy + \lambda(x + y - 4)$$

Then

$$\begin{aligned} \frac{\partial F}{\partial x} &= 3x^2 + y + \lambda = 0 \\ \frac{\partial F}{\partial y} &= 3y^2 + x + \lambda = 0 \\ \frac{\partial F}{\partial \lambda} &= x + y - 4 = 0 \end{aligned}$$

Solving the first two equations for λ and substituting, we get

$$\begin{aligned} \lambda &= -(3x^2 + y) \\ \lambda &= -(3y^2 + x) \\ 3x^2 + y &= 3y^2 + x \end{aligned}$$

Solving $x + y - 4 = 0$ for y gives $y = 4 - x$. Substituting for y in the equation above, we get

$$\begin{aligned} 3x^2 + (4 - x) &= 3(4 - x)^2 + x \\ 3x^2 + 4 - x &= 48 - 24x + 3x^2 + x \\ 22x = 44 \quad &\text{or} \quad x = 2 \end{aligned}$$

Thus when $x + y - 4 = 0$, $x = 2$ and $y = 2$ give the minimum value $z = 20$.

CHECKPOINT

Find the minimum value of $f(x, y) = x^2 + y^2 - 4xy$, subject to the constraint $x + y = 10$, by:

1. forming the objective function $F(x, y, \lambda)$;
2. finding $\dfrac{\partial F}{\partial x}$, $\dfrac{\partial F}{\partial y}$, and $\dfrac{\partial F}{\partial \lambda}$;
3. setting the three partial derivatives (from question 2) equal to 0, and solving the equations simultaneously for x and y;
4. finding the value of $f(x, y)$ at the critical values of x and y.

We can also use Lagrange multipliers to find the maxima and minima of functions of three (or more) variables, subject to two (or more) constraints. The method involves using two multipliers, one for each constraint, to form an objective function $F = f + \lambda g_1 + \mu g_2$. We leave further discussion for more advanced courses.

We can easily extend the method to functions of three or more variables, as the following example shows.

EXAMPLE 3

Find the minimum value of the function $w = x + y^2 + z^2$, subject to the constraint $x + y + z = 1$.

Solution

The function to be minimized is $f(x, y, z) = x + y^2 + z^2$. The constraint is $g(x, y, z) = 0$, where $g(x, y, z) = x + y + z - 1$.
The objective function is

$$F(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z)$$

or

$$F(x, y, z, \lambda) = x + y^2 + z^2 + \lambda(x + y + z - 1)$$

Then

$$\frac{\partial F}{\partial x} = 1 + \lambda = 0$$

$$\frac{\partial F}{\partial y} = 2y + \lambda = 0$$

$$\frac{\partial F}{\partial z} = 2z + \lambda = 0$$

$$\frac{\partial F}{\partial \lambda} = x + y + z - 1 = 0$$

Solving the first three equations simultaneously gives

$$\lambda = -1 \qquad y = \frac{1}{2} \qquad z = \frac{1}{2}$$

Substituting these values in the fourth equation (which is the constraint), we get $x + \frac{1}{2} + \frac{1}{2} - 1 = 0$, so $x = 0$, $y = \frac{1}{2}$, $z = \frac{1}{2}$. Thus $w = \frac{1}{2}$ is the minimum value because other values of x, y, and z that satisfy $x + y + z = 1$ give larger values for w.

Let us now return to the utility problem posed in the Application Preview.

EXAMPLE 4

Find x and y that maximize the utility function $U = x^2y^2$ subject to the budget constraint $2x + 4y = 40$.

Solution

First we rewrite the constraint as $2x + 4y - 40 = 0$. Then the objective function is

$$F(x, y, \lambda) = x^2y^2 + \lambda(2x + 4y - 40).$$

$$\frac{\partial F}{\partial x} = 2xy^2 + 2\lambda, \qquad \frac{\partial F}{\partial y} = 2x^2y + 4\lambda, \qquad \frac{\partial F}{\partial \lambda} = 2x + 4y - 40$$

Setting these partial derivatives equal to 0 and solving gives

$$-\lambda = xy^2 = x^2y/2, \quad \text{or} \quad xy^2 - x^2y/2 = 0$$

so

$$xy(y - x/2) = 0$$

yields $x = 0$, $y = 0$, or $x = 2y$. Neither $x = 0$ nor $y = 0$ maximizes utility. If $x = 2y$, we have

$$0 = 4y + 4y - 40$$

Thus $y = 5$ and $x = 10$.

Testing values near $x = 10$, $y = 5$ shows that these values maximize utility at $U = 2500$.

Figure 14.13 shows the budget constraint $2x + 4y = 40$ from Example 4 graphed with the indifference curves for $U = x^2y^2$ that correspond to $U = 500$, $U = 2500$, and $U = 5000$.

Whenever an indifference curve intersects the budget constraint, that utility level is attainable within the budget. Note that the highest attainable utility (such as $U = 2500$, found in Example 4) corresponds to the indifference curve that touches the budget constraint at exactly one point—that is, the curve that has the budget constraint as a tangent line. Note also that utility levels greater than $U = 2500$ are not attainable within the budget because the indifference curve "misses" the budget constraint line (as for $U = 5000$).

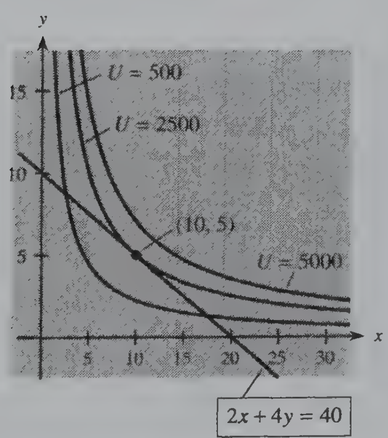


Figure 14.13

EXAMPLE 5

Suppose that the Cobb-Douglas production function for a certain manufacturer gives the number of units of production z according to

$$z = f(x, y) = 100x^{4/5}y^{1/5}$$

where x is the number of units of labor and y is the number of units of capital. Suppose further that labor costs \$160 per unit, capital costs \$200 per unit, and the total cost for capital and labor is limited to \$100,000, so that production is constrained by

$$160x + 200y = 100{,}000$$

Find the number of units of labor and the number of units of capital that maximize production.

Solution

The objective function is

$$F(x, y, \lambda) = 100x^{4/5}y^{1/5} + \lambda(160x + 200y - 100{,}000)$$

$$\frac{\partial F}{\partial x} = 80x^{-1/5}y^{1/5} + 160\lambda, \qquad \frac{\partial F}{\partial y} = 20x^{4/5}y^{-4/5} + 200\lambda$$

$$\frac{\partial F}{\partial \lambda} = 160x + 200y - 100{,}000$$

Setting these partial derivatives equal to 0 and solving gives

$$\lambda = \frac{-80x^{-1/5}y^{1/5}}{160} = \frac{-20x^{4/5}y^{-4/5}}{200} \qquad \text{or} \qquad \frac{y^{1/5}}{2x^{1/5}} = \frac{x^{4/5}}{10y^{4/5}}$$

This means $5y = x$. Using this in $\frac{\partial F}{\partial \lambda} = 0$ gives

$$\begin{aligned} 160(5y) + 200y - 100{,}000 &= 0 \\ 1000y &= 100{,}000 \\ y &= 100 \\ x = 5y &= 500 \end{aligned}$$

Thus production is maximized at $z = 100(500)^{4/5}(100)^{1/5} \approx 36{,}239$ when $x = 500$ (units of labor) and $y = 100$ (units of capital). See Figure 14.14.

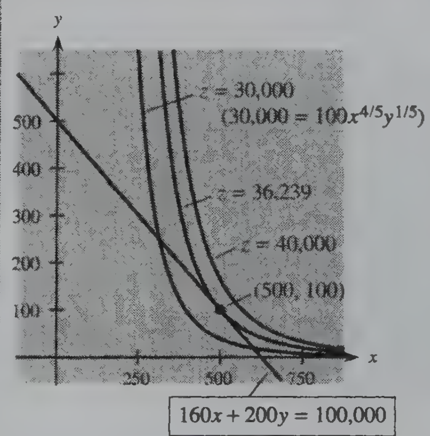


Figure 14.14

In problems of this type, economists call the value of $-\lambda$ the **marginal productivity of money.** In this case

$$-\lambda = \frac{y^{1/5}}{2x^{1/5}} = \frac{(100)^{0.2}}{2(500)^{0.2}} \approx 0.362$$

This means that each additional dollar spent on production results in approximately 0.362 additional unit produced.

Finally, Figure 14.14 shows the graph of the constraint together with some production function curves that correspond to different production levels.

CHECKPOINT SOLUTIONS

1. $F(x, y, \lambda) = x^2 + y^2 - 4xy + \lambda(10 - x - y)$
2. $\dfrac{\partial F}{\partial x} = 2x - 4y - \lambda, \qquad \dfrac{\partial F}{\partial y} = 2y - 4x - \lambda, \qquad \dfrac{\partial F}{\partial \lambda} = 10 - x - y$
3. $0 = 2x - 4y - \lambda$ (1)
   $0 = 2y - 4x - \lambda$ (2)
   $0 = 10 - x - y$ (3)

   From equations (1) and (2) we have the following:

   $$\lambda = 2x - 4y \quad \text{and} \quad \lambda = 2y - 4x, \quad \text{so}$$
   $$2x - 4y = 2y - 4x$$
   $$6x = 6y, \quad \text{or} \quad x = y$$

   Using $x = y$ in equation (3) gives $0 = 10 - x - x$, or $2x = 10$. Thus $x = 5$ and $y = 5$.
4. $f(5, 5) = 25 + 25 - 100 = -50$ is the minimum because other values that satisfy the constraint give larger z-values.

EXERCISE 14.5

1. Find the minimum value of $z = x^2 + y^2$ subject to the condition $x + y = 6$.
2. Find the minimum value of $z = 4x^2 + y^2$ subject to the constraint $x + y = 5$.
3. Find the minimum value of $z = 3x^2 + 5y^2 - 2xy$ subject to the constraint $x + y = 5$.
4. Find the maximum value of $z = 2xy - 3x^2 - 5y^2$ subject to the constraint $x + y = 5$.
5. Find the maximum value of $z = x^2y$ subject to $x + y = 6, x \geq 0, y \geq 0$.
6. Find the maximum value of the function $z = x^3y^2$ subject to $x + y = 10, x \geq 0, y \geq 0$.
7. Find the maximum value of the function $z = 2xy - 2x^2 - 4y^2$ subject to the condition $x + 2y = 8$.
8. Find the minimum value of $z = 2x^2 + y^2 - xy$ subject to the constraint $2x + y = 8$.
9. Find the minimum value of $z = x^2 + y^2$ subject to the condition $2x + y + 1 = 0$.
10. Find the minimum value of $z = x^2 + y^2$ subject to the condition $xy = 1$.
11. Find the minimum value of $w = x^2 + y^2 + z^2$ subject to the constraint $x + y + z = 3$.
12. Find the minimum value of $w = x^2 + y^2 + z^2$ subject to the condition $2x - 4y + z = 21$.
13. Find the maximum value of $w = xz + y$ subject to the constraint $x^2 + y^2 + z^2 = 1$.
14. Find the maximum value of $w = x^2yz$ subject to the constraint $4x + y + z = 4, x \geq 0, y \geq 0$, and $z \geq 0$.

Applications

15. ***Utility*** Suppose that the utility function for two commodities is given by $U = x^2y$ and that the budget constraint is $3x + 6y = 18$. What values of x and y will maximize utility?

16. ***Utility*** Suppose that the budget constraint in Problem 15 is $5x + 20y = 80$. What values of x and y will maximize $U = x^2y$?

17. ***Utility*** Suppose that the utility function for two products is given by $U = x^2y$, and the budget constraint is $2x + 3y = 120$. Find the values of x and y that maximize utility. Check by graphing the budget constraint with the indifference curve for maximum utility and with two other indifference curves.

18. ***Utility*** Suppose that the utility function for two commodities is given by $U = x^2y^3$, and the budget constraint is $10x + 15y = 250$. Find the values of x and y that maximize utility. Check by graphing the budget constraint with the indifference curve for maximum utility and with two other indifference curves.

19. ***Production*** A company has the Cobb-Douglas production function

$$z = 400x^{0.6}y^{0.4}$$

where x is the number of units of labor and y is the number of units of capital. Suppose labor costs \$150 per unit, capital costs \$100 per unit, and the total cost of labor and capital is limited to \$100,000.
(a) Find the number of units of labor and the number of units of capital that maximize production.
(b) Find the marginal productivity of money and interpret it.
(c) Graph the constraint with the optimal value for production and with two other z-values (one smaller than the optimal value and one larger).

20. ***Production*** Suppose a company has the Cobb-Douglas production function

$$z = 100^{0.75}y^{0.25}$$

where x is the number of units of labor and y is the number of units of capital. Suppose further that labor costs \$90 per unit, capital costs \$150 per unit, and the total costs of labor and capital are limited to \$90,000.
(a) Find the number of units of labor and the number of units of capital that maximize production.
(b) Find the marginal productivity of money and interpret it.
(c) Graph the constraint with the optimal value for production and with two other z-values (one smaller than the optimal value and one larger).

21. ***Cost*** A firm has two plants, X and Y. Suppose that the cost of producing x units at plant X is $x^2 + 1200$ and the cost of producing y units of the same product at plant Y is given by $3y^2 + 800$. If the firm has an order for 1200 units, how many should it produce at each plant to fill this order and minimize the cost of production?

22. ***Cost*** Suppose that the cost of producing x units at plant X is $(3x + 4)x$ and that the cost of producing y units of the same product at plant Y is $(2y + 8)y$. If the firm that owns the plants has an order for 149 units, how many should it produce at each plant to fill this order and minimize its cost of production?

23. ***Revenue*** On the basis of past experience a company has determined that its sales revenue is related to its advertising according to the formula $s = 20x + y^2 + 4xy$, where x is the amount spent on radio advertising and y is the amount spent on television advertising. If the company plans to spend \$30,000 on these two means of advertising, how much should it spend on each method to maximize its sales revenue?

24. ***Manufacturing*** Find the dimensions x, y, and z of the rectangular box with the largest volume that satisfies

$$3x + 4y + 12z = 12$$

25. ***Manufacturing*** Find the dimensions of the box with square base, open top, and volume 500,000 cubic centimeters that requires the least materials.

26. ***Manufacturing*** Show that a box with a square base, an open top, and a fixed volume requires the least material to build if it has a height equal to one-half the length of one side of the base.

KEY TERMS AND FORMULAS

Section	Key Terms	Formula
14.1	Function of two variables	
	Variables: independent, dependent	
	Domain	
	Coordinate planes	
	Utility	
	Indifference curve	
	Indifference map	
14.2	First-order partial derivative	
	With respect to x	$z_x = \frac{\partial z}{\partial x}$
	With respect to y	$z_y = \frac{\partial z}{\partial y}$
	Higher-order partial derivatives	
	Second partial derivatives	z_{xx}, z_{yy}, z_{xy}, and z_{yx}
14.3	Joint cost function	$C = Q(x, y)$
	Marginal cost	
	Marginal productivity	
	Demand function	
	Marginal demand function	
	Competitive products	
	Complementary products	
14.4	Critical values for maxima and minima	Solve simultaneously $\begin{cases} z_x = 0 \\ z_y = 0 \end{cases}$.
	Test for critical values	Use $D(x, y) = (z_{xx})(z_{yy}) - (z_{xy})^2$.
	Linear regression	$\hat{y} = a + bx$
		$b = \frac{\sum x \cdot \sum y - n\sum xy}{(\sum x)^2 - n\sum x^2}$
		$a = \frac{\sum y - b\sum x}{n}$
14.5	Maxima and minima subject to constraints	
	Lagrange multipliers	
	Objective function	$F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$

REVIEW EXERCISES

Section 14.1

1. What is the domain of $z = \frac{3}{2x - y}$?
2. What is the domain of $z = \frac{3x + 2\sqrt{y}}{x^2 + y^2}$?
3. If $w(x, y, z) = x^2 - 3yz$, find $w(2, 3, 1)$.
4. If $Q(K, L) = 70K^{2/3}L^{1/3}$, find $Q(64{,}000, 512)$.

Section 14.2

5. Find $\frac{\partial z}{\partial x}$ if $z = 5x^3 + 6xy + y^2$.

6. Find $\frac{\partial z}{\partial y}$ if $z = 12x^5 - 14x^3y^3 + 6y^4 - 1$.

In Problems 7–12, find z_x and z_y.

7. $z = 4x^2y^3 + \frac{x}{y}$
8. $z = \sqrt{x^2 + 2y^2}$
9. $z = (xy + 1)^{-2}$
10. $z = e^{x^2y^3}$
11. $z = e^{xy} + y \ln x$
12. $z = e^{\ln xy}$
13. Find the partial derivative of $f(x, y) = 4x^3 - 5xy^2 + y^3$ with respect to x at the point $(1, 2, -8)$.
14. Find the slope of the tangent in the x-direction to the surface $z = 5x^4 - 3xy^2 + y^2$ at $(1, 2, -3)$.

In Problems 15–18, find the second partials.

(a) z_{xx} (b) z_{yy} (c) z_{xy} (d) z_{yx}

15. $z = x^2y - 3xy$
16. $z = 3x^3y^4 - \frac{x^2}{y^2}$
17. $z = x^2e^{y^2}$
18. $z = \ln(xy + 1)$

Section 14.4

19. Test $z = 16 - x^2 - xy - y^2 + 24y$ for maxima and minima.
20. Test $z = x^3 + y^3 - 3xy$ for maxima and minima.

Section 14.5

21. Find the minimum value of $z = 4x^2 + y^2$ subject to the constraint $x + y = 10$.
22. Find the maximum value of $z = x^4y^2$ subject to the constraint $x + y = 9, x \geq 0, y \geq 0$.

Applications

Section 14.1

23. ***Utility*** Suppose that the utility function for two goods X and Y is given by $U = x^2y$, and a consumer purchases 6 units of X and 15 units of Y. If the consumer purchases 60 units of Y, how many units of X must be purchased to retain the same level of utility?

24. ***Utility*** Suppose that an indifference curve for two products, X and Y, has the equation $xy = 1600$. If 80 units of X are purchased, how many units of Y must be purchased?

Section 14.2

25. ***Concorde sonic booms*** The width of the region on the ground on either side of the path of France's Concorde jet in which people hear the sonic boom is given by

$$w = f(T, h, d) = 2\sqrt{Th/d}$$

where T is the air temperature at ground level in kelvins (K), h is the Concorde's altitude in kilometers, and d is the vertical temperature gradient (the temperature drop in kelvins per kilometer).*

(a) Suppose the Concorde approaches Washington, D.C., from Europe on a course that takes it south of Nantucket Island at an altitude of 16.8 km. If the surface temperature is 293 K and the vertical temperature gradient is 5 K/km, how far south of Nantucket must the plane pass to keep the sonic boom off the island?
(b) Interpret $f(287, 17.1, 4.9) \approx 63.3$.
(c) Find $\frac{\partial f}{\partial h}(293, 16.8, 5)$ and interpret the result.
(d) Find $\frac{\partial f}{\partial d}(293, 16.8, 5)$ and interpret the result.

Section 14.3

26. ***Cost*** The joint cost function for two products is $C(x, y) = x^2\sqrt{y^2 + 13}$. Find the marginal cost with respect to
(a) x if 20 units of x and 6 units of y are produced.
(b) y if 20 units of x and 6 units of y are produced.

27. ***Production*** Suppose that the production function for a company is given by

$$Q = 80K^{1/4}L^{3/4}$$

where Q is the output (in hundreds of units), K is the capital expenditures (in thousands of dollars), and L is the work-hours. Find $\partial Q/\partial K$ and $\partial Q/\partial L$ when expenditures are \$625,000 and total work-hours are 4096. Interpret the results.

28. ***Marginal demand*** The demand functions for two related products, product A and product B, are given by

$$q_A = 400 - 2p_A - 3p_B$$
$$q_B = 300 - 5p_A - 6p_B$$

(a) Find the marginal demand of q_A with respect to p_A.
(b) Find the marginal demand of q_B with respect to p_B.
(c) Are the products complementary or competitive?

*Balachandra, N. K., W. L. Donn, and D. H. Rind, "Concorde Sonic Booms as an Atmospheric Probe," *Science*, 1 July 1977, Vol. 197, p. 47.

29. ***Marginal demand*** Suppose that the demand functions for two related products, A and B, are given by

$$q_A = 800 - 40p_A - \frac{2}{p_B + 1}$$
$$q_B = 1000 - \frac{10}{p_A + 4} - 30p_B$$

Determine whether the products are competitive or complementary.

Section 14.4

30. ***Profit*** The profit from the sale of two products is given by $P(x, y) = 40x + 80y - x^2 - y^2$, where x is the number of units of product 1 and y is the number of units of product 2. Selling how much of each product will maximize profit?

Section 14.5

31. ***Utility*** If the utility function for two commodities is $U = x^2y$, and the budget constraint is $4x + 5y = 60$, find the values of x and y that maximize utility.

32. ***Production*** Suppose a company has the Cobb-Douglas production function

$$z = 300x^{2/3}y^{1/3}$$

where x is the number of units of labor and y is the number of units of capital. Suppose labor costs are \$50 per unit, capital costs are \$50 per unit, and total costs are limited to \$75,000.
(a) Find the number of units of labor and the number of units of capital that maximize production.
(b) Find the marginal productivity of money and interpret your result.
(c) Graph the constraint with the production function when $z = 180,000$, $z = 300,000$, and when the z-value is optimal.

33. ***Taxes*** The following data show U.S. national personal income and personal taxes for selected years.
(a) Write the linear regression equation that best fits these data.
(b) Use the equation found in (a) to predict the taxes when national personal income reaches \$7000 billion.

Income (x) (billions)	*Taxes (y) (billions)*
\$2285.7	\$312.4
2560.4	360.2
2718.7	371.4
2891.7	369.3
3205.5	395.5
3439.6	437.7
3647.5	459.9
3877.3	514.2
4172.8	532.0
4489.3	594.9
4791.6	624.8
4968.5	624.8
5264.2	650.5
5480.1	689.9
5753.1	731.4
6150.8	795.1
6495.2	886.9

Source: Bureau of Economic Analysis, U.S. Commerce Dept.

34. ***Supply and demand*** The table gives the number of color television sets (in thousands) sold in 15 different years, along with the corresponding average price per set for the year. Use linear regression to find the best linear equation defining the demand function $q = f(p)$.

Price (p) (dollars)	*Quantity Sold (q) (thousands)*
471.56	9793
487.79	10236
487.32	9107
510.78	7700
504.39	6485
466.29	8411
449.81	10071
509.86	7908
524.96	6349
514.10	4822
515.43	5962
520.82	5981
525.01	5777
462.32	5892
560.09	2646

Source: U.S. Bureau of the Census, *Statistical Abstract of the United States*, Washington, D.C.

CHAPTER TEST

1. Consider the function $f(x, y) = \dfrac{2x + 3y}{\sqrt{x^2 - y}}$.
   (a) Find the domain of $f(x, y)$.
   (b) Evaluate $f(-4, 12)$.
2. Find all first and second partial derivatives of
$$z = f(x, y) = 5x - 9y^2 + 2(xy + 1)^5$$
3. Let $z = 6x^2 + x^2y + y^2 - 4y + 9$. Find the pairs (x, y) that are critical points for z, and then classify each as a relative maximum, a relative minimum, or a saddle point.
4. Suppose a company's monthly production value Q, in thousands of dollars, is given by the Cobb-Douglas production function
$$Q = 10K^{0.45}L^{0.55}$$
where K is thousands of dollars of capital investment per month and L is the total hours of labor per month. Capital investment is currently \$10,000 per month and monthly work-hours of labor total 1590.
   (a) Find the monthly production value (to the nearest thousand dollars).
   (b) Find the marginal productivity with respect to capital investment, and interpret your result.
   (c) Find the marginal productivity with respect to total hours of labor, and interpret your result.
5. The monthly payment R on a loan is a function of the amount borrowed, A, in thousands of dollars; the length of the loan, n, in years; and the annual interest rate, r, as a percent. Thus $R = f(A, n, r)$. In (a) and (b), write a sentence that explains the practical meaning of each mathematical statement.
   (a) $f(94.5, 25, 7) = \$667.91$
   (b) $\dfrac{\partial f}{\partial r}(94.5, 25, 7) = \$49.76$
   (c) Would $\dfrac{\partial f}{\partial n}(94.5, 25, 7)$ be positive, negative, or zero? Explain.
6. Let $f(x, y) = 2e^{x^2y^2}$. Find $\dfrac{\partial^2 f}{\partial x\, \partial y}$.
7. Suppose the demand functions for two products are
$$q_1 = 300 - 2p_1 - 5p_2 \quad \text{and} \quad q_2 = 150 - 4p_1 - 7p_2$$
where q_1 and q_2 represent quantities demanded and p_1 and p_2 represent prices. What calculations enable us to decide whether the products are competitive or complementary? Are these products competitive or complementary?
8. Suppose a store sells two brands of disposable cameras and the profit for these is a function of their two selling prices. The type 1 camera sells for \$$x$, the type 2 sells for \$$y$, and profit is given by
$$P = 915x - 30x^2 - 45xy + 975y - 30y^2 - 3500$$
Find the selling prices that maximize profit.
9. Find x and y that maximize the utility function $U = x^3y$ subject to the budget constraint $30x + 20y = 8000$.
10. For a middle-income family, the estimated annual expenditures associated each year of raising a child through age 11 are given in the table below.
   (a) Find the linear regression line for these data.
   (b) Use the regression equation to project the annual expenditure associated with raising a child during his or her 14th year.
   (c) If this linear model had the form $E = f(x)$, where E is the expenditures and x is the age, would $f(35)$ make sense? Explain.

Age	*Expenditures*	*Age*	*Expenditures*
0	\$7,880	6	\$11,020
1	8,270	7	11,590
2	8,700	8	12,200
3	9,380	9	12,780
4	9,870	10	13,450
5	10,390	11	14,150

Source: Family Economics Research Group, U.S. Dept. of Agriculture, 1996

Extended Applications Group Projects

I. Advertising

To model sales of its tires, the manufacturer of GRIPPER tires used the quadratic equation $S = a_0 + a_1x + a_2x^2 + b_1y$, where S is regional sales in millions of dollars, x is TV advertising expenditures in millions of dollars, and y is other promotional expenditures in millions of dollars. (See the Extended Application/Group Project "Marginal Return to Sales," on page 729.)

Although this model represents the relationship between advertising and sales dollars for small changes in advertising expenditures, it is clear to the vice president of advertising that it does not apply to large expenditures for TV advertising on a national level. He knows from experience that increased expenditures for TV advertising do result in more sales, but at a decreasing rate of return for the product.

The vice president is aware that some advertising agencies model the relationship between advertising and sales by the function

$$S = b_0 + b_1(1 - e^{-ax}) + c_1y$$

where $a > 0$, S is sales in millions of dollars, x is TV advertising expenditures in millions of dollars, and y is other promotional expenditures in millions of dollars.* The equation

$$S_n = 24.58 + 325.18(1 - e^{-x/14}) + b_1y$$

has the form mentioned previously as being used by some advertising agencies. For TV advertising expenditures up to $20 million, this equation closely approximates

$$S_1 = 30 + 20x - 0.4x^2 + b_1y$$

which, in the Extended Application/Group Project on page 729, was used with fixed promotional expenses to describe advertising and sales in Region 1.

To help the vice president decide whether this is a better model for large expenditures, answer the following questions.

1. What is $\partial S_1/\partial x$? Does this indicate that sales might actually decline after some amount is spent on TV advertising? If so, what is this amount?
2. Does the quadratic model $S_1(x, y)$ indicate that sales will become negative after some amount is spent on TV advertising? Does this model cease to be useful in predicting sales after a certain point?
3. What is $\partial S_n/\partial x$? Does this indicate that sales will continue to rise if additional money is devoted to TV advertising? Is S_n growing at a rate that is increasing or decreasing when promotional sales are held constant? Is S_n a better model for large expenditures?

*Mansfield, Edwin, *Managerial Economics* (New York: Norton, 1990).

4. If this model does describe the relationship between advertising and sales, and if promotional expenditures are held constant at y_0, is there an upper limit to the sales, even if an unlimited amount of money is spent on TV advertising? If so, what is it?

II. Competitive Pricing

Often retailers sell different brands of competing products. Depending on the joint demand for the products, the retailer may be able to set prices that regulate demand and, therefore, influence profits.

Suppose HOME-ALL, Inc., a national chain of home improvement retailers, sells two competing brands of interior flat paint, En-Dure 100 and Croyle & James, which the chain purchases for \$8 per gallon and \$10 per gallon, respectively. HOME-ALL's research department has determined the following two monthly demand equations for these paints:

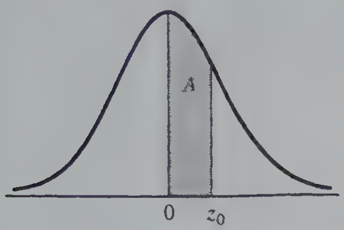
$$D = 120 - 40d + 30c \qquad \text{and} \qquad C = 680 + 30d - 40c$$

where D is hundreds of gallons of En-Dure 100 demanded at \$$d$ per gallon and C is hundreds of gallons of Croyle & James demanded at \$$c$ per gallon. For what prices should HOME-ALL sell these paints in order to maximize its monthly profit on these items?

To answer this question, complete the following.

1. Recall that revenue is a product's selling price per item times the number of items sold. With this in mind, formulate HOME-ALL's total revenue function for the two paints as a function of their prices.
2. Form HOME-ALL's profit function for the two paints (in terms of their selling prices).
3. Determine the price of each type of paint that will maximize HOME-ALL's profit.
4. Write a brief report to management that details your pricing recommendations and justifies them.

Appendix



Areas Under the Standard Normal Curve

The value of A is the area under the standard normal curve between $z = 0$ and $z = z_0$, for $z_0 \geq 0$. Areas for negative values of z_0 are obtained by symmetry.

z_0	A	z_0	A	z_0	A	z_0	A
.00	.0000	.36	.1406	.72	.2642	1.08	.3599
.01	.0040	.37	.1443	.73	.2673	1.09	.3621
.02	.0080	.38	.1480	.74	.2704	1.10	.3643
.03	.0120	.39	.1517	.75	.2734	1.11	.3665
.04	.0160	.40	.1554	.76	.2764	1.12	.3686
.05	.0199	.41	.1591	.77	.2794	1.13	.3708
.06	.0239	.42	.1628	.78	.2823	1.14	.3729
.07	.0279	.43	.1664	.79	.2852	1.15	.3749
.08	.0319	.44	.1700	.80	.2881	1.16	.3770
.09	.0359	.45	.1736	.81	.2910	1.17	.3790
.10	.0398	.46	.1772	.82	.2939	1.18	.3810
.11	.0438	.47	.1808	.83	.2967	1.19	.3830
.12	.0478	.48	.1844	.84	.2996	1.20	.3849
.13	.0517	.49	.1879	.85	.3023	1.21	.3869
.14	.0557	.50	.1915	.86	.3051	1.22	.3888
.15	.0596	.51	.1950	.87	.3079	1.23	.3907
.16	.0636	.52	.1985	.88	.3106	1.24	.3925
.17	.0675	.53	.2019	.89	.3133	1.25	.3944
.18	.0714	.54	.2054	.90	.3159	1.26	.3962
.19	.0754	.55	.2088	.91	.3186	1.27	.3980
.20	.0793	.56	.2123	.92	.3212	1.28	.3997
.21	.0832	.57	.2157	.93	.3238	1.29	.4015
.22	.0871	.58	.2190	.94	.3264	1.30	.4032
.23	.0910	.59	.2224	.95	.3289	1.31	.4049
.24	.0948	.60	.2258	.96	.3315	1.32	.4066
.25	.0987	.61	.2291	.97	.3340	1.33	.4082
.26	.1026	.62	.2324	.98	.3365	1.34	.4099
.27	.1064	.63	.2357	.99	.3389	1.35	.4115
.28	.1103	.64	.2389	1.00	.3413	1.36	.4131
.29	.1141	.65	.2422	1.01	.3438	1.37	.4147
.30	.1179	.66	.2454	1.02	.3461	1.38	.4162
.31	.1217	.67	.2486	1.03	.3485	1.39	.4177
.32	.1255	.68	.2518	1.04	.3508	1.40	.4192
.33	.1293	.69	.2549	1.05	.3531	1.41	.4207
.34	.1331	.70	.2580	1.06	.3554	1.42	.4222
.35	.1368	.71	.2612	1.07	.3577	1.43	.4236

Areas Under the Standard Normal Curve (Continued)

z_0	A	z_0	A	z_0	A	z_0	A
1.44	.4251	1.93	.4732	2.42	.4922	2.91	.4982
1.45	.4265	1.94	.4738	2.43	.4925	2.92	.4983
1.46	.4279	1.95	.4744	2.44	.4927	2.93	.4983
1.47	.4292	1.96	.4750	2.45	.4929	2.94	.4984
1.48	.4306	1.97	.4756	2.46	.4931	2.95	.4984
1.49	.4319	1.98	.4762	2.47	.4932	2.96	.4985
1.50	.4332	1.99	.4767	2.48	.4934	2.97	.4985
1.51	.4345	2.00	.4773	2.49	.4936	2.98	.4986
1.52	.4357	2.01	.4778	2.50	.4938	2.99	.4986
1.53	.4370	2.02	.4783	2.51	.4940	3.00	.4987
1.54	.4382	2.03	.4788	2.52	.4941	3.01	.4987
1.55	.4394	2.04	.4793	2.53	.4943	3.02	.4987
1.56	.4406	2.05	.4798	2.54	.4945	3.03	.4988
1.57	.4418	2.06	.4803	2.55	.4946	3.04	.4988
1.58	.4430	2.07	.4808	2.56	.4948	3.05	.4989
1.59	.4441	2.08	.4812	2.57	.4949	3.06	.4989
1.60	.4452	2.09	.4817	2.58	.4951	3.07	.4989
1.61	.4463	2.10	.4821	2.59	.4952	3.08	.4990
1.62	.4474	2.11	.4826	2.60	.4953	3.09	.4990
1.63	.4485	2.12	.4830	2.61	.4955	3.10	.4990
1.64	.4495	2.13	.4834	2.62	.4956	3.11	.4991
1.65	.4505	2.14	.4838	2.63	.4957	3.12	.4991
1.66	.4515	2.15	.4842	2.64	.4959	3.13	.4991
1.67	.4525	2.16	.4846	2.65	.4960	3.14	.4992
1.68	.4535	2.17	.4850	2.66	.4961	3.15	.4992
1.69	.4545	2.18	.4854	2.67	.4962	3.16	.4992
1.70	.4554	2.19	.4857	2.68	.4963	3.17	.4992
1.71	.4564	2.20	.4861	2.69	.4964	3.18	.4993
1.72	.4573	2.21	.4865	2.70	.4965	3.19	.4993
1.73	.4582	2.22	.4868	2.71	.4966	3.20	.4993
1.74	.4591	2.23	.4871	2.72	.4967	3.21	.4993
1.75	.4599	2.24	.4875	2.73	.4968	3.22	.4994
1.76	.4608	2.25	.4878	2.74	.4969	3.23	.4994
1.77	.4616	2.26	.4881	2.75	.4970	3.24	.4994
1.78	.4625	2.27	.4884	2.76	.4971	3.25	.4994
1.79	.4633	2.28	.4887	2.77	.4972	3.26	.4994
1.80	.4641	2.29	.4890	2.78	.4973	3.27	.4995
1.81	.4649	2.30	.4893	2.79	.4974	3.28	.4995
1.82	.4656	2.31	.4896	2.80	.4974	3.29	.4995
1.83	.4664	2.32	.4898	2.81	.4975	3.30	.4995
1.84	.4671	2.33	.4901	2.82	.4976	3.31	.4995
1.85	.4678	2.34	.4904	2.83	.4977	3.32	.4996
1.86	.4686	2.35	.4906	2.84	.4977	3.33	.4996
1.87	.4693	2.36	.4909	2.85	.4978	3.34	.4996
1.88	.4700	2.37	.4911	2.86	.4979	3.35	.4996
1.89	.4706	2.38	.4913	2.87	.4980	3.36	.4996
1.90	.4713	2.39	.4916	2.88	.4980	3.37	.4996
1.91	.4719	2.40	.4918	2.89	.4981	3.38	.4996
1.92	.4726	2.41	.4920	2.90	.4981	3.39	.4997

Areas Under the Standard Normal Curve (Continued)

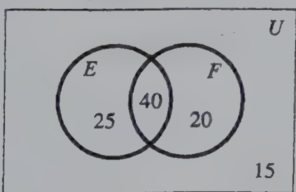
z_0	A	z_0	A	z_0	A	z_0	A
3.40	.4997	3.52	.4998	3.64	.4999	3.76	.4999
3.41	.4997	3.53	.4998	3.65	.4999	3.77	.4999
3.42	.4997	3.54	.4998	3.66	.4999	3.78	.4999
3.43	.4997	3.55	.4998	3.67	.4999	3.79	.4999
3.44	.4997	3.56	.4998	3.68	.4999	3.80	.4999
3.45	.4997	3.57	.4998	3.69	.4999	3.81	.4999
3.46	.4997	3.58	.4998	3.70	.4999	3.82	.4999
3.47	.4997	3.59	.4998	3.71	.4999	3.83	.4999
3.48	.4998	3.60	.4998	3.72	.4999	3.84	.4999
3.49	.4998	3.61	.4999	3.73	.4999	3.85	.4999
3.50	.4998	3.62	.4999	3.74	.4999	3.86	.4999
3.51	.4998	3.63	.4999	3.75	.4999		

Answers

Below are the answers to odd-numbered Section Exercises and all the Chapter Review and Chapter Test problems.

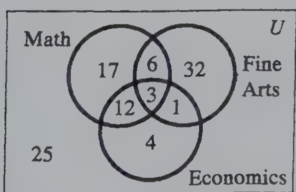
Exercise 0.1 (page 7)

1. $\in$ 3. $\in$ 5. $\notin$ 7. $\{1, 2, 3, 4, 5, 6, 7\}$
9. $\{x: x$ is a natural number greater than 2 and less than 8$\}$
11. yes 13. no 15. $D \subseteq C$ 17. $D \subseteq A$
19. $A \subseteq B$ or $B \subseteq A$ 21. yes 23. no
25. A and B, B and D, C and D
27. $A \cap B = \{4, 6\}$ 29. $A \cap B = \varnothing$
31. $A \cup B = \{1, 2, 3, 4, 5\}$
33. $A \cup B = \{1, 2, 3, 4\} = B$
35. $A' = \{4, 6, 9, 10\}$
37. $A \cap B' = \{1, 2, 5, 7\}$ 39. $(A \cup B)' = \{6, 9\}$
41. $A' \cup B' = \{1, 2, 4, 5, 6, 7, 9, 10\}$
43. $\{1, 2, 3, 5, 7, 9\}$ 45. $\{4, 6, 8, 10\}$
47. $A - B = \{1, 7\}$ 49. $A - B = \varnothing$ or $\{\ \}$
51. (a) $L = \{94, 95, 96, 97\}$;
$H = \{92, 93, 94, 95, 96, 97\}$;
$C = \{90, 91, 95, 96, 97\}$
(b) $L \subseteq H$
(c) C' is the set of years when the percentage change from low to high was 25% or less.
(d) $\{90, 91, 92, 93, 94\}$ = the set of years when the high was 3300 or less or the percentage change was 25% or less.
(e) $\{90, 91\}$ = the set of years when the low was 3300 or less and the percentage change exceeded 25%.
53. (a) 130 (b) 840 (c) 520
55. (a)



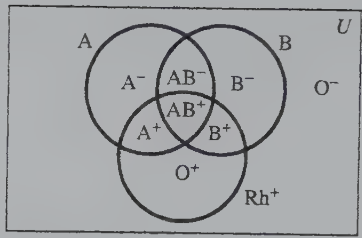
(b) 40
(c) 85
(d) 25

57.



(a) 25
(b) 43
(c) 53

59. (a) and (b)



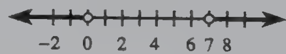
Exercise 0.2 (page 15)

1. (a) irrational (b) rational, integer
(c) rational, integer, natural (d) meaningless
3. (a) Commutative (b) Distributive
(c) Multiplicative identity
5. $<$ 7. $<$ 9. $>$ 11. 11 13. 4
15. 2 17. $\frac{-4}{3}$ 19. 3 21. $\frac{17}{11}$ 23. entire line
25. $(1, 3]$ 27. $(2, 10)$ 29. $x \le 5$ 31. $x > 4$
33. $(-3, 4)$

35. $(4, +\infty)$

37. $[-1, +\infty)$

39. $(-\infty, 0) \cup (7, +\infty)$



41. -0.000038585
43. 9122.387471 45. 3240.184509
47. (a) \$1088.91 (b) \$258.62 (c) \$627.20
49. (a) Formula (2) is more accurate; 1996: \$24.54; 1997: \$26.10
(b) \$40.29

Exercise 0.3 (page 20)

1. -64 3. -16 5. $\frac{1}{9}$ 7. $-\frac{9}{4}$ 9. 6^8
11. $\frac{1}{10}$ 13. 3^9 15. $(\frac{3}{2})^2 = \frac{9}{4}$ 17. $1/x^6$
19. x/y^2 21. x^7 23. $x^{-2} = 1/x^2$ 25. x^4
27. y^{12} 29. x^{12} 31. x^2y^2 33. $16/x^4$
35. $x^8/(16y^4)$ 37. $-16a^2/b^2$ 39. $2/(xy^2)$
41. $1/(x^9y^6)$ 43. $(a^{18}c^{12})/b^6$
45. (a) $1/(2x^4)$ (b) $1/(16x^4)$ (c) $1/x^4$ (d) 8
47. x^{-1} 49. $8x^3$ 51. $\frac{1}{4}x^{-2}$ 53. $-\frac{1}{8}x^3$
55. 2.0736 57. 0.1316872428
59. $S = \$2114.81$; $I = \$914.81$
61. $S = \$9607.70$; $I = \$4607.70$
63. \$7806.24

65. (a) 15, 18, 20, 22 (b) \$0.53, \$0.76, \$0.96, \$1.22 (c) \$3.13
67. (a) 46.7, 71.7, 3652.5 all in billions of \$ (b) World War II had just ended.
69. (a) 10 (b) \$110.5 billion (c) \$872.5 billion (d) \$2564.4 billion

Exercise 0.4 (page 29)

1. $\frac{16}{3}$ 3. -8 5. 8 7. not real 9. $\frac{9}{4}$
11. (a) 4 (b) $\frac{1}{4}$ 13. $(6.12)^{4/9} \approx 2.237$
15. $m^{3/2}$ 17. $(m^2n^5)^{1/4}$ 19. $\sqrt[4]{x^7}$ 21. $-1/(4\sqrt[4]{x^5})$
23. $y^{3/4}$ 25. $z^{19/4}$ 27. $1/y^{5/2}$ 29. x
31. $1/y^{21/10}$ 33. $x^{1/2}$ 35. $1/x$ 37. $8x^2$
39. $8x^2y^2\sqrt{2y}$ 41. $2x^2y\sqrt[3]{5x^2y^2}$ 43. $6x^2y\sqrt{x}$
45. $42x^3y^2\sqrt{x}$ 47. $2xy^5/3$ 49. $2b\sqrt[4]{b}/(3a^2)$
51. $1/9$ 53. 7 55. $\sqrt{6}/3$ 57. $\sqrt{mx}/x$
59. $\sqrt[3]{mx^2}/x^2$ 61. $-\frac{2}{3}x^{-2/3}$ 63. $3x^{3/2}$
65. $(3\sqrt{x})/2$ 67. $1/(2\sqrt{x})$
69. (a) $10^{8.5} = 10^{17/2} = \sqrt{10^{17}}$ (b) 316,227,766 (c) 14.125
71. 74 kg 73. 39,491 75. (a) 10 (b) 259

Exercise 0.5 (page 39)

1. (a) 2 (b) -1 (c) 10 (d) one
3. (a) 5 (b) -14 (c) 0 (d) several
5. (a) 5 (b) 0 (c) 2 (d) -5
7. -12 9. $\frac{-7}{31}$ 11. $21pq - 2p^2$
13. $m^2 - 7n^2 - 3$ 15. $3q + 12$
17. $x^2 - 1$ 19. $35x^5$ 21. $3rs$
23. $2ax^4 + a^2x^3 + a^2bx^2$ 25. $6y^2 - y - 12$
27. $2 - 5x^2 + 2x^4$ 29. $16x^2 + 24x + 9$
31. $x^4 - x^2 + \frac{1}{4}$ 33. $4x^2 - 1$ 35. $0.01 - 16x^2$
37. $x^3 - 8$ 39. $x^8 + 3x^6 - 10x^4 + 5x^3 + 25x$
41. $3 + m + 2m^2n$ 43. $8x^3y^2/3 + 5/(3y) - 2x^2/(3y)$
45. $x^3 + 3x^2 + 3x + 1$ 47. $8x^3 - 36x^2 + 54x - 27$
49. $0.1x^2 - 1.995x - 0.1$
51. $x^2 - 2x + 5 - 11/(x + 2)$
53. $x^2 + 3x - 1 + (-4x + 2)/(x^2 + 1)$
55. $x + 2x^2$ 57. $x - x^{1/2} - 2$ 59. $x - 9$
61. $4x^2 + 4x$
63. (a) $9x^2 - 21x + 13$ (b) 5 65. $55x$
67. (a) $4000 - x$ (b) $0.10x$ (c) $0.08(4000 - x)$ (d) $0.10x + 0.08(4000 - x)$
69. $(15 - 2x)(10 - 2x)x$
71. (a) = A2 − 1 (b) width = 50 − length (c) = 50 − A2 (d) = A2*B2 (e) length = 33, width = 17

73. (a)

	A	B
1	Year	Tax load
2	1995	37.04688
3	1996	37.29315
4	1997	37.53942
5	1998	37.78569
6	1999	38.03196
7	2000	38.27823
8	2001	38.5245
9	2002	38.77077
10	2003	39.01704
11	2004	39.26331
12	2005	39.50958
13	2006	39.75585
14	2007	40.00212
15	2008	40.24839
16	2009	40.49466
17	2010	40.74093

(b) 2007

Exercise 0.6 (page 46)

1. $3b(3a - 4a^2 + 6b)$ 3. $2x(2x + 4y^2 + y^3)$
5. $(7x^2 + 2)(x - 2)$ 7. $(6 + y)(x - m)$
9. $(x + 2)(x + 6)$ 11. $(x - 3)(x + 2)$
13. $(7x + 4)(x - 2)$ 15. $(x - 5)^2$
17. $(7a + 12b)(7a - 12b)$
19. (a) $(3x - 1)(3x + 8)$ (b) $(9x + 4)(x + 2)$
21. $x(4x - 1)$ 23. $(x^2 - 5)(x + 4)$
25. $2(x - 7)(x + 3)$ 27. $2x(x - 2)^2$
29. $(2x - 3)(x + 2)$ 31. $3(x + 4)(x - 3)$
33. $2x(x + 2)(x - 2)$ 35. $(5x + 2)(2x + 3)$
37. $(5x - 1)(2x - 9)$
39. $(y^2 + 4x^2)(y + 2x)(y - 2x)$
41. $(x + 2)^2(x - 2)^2$
43. $(2x + 1)(2x - 1)(x + 1)(x - 1)$
45. $(x + 1)^3$ 47. $(x - 4)^3$
49. $(x - 4)(x^2 + 4x + 16)$
51. $(3 + 2x)(9 - 6x + 4x^2)$ 53. $x + 1$
55. $1 + x$ 57. $7x - 3x^3$ 59. $P(1 + rt)$
61. $m(c - m)$
63. (a) $p(10{,}000 - 100p)$; $x = 10{,}000 - 100p$ (b) 6200

Exercise 0.7 (page 53)

1. $2y^3/z$ 3. $\frac{1}{3}$ 5. $(x - 1)/(x - 3)$
7. $(2xy^2 - 5)/(y + 3)$ 9. $20x/y$ 11. $\frac{32}{3}$
13. $2x^2 - 7x + 6$
15. $-(x + 1)(x + 3)/[(x - 1)(x - 3)]$
17. $15bc^2/2$ 19. $5y/(y - 3)$
21. $\dfrac{-x(x-3)(x+2)}{x+3}$ 23. $\dfrac{1}{x+1}$ 25. $\dfrac{4a-4}{a(a-2)}$

27. $\dfrac{-x^2+x+1}{x+1}$ 29. $\dfrac{16a+15a^2}{12(x+2)}$ 31. $\dfrac{79x+9}{30(x-2)}$

33. $\dfrac{-4y}{(x-2y)^2(x+2y)}$ 35. $\dfrac{9x+4}{(x-2)(x+2)(x+1)}$

37. $(7x-3x^3)/\sqrt{3-x^2}$ 39. $\frac{1}{6}$ 41. xy

43. $\dfrac{x+1}{x^2}$ 45. $\dfrac{1}{\sqrt{a}}=\dfrac{\sqrt{a}}{a}$ 47. $\dfrac{x-2}{(x-3)\sqrt{x^2+9}}$

49. (a) -12 (b) $\frac{25}{36}$ 51. $2b-a$

53. $\dfrac{x^3+y^3}{x^2y^2(x+y)}=\dfrac{x^2-xy+y^2}{x^2y^2}$

55. $(1-2\sqrt{x}+x)/(1-x)$ 57. $1/(\sqrt{x+h}+\sqrt{x})$

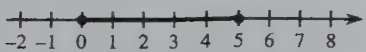
59. $(bc+ac+ab)/abc$

61. (a) $\dfrac{0.1x^2+55x+4000}{x}$ (b) $0.1x^2+55x+4000$

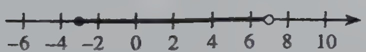
63. $\dfrac{t^2+9t}{(t+3)^2}$

Chapter 0 Review Exercises (page 57)

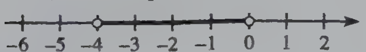
1. yes 2. no 3. no 4. $\{1, 2, 3, 4, 9\}$
5. $\{5, 6, 7, 8, 10\}$ 6. $\{1, 2, 3, 4, 9\}$
7. yes, $(A' \cup B')' = \{1, 3\} = A \cap B$
8. (a) commutative property of addition
(b) associative property of multiplication
(c) distributive law
9. (a) irrational (b) rational, integer
(c) meaningless
10. (a) $>$ (b) $<$ (c) $>$ 11. 6
12. 142 13. 10 14. 5/4 15. 9 16. -29
17. 13/4 18. -10.62857888
19. (a) $[0, 5]$, closed



(b) $[-3, 7)$, half-open

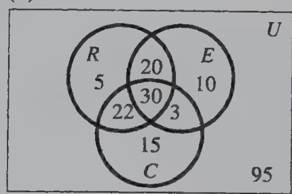


(c) $(-4, 0)$, open



20. (a) $-1 < x < 16$ (b) $-12 \le x \le 8$
(c) $x < -1$
21. (a) 1 (b) $2^{-2} = 1/4$ (c) 4^6 (d) 7
22. (a) $1/x^2$ (b) x^{10} (c) x^9 (d) $1/y^8$ (e) y^6
23. $-x^2y^2/36$ 24. $9y^8/(4x^4)$ 25. $y^2/(4x^4)$
26. $-x^8z^4/y^4$ 27. $3x/(y^7z)$ 28. $x^5/(2y^3)$
29. (a) 4 (b) 2/7 (c) 1.1
30. (a) $x^{1/2}$ (b) $x^{2/3}$ (c) $x^{-1/4}$
31. (a) $\sqrt[3]{x^2}$ (b) $1/\sqrt{x}=\sqrt{x}/x$ (c) $-x\sqrt{x}$
32. (a) $5y\sqrt{2x}/2$ (b) $\sqrt[3]{x^2y}/x^2$ 33. $x^{5/6}$ 34. y
35. $x^{17/4}$ 36. $x^{11/3}$ 37. $x^{2/5}$ 38. x^2y^8

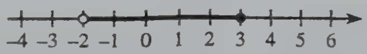
39. $2xy^2\sqrt{3xy}$ 40. $25x^3y^4\sqrt{2y}$ 41. $6x^2y^4\sqrt[3]{5x^2y^2}$
42. $8a^2b^4\sqrt{2a}$ 43. $2xy$ 44. $4x\sqrt{3xy}/(3y^4)$
45. $-x-2$ 46. $-x^2-x$
47. $4x^3+xy+4y-4$ 48. $24x^5y^5$
49. $3x^2-7x+4$ 50. $3x^2+5x-2$
51. $4x^2-7x-2$ 52. $6x^2-11x-7$
53. $4x^2-12x+9$ 54. $16x^2-9$
55. $2x^4+2x^3-5x^2+x-3$
56. $8x^3-12x^2+6x-1$
57. x^3-y^3 58. $(2/y)-(3xy/2)-3x^2$
59. $3x^2+2x-3+(-3x+7)/(x^2+1)$
60. $x^3-x^2+2x+7+21/(x-3)$
61. x^2-x 62. $2x-a$ 63. $x^3(2x-1)$
64. $2(x^2+1)^2(1+x)(1-x)$ 65. $(2x-1)^2$
66. $(4+3x)(4-3x)$ 67. $2x^2(x+2)(x-2)$
68. $(x-7)(x+3)$ 69. $(3x+2)(x-1)$
70. $(4x+3)(3x-8)$ 71. $(2x+3)^2(2x-3)^2$
72. $x^{2/3}+1$ 73. (a) $\dfrac{x}{(x+2)}$ (b) $\dfrac{2xy(2-3xy)}{2x-3y}$
74. $\dfrac{x^2-4}{x(x+4)}$ 75. $\dfrac{(x+3)}{(x-3)}$ 76. $\dfrac{x^2(3x-2)}{(x-1)(x+2)}$
77. $(6x^2+9x-1)/(6x^2)$ 78. $\dfrac{4x-x^2}{4(x-2)}$
79. $-\dfrac{x^2+2x+2}{x(x-1)^2}$ 80. $\dfrac{x(x-4)}{(x-2)(x+1)(x-3)}$
81. $\dfrac{(x-1)^3}{x^2}$ 82. $\dfrac{1-x}{1+x}$ 83. $3(\sqrt{x}+1)$
84. $2/(\sqrt{x}+\sqrt{x-4})$
85. (a)



R: recognized
E: exercise
C: community involvement

(b) 10 (c) 100
86. (a) 4115.27 (b) \$66,788.69
87. (a) $10{,}000\left[\dfrac{(0.0065)(1.0065)^n}{(1.0065)^n-1}\right]$
(b) \$243.19 (for both)
88. (a) $5 = k\sqrt[3]{A}$ (b) $\sqrt[3]{2.25} \times 1.31$
89. (a) $\dfrac{5400p}{100-p}$
(b) \$0. It costs nothing if no effort is made to remove pollution.
(c) \$264,600
(d) Undefined. Removing 100% would be impossible, and the cost of getting close would be enormous.

Chapter 0 Test (page 59)

1. (a) $\{3, 4, 6, 8\}$ (b) $\{3, 4\}$; $\{3, 6\}$; or $\{4, 6\}$
(c) $\{6\}$ or $\{8\}$
2. 21
3. (a) 8 (b) 1 (c) $\frac{1}{2}$ (d) -10 (e) 30
(f) $\frac{5}{6}$ (g) $\frac{2}{3}$ (h) -3
4. (a) $\sqrt[4]{x}$ (b) $\dfrac{1}{\sqrt[4]{x^3}}$ 5. (a) $\dfrac{1}{x^5}$ (b) $\dfrac{x^{21}}{y^6}$
6. (a) $\dfrac{\sqrt{5x}}{5}$ (b) $2a^2b^2\sqrt{6ab}$ (c) $\dfrac{1-2\sqrt{x}+x}{1-x}$
7. (a) 5 (b) -8 (c) -5
8. $(-2, 3]$ 

9. (a) $2x^2(4x-1)$ (b) $(x-12)(x+2)$
(c) $(3x-2)(2x-3)$ (d) $2x^3(1+4x)(1-4x)$
10. (c); -2
11. $2x + 1 + \dfrac{2x-6}{x^2-1}$
12. (a) $19y - 45$ (b) $-6t^6 + 9t^9$
(c) $4x^3 - 21x^2 + 13x - 2$ (d) $-18x^2 + 15x - 2$
(e) $4m^2 - 28m + 49$ (f) $\dfrac{x^4}{3x+9}$ (g) $\dfrac{x^7}{81}$
(h) $\dfrac{6-x}{x-8}$ (i) $\dfrac{x^2-4x-3}{x(x-3)(x+1)}$
13. $\dfrac{y-x}{y+xy^2}$
14. (a) 0 (b) 175
15. \$4875.44 (nearest cent)

Exercise 1.1 (page 73)

1. $x = -9/4$ 3. $x = 0$ 5. $x = 4/5$
7. $x = -32$ 9. $x = 16/5$ 11. $x = -29/2$
13. $x = 17/13$ 15. $x = 13/5$ 17. $x = -1/3$
19. $x = 3$ 21. $x = 74$ 23. No solution
25. $x = 5/4$ 27. $x \approx -0.279$
29. $x \approx -1147.362$ 31. $y = \frac{3}{4}x - \frac{15}{4}$
33. $y = -6x + \frac{22}{3}$ 35. $P = I/(rt)$
37. 96 39. (a) 132 pounds (b) 73.6 inches
41. (a) 59.2% (approx.) (b) $t \approx 107$, in 2082
43. $t \approx 26$ in 2014 45. 9000
47. (a) $T = (7n - 52)/12$ (b) 61°
49. \$4000 51. \$90,000 at 9%; \$30,000 at 13%
53. \$2160/month; 8% increase
55. lost \$40 57. \$307 59. \$246

Exercise 1.2 (page 85)

1. yes 3. no
5. yes; each x-value has exactly one y-value;
$D = \{1, 2, 3, 8, 9\}$, $R = \{-4, 5, 16\}$
7. (a) -10 (b) 6 (c) -34 (d) 2.8
9. (a) -3 (b) 1 (c) 13 (d) 6
11. (a) 63/8 (b) 6 (c) -6
13. (a) no, $f(2+1) = f(3) = 13$ but $f(2) + f(1) = 10$
(b) $1 + x + h + x^2 + 2xh + h^2$
(c) no, $f(x) + f(h) = 2 + x + h + x^2 + h^2$
(d) no, $f(x) + h = 1 + x + x^2 + h$
(e) $1 + 2x + h$
15. (a) $-2x^2 - 4xh - 2h^2 + x + h$
(b) $-4x - 2h + 1$
17. The vertical-line test shows that graph (a) represents a function of x, but graph (b) does not.
19. (a) 10 (b) 6
21. (a) $b = a^2 - 4a$ (b) $(1, -3)$, yes
(c) $(3, -3)$, yes (d) $x = 0, x = 4$, yes
23. (a) $3x + x^3$ (b) $3x - x^3$ (c) $3x^4$ (d) $\dfrac{3}{x^2}$
25. (a) $\sqrt{2x} + x^2$ (b) $\sqrt{2x} - x^2$
(c) $x^2\sqrt{2x}$ (d) $\dfrac{\sqrt{2x}}{x^2}$
27. (a) $-8x^3$ (b) $1 - 2(x-1)^3$
(c) $[(x-1)^3 - 1]^3$ (d) $(x-1)^6$
29. (a) $2\sqrt{x^4+5}$ (b) $16x^2 + 5$
(c) $2\sqrt{2\sqrt{x}}$ (d) $4x$
31. D: all reals; R: reals $y \geq 4$
33. D: reals $x \geq -4$: R: reals $y \geq 0$
35. $x \geq 1, x \neq 2$ 37. $-7 \leq x \leq 7$
39. (a) $f(20) = 103{,}000$ means that if \$103,000 is borrowed, it can be repaid in 20 years (of \$800-per-month payments).
(b) no; $f(5+5) = f(10) = 69{,}000$, but $f(5) + f(5) = 80{,}000$
41. (a) $f(1950) = 16.5$ means that in 1950 there were 16.5 workers supporting each Social Security beneficiary.
(b) 3.4
(c) The parts of the graph that correspond to data prior to 1995 would be the same, but data on this graph beyond 1995 are predictions for the future and only might be accurate.
(d) Domain: $1950 \leq t \leq 2050$
Range: $1.9 \leq n \leq 16.5$
43. (a) $s \geq 0$
(b) $f(10) \approx -29.33$ means that if the air temperature is -5°F and there is a 10 mph wind, then the temperature feels like -29.33°F.
(c) $f(0) = 45.694$ from the formula, but $f(0)$ should equal the air temperature, -5°F.
45. (a) yes (b) $t \neq -2$ (c) $t \geq 0$
47. (a) yes (b) all reals
(c) D: $32° \leq F \leq 212°$; R: $0° \leq C \leq 100°$
(d) 4.44°C

49. (a) $0 \le p < 100$
(b) \$5972.73; to remove 45% of the particulate pollution would cost \$5972.73.
(c) \$65,700; to remove 90% of the particulate pollution would cost \$65,700.
(d) \$722,700; to remove 99% of the particulate pollution would cost \$722,700.
(e) \$1,817,700; to remove 99.6% of the particulate pollution would cost \$1,817,700.

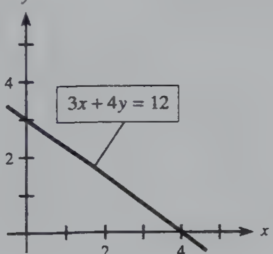
51. (a) yes (b) $A(2) = 96$; $A(30) = 600$
(c) $0 < x < 50$

53. (a) $(p \circ x)(t) = 180(1000 + 10t) - \dfrac{(1000 + 10t)^2}{100} - 200$
(b) $x = 1150$, $p = \$193,575$

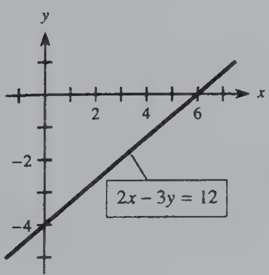
55. $L = 2x + 3200/x$ 57. $R = (30 + x)(10 - 0.20x)$

Exercise 1.3 (page 99)

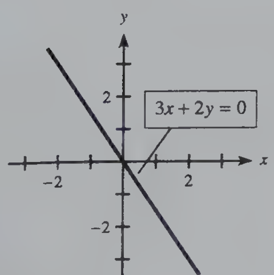
1. x-intercept 4
y-intercept 3



3. x-intercept 6
y-intercept -4



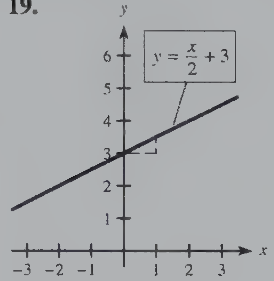
5. x-intercept 0
y-intercept 0



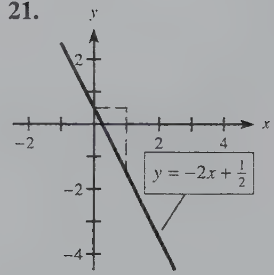
7. $m = -5$ 9. $m = 0$
11. $m = 7/3$, $b = -1/4$ 13. $m = 0$, $b = 3$

15. no slope, no y-intercept
17. $m = -2/3$, $b = 2$

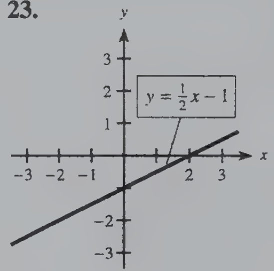
19.



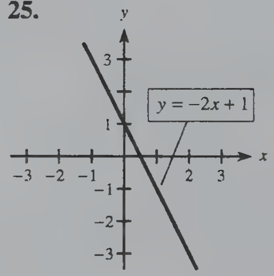
21.



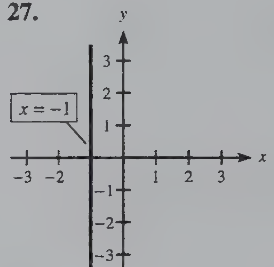
23.



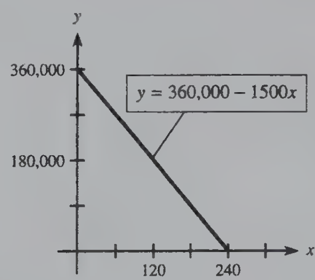
25.



27.

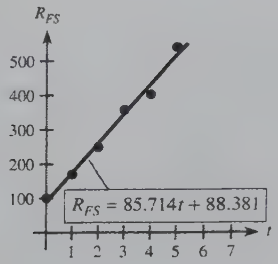


29. $y = 2x - 4$ 31. $-x + 13y = 32$ 33. $y = 0$
35. perpendicular
37. neither; same line
39. $y = -\frac{3}{5}x - \frac{41}{5}$ 41. $y = -\frac{6}{5}x + \frac{23}{5}$
43. (a)

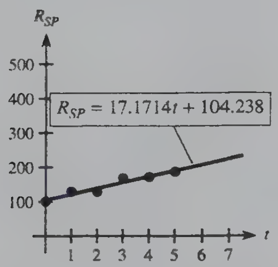


(b) 240 months
(c) After 60 months, the value of the building is \$270,000.

45. **(a)**



(b)



(c) At $t = 0$, $R_{FS} = 88.381$ and $R_{SP} = 104.238$. These are different from 100 because these equations are a good fit to the data but are not perfect; the data are not exactly linear.

(d) These equations show past performance; future sales, market conditions, profits, and confidence cannot be measured and are not part of these equations.

47. **(a)** $m = 0.1369$ $b = -5.091255$

(b) The y-intercept indicates that when there were 0 terminals, the amount transacted was negative. This is impossible. The model must be restricted to when both $x > 0$ and $y \geq 0$.

(c) The slope means that the transaction amount increases by \$0.1369 (billion) when the number of ATMs increases by 1 (thousand).

49. $y = 4.95 + 0.0838x$

51. **(a)** $f = 0.838m - 1387.4$ **(b)** \$23,752.60

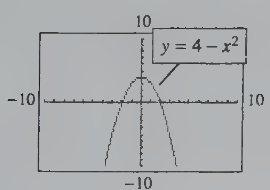
53. $p = 85{,}000 - 1700x$

55. $R = 3.2t - 0.2$

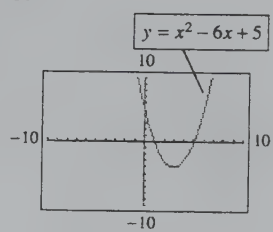
57. $y = 0.48x - 71$

Exercise 1.4 *(page 108)*

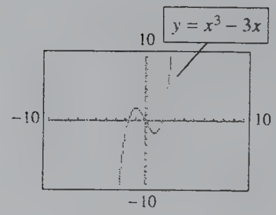
1.



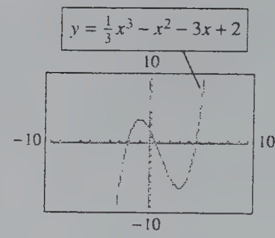
3.



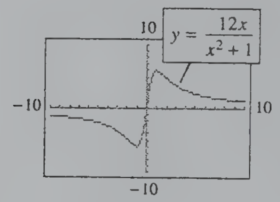
5.



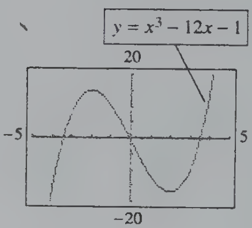
7.



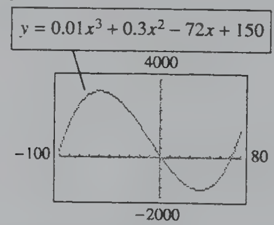
9.



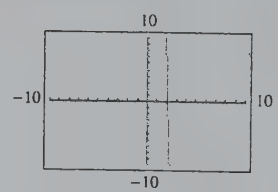
11.



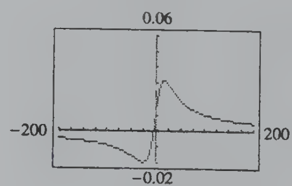
13. **(a)** $y = 0.01\,x^3 + 0.3x^2 - 72x + 150$



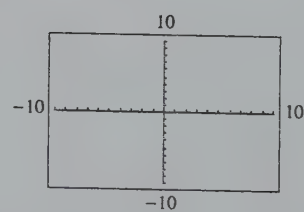
(b) Standard window



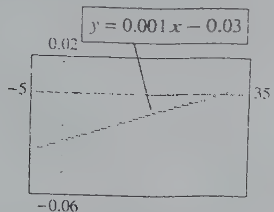
15. **(a)** $y = \dfrac{x + 15}{x^2 + 400}$



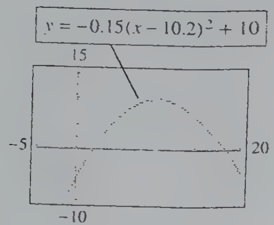
(b) Standard window



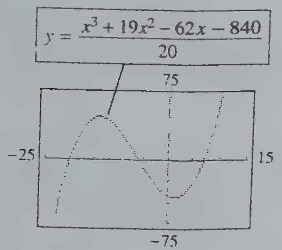
17. (a) The equation is linear, so the graph will be a line. Use the intercepts to determine a window.
(b) Window: x-min $= -5$, x-max $= 35$, y-min $= -0.06$, y-max $= 0.02$
(c) $y = 0.001x - 0.03$



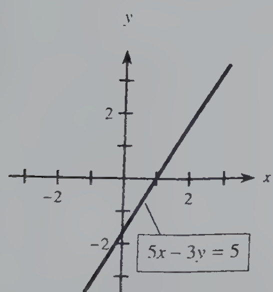
19.



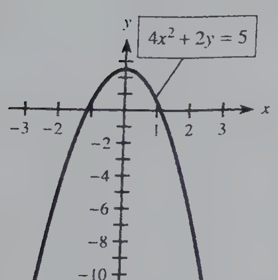
21. $y = \dfrac{(x^3 + 19x^2 - 62x - 840)}{20}$



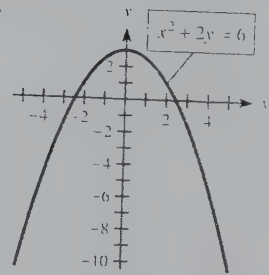
23.



25.

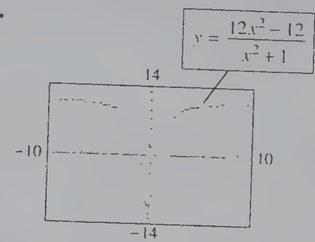


27.

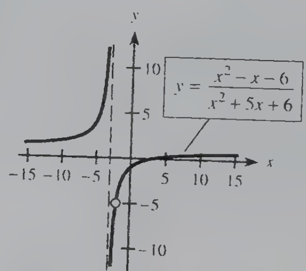


29. $f(1) = 0, f(-3/2) = -65/8$

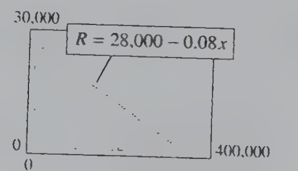
31.



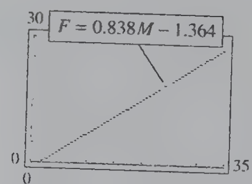
33.



35. (a) $(-1.11, 8.11)$ **(b)** $(-1.11, 8.11)$
37. (a) 1.000, 4.000 **(b)** 1.000, 4.000
39.

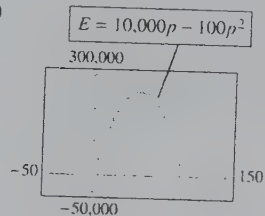


41. (a)



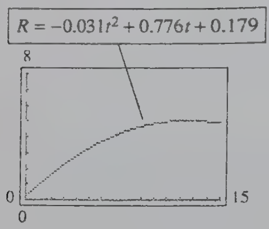
(b) The coordinates mean that when a male's salary is \$50 thousand, a female's is \$40.536 thousand.

43. (a)

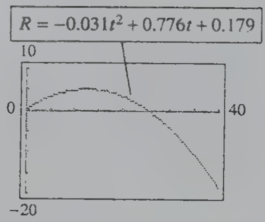


(b) $E \geq 0$ when $0 \leq p \leq 100$

45. **(a)**

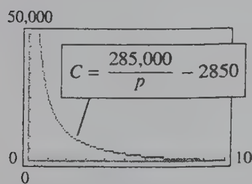


(b)



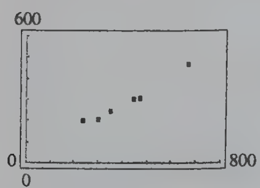
(c) From 1980 to 1995 revenues were rising to what appears to be a maximum. From 1980 to 2020 revenues rose then declined below 0. This model cannot be valid for this period, because revenue cannot be negative.

47. **(a)**

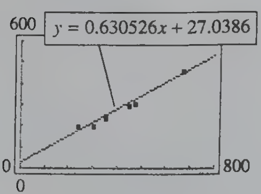


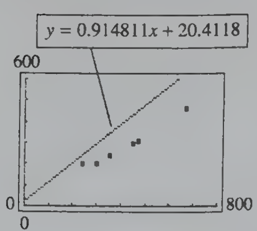
(b) Near $p=0$, cost grows without bound.
(c) The coordinates of the point mean that obtaining stream water with 1% of the current pollution levels would cost $282,150.
(d) The p-intercept means that stream water with 100% of the current pollution levels would cost $0.

49. **(a)**



(b)





The first equation fits better.

(c) predicts $325 (nearest dollar), actual is $305.

Exercise 1.5 *(page 121)*

1. $x=2, y=2$ **3.** no solution **5.** $x=2, y=5$
7. $x=14/11, y=6/11$ **9.** $x=10/3, y=2$
11. $x=2, y=1$ **13.** $x=1, y=1$ **15.** no solution
17. $x=-52/7, y=-128/7$ **19.** dependent
21. $x=1, y=7$
23. $x=4, y=2$ **25.** $x=-1, y=1$
27. $x=-17, y=7, z=5$
29. $x=4, y=12, z=-1$
31. $x=44, y=-9, z=-1/2$
33. $68,000 at 18%; $77,600 at 10%
35. $13,500 at 10%; $10,000 at 12%
37. $A=4$ oz, $B=6\frac{2}{3}$ oz **39.** $A=4550, B=1500$
41. 7 cc of 20%; 3 cc of 5%
43. 10,000 at $20; 6000 at $30 **45.** 80 cc
47. 5 oz of A, 1 oz of B, 5 oz of C
49. $A=200, B=100, C=200$

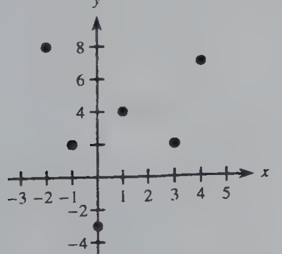
Exercise 1.6 *(page 131)*

1. **(a)** $C(x)=17x+3400$ **(b)** $6800
3. **(a)** $R(x)=34x$ **(b)** $10,200
5. **(a)** $P(x)=17x-3400$ **(b)** $1700
7. **(a)** $P(x)=37x-1850$ **(b)** −$740, loss of $740
(c) 50
9. **(a)** $m=5, b=250$
(b) $\overline{MC}=5$ means each additional unit produced costs $5.
(c) 250
(d) Slope = marginal cost; c-intercept = fixed costs
(e) 5, 5
11. **(a)** 27
(b) $\overline{MR}=27$ means each additional unit sold brings in $27.
(c) 27, 27
13. **(a)** $P(x)=22x-250$
(b) 22
(c) $\overline{MP}=22$
(d) Each unit sold adds $22 to profits at all levels of production, so produce and sell as much as possible.
15. $P=58x-8500, \overline{MP}=58$

17. **(a)** Revenue passes through the origin. **(b)** \$2000 **(c)** 400 units **(d)** $\overline{MC} = 2.5$; $\overline{MR} = 7.5$
19. 33
21. **(a)** $R(x) = 12x$; $C(x) = 8x + 1600$ **(b)** 400 units
23. **(a)** $P(x) = 4x - 1600$
(b) $x = 400$ units to break even
25. **(a)** $R(x) = 54.90x$ **(b)** $C(x) = 14.90x + 20{,}200$
(c) 505
27. demand decreases
29. **(a)** 650 (approx) **(b)** 300 **(c)** shortage
31. 16 demanded, 25 supplied; surplus
33. $p = -2q/3 + 1060$ **35.** $p = 0.0001q + 0.5$
37. **(a)** demand falls; supply rises **(b)** (30, \$25)
39. **(a)** $q = 20$ **(b)** $q = 40$
(c) shortage, 20 units short
41. shortage **43.** $q = 20, p = \$18$
45. $q = 10, p = \$180$ **47.** $q = 100, p = \$325$
49. **(a)** \$15 **(b)** $q = 100, p = \$100$
(c) $q = 50, p = \$110$ **(d)** yes
51. $q = 8$; $p = \$188$ **53.** $q = 500, p = 40$
55. $q = 1200, p = \$15$

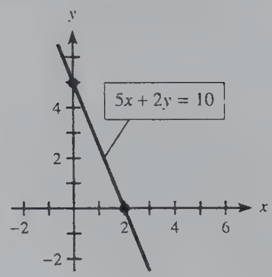
Chapter 1 Review Exercises *(page 137)*

1. $x = 7$ **2.** $x = \frac{31}{3}$ **3.** $x = -13$
4. $-\frac{29}{8}$ **5.** $x = -\frac{1}{9}$ **6.** $x = 10.05$
7. $x = 8$ **8.** $y = -\frac{2}{3}x - \frac{4}{3}$ **9.** no solution
10. yes **11.** no **12.** yes
13. D: reals $x \le 9$; R: reals $y \ge 0$
14. **(a)** 2 **(b)** 37 **(c)** 29/4
15. **(a)** 0 **(b)** 9/4 **(c)** 10.01
16. $9 - 2x - h$ **17.** yes **18.** no **19.** 4
20. $x = 0, x = 4$
21. **(a)** 7 **(b)** $x = -1, x = 3$
(c)

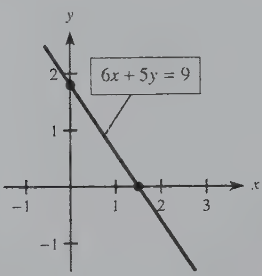


22. **(a)** $x^2 + 3x + 5$ **(b)** $(3x + 5)/x^2$
(c) $3x^2 + 5$ **(d)** $9x + 20$

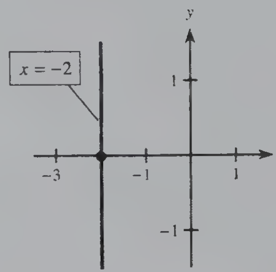
23. x: 2, y: 5



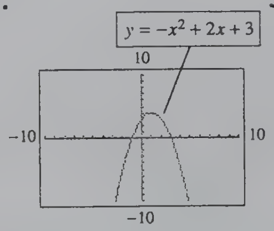
24. x: 3/2, y: 9/5



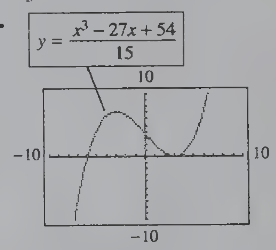
25. x: −2, y: none



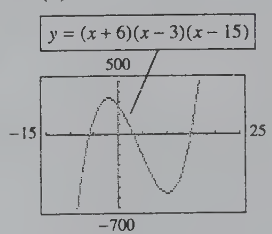
26. $m = 1$ **27.** undefined
28. $m = -\frac{2}{5}, b = 2$ **29.** $m = -\frac{4}{3}, b = 2$
30. $y = 4x + 2$ **31.** $y = -\frac{1}{2}x + 3$
32. $y = \frac{2}{5}x + \frac{9}{5}$ **33.** $y = -\frac{11}{8}x + \frac{17}{4}$ **34.** $x = -1$
35. $y = 4x + 2$ **36.** $y = \frac{4}{3}x + \frac{10}{3}$
37.



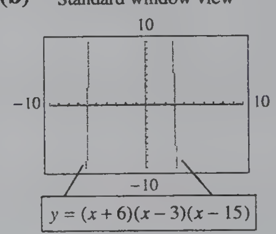
38.



39. **(a)**

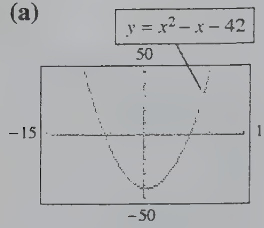


(b) Standard window view

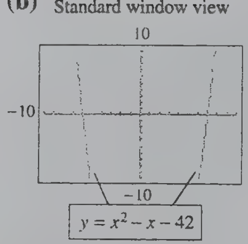


(c) The graph in (a) shows the complete graph. The graph in (b) shows a piece that rises toward the high point and a piece between the high and low points.

40. **(a)**

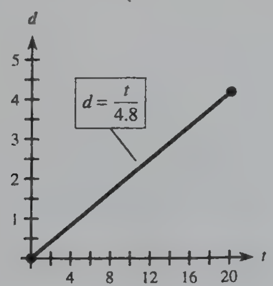


(b)



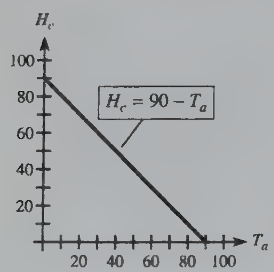
(c) The graph in (a) is a complete graph. The one in (b) shows pieces that fall toward the minimum point and rise from it.

41. reals $x \geq -3$ with $x \neq 0$ **42.** $x = 2, y = 1$
43. $x = 10, y = -1$ **44.** $x = 3, y = -2$
45. no solution **46.** $x = 10, y = -71$
47. $x = 1, y = -1, z = 2$
48. $x = 11, y = 10, z = 9$ **49.** 95%
50. 40,000 mi. He would normally drive more than 40,000 miles in 5 years, so he should buy diesel.
51. **(a)** yes **(b)** no **(c)** 4
52. **(a)** \$565.44
(b) The monthly payment on a \$70,000 loan is \$494.75.
53. **(a)** $(p \circ q)(t) = 180(1000 + 10t) - \dfrac{(1000 + 10t)^2}{100} - 200$
(b) $x = 1150, p = \$193{,}575$
54. $(W \circ L)(t) = k\left(50 - \dfrac{(t - 20)^2}{10}\right)^3$
55. **(a)**



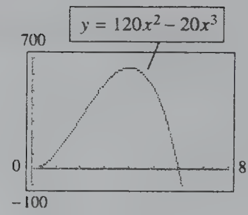
(b) When the time between lightning flashes is 9.6 seconds the storm is 2 miles away.

56.



57. $P = 58x - 8500$
58. $F = \frac{9}{5}C + 32$ or $C = \frac{5}{9}(F - 32)$

59. **(a)**

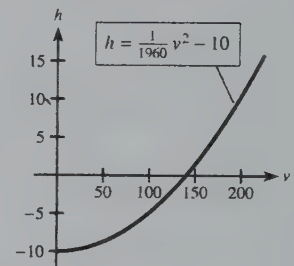


(b) $0 \leq x \leq 6$

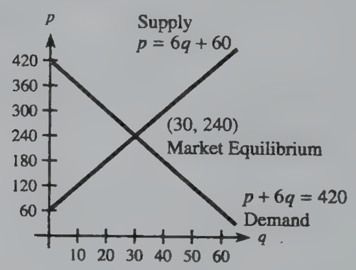
60. $v^2 = 1960(h + 10)$

$$\frac{v^2}{1960} = h + 10$$

$$h = \frac{1}{1960}v^2 - 10$$



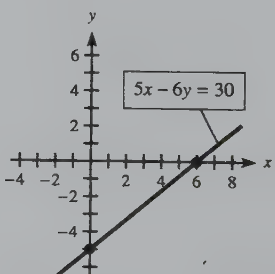
61. \$100,000 at 9.5%; \$50,000 at 11%
62. 2.8 liters of 20%; 1.2 liters of 70%
63. **(a)** 12 supplied; 14 demanded **(b)** shortage
(c) increase
64.



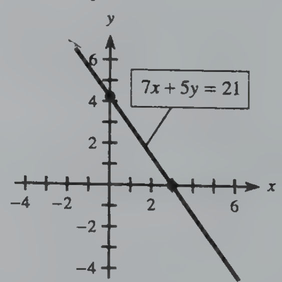
65. **(a)** 38.80 **(b)** 61.30 **(c)** 22.50 **(d)** 200
66. **(a)** $C(x) = 22x + 1500$ **(b)** $R(x) = 52x$
(c) $P(x) = 30x - 1500$ **(d)** $\overline{MC} = 22$
(e) $\overline{MR} = 52$ **(f)** $\overline{MP} = 30$ **(g)** $x = 50$
67. $q = 300, p = \$150$ **68.** $q = 700, p = 80$

Chapter 1 Test (page 141)

1. $x = 18/7$ **2.** $x = -3/7$
3. $x = -38$ **4.** $5 - 4x - 2h$
5. x: 6 y: −5



6. x: 3 y: 21/5



7. **(a)** Domain: $x \geq -4$ **(b)** $2\sqrt{7}$ **(c)** 6
Range: $f(x) \geq 0$

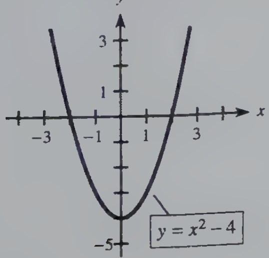
8. $y = -\frac{3}{2}x + \frac{1}{2}$ 9. $m = -\frac{5}{4}$; $b = \frac{15}{4}$
10. (a) $x = -3$ (b) $y = -4x - 13$
11. (a) No; a vertical line intersects the curve twice.
(b) Yes; there is one y-value for each x-value.
(c) No; one value of x gives two y-values.
12. $x = -2$, $y = 2$
13. (a) $5x^3 + 2x^2 - 3x$ (b) $x + 2$ (c) $5x^2 + 7x + 2$
14. (a) 30 (b) $P = 8x - 1200$ (c) 150
(d) 8; the sale of each additional unit gives \$8 more profit.
15. (a) $R = 50x$
(b) 19,000; it costs \$19,000 to produce 100 units.
(c) 450
16. $q = 200$, $p = \$2500$
17. (a) 360,000; original value of the building
(b) -1500; building depreciates \$1500 per month.
18. 400 19. 12,000 at 9%, 8000 at 6%

Exercise 2.1 (page 156)

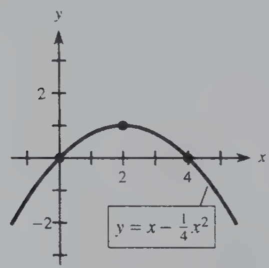
1. $x^2 + 5x - 3 = 0$ 3. $x^2 + 2x - 1 = 0$
5. $y^2 + 3y - 2 = 0$ 7. $\frac{3}{2}, -\frac{3}{2}$ 9. 0, 1
11. $-7, 3$ 13. $\frac{1}{2}$ 15. $8, -4$ 17. $-4, -2$
19. $-6, 2$ 21. $4, -3/4$ 23. 8, 1 25. 1/2
27. (a) $2 + 2\sqrt{2}, 2 - 2\sqrt{2}$ (b) $4.83, -0.83$
29. no real solutions
31. (a) $-\frac{7}{4}, \frac{3}{4}$ (b) $-1.75, 0.75$
33. (a) $(1 \pm \sqrt{31})/5$ (b) $1.31, -0.91$
35. $\sqrt{7}, -\sqrt{7}$ 37. no real solutions 39. $1, -9$
41. $-9, -10$ 43. $-\frac{2}{7} \approx -0.29$
45. $-2, 5$ 47. $-300, 100$ 49. $0.69, -0.06$
51. $x = 20$ or $x = 70$ 53. $x = 10$ or $x = 345\frac{5}{9}$
55. (a) $4\sqrt{41} \approx 25.6$ (b) $4\sqrt{161} \approx 50.8$
(c) 25.2; K_c is approximately doubled
57. 59.7 mph 59. (a) 1.93, 13.99 (b) 1994
61. $x = 16.59$; 1991 63. \$80

Exercise 2.2 (page 164)

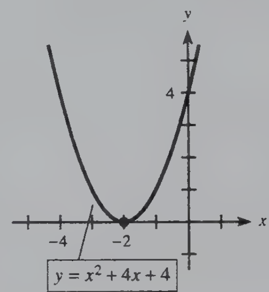
1. $(-1, -\frac{1}{2})$; min 3. (1, 9); max
5. (a) $x = 3$ (b) $f(3) = 9$
7. (a) $x = -1$ (b) $f(-1) = -4$
9. min $(0, -4)$; zeros $(-2, 0)$, $(2, 0)$



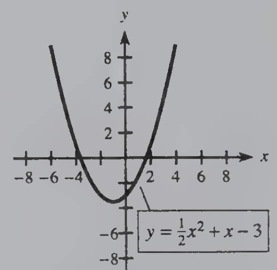
11. max (2, 1); zeros (0, 0), (4, 0)



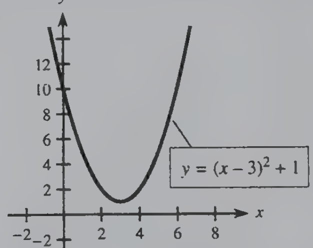
13. min $(-2, 0)$; zero $(-2, 0)$



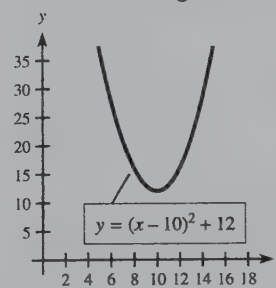
15. min $(-1, -3\frac{1}{2})$; zeros $(-1 + \sqrt{7}, 0)$, $(-1 - \sqrt{7}, 0)$



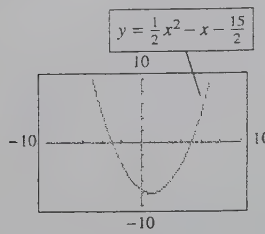
17. (a) 3 units to the right and 1 unit up
(b)



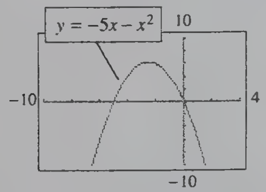
19. (a) 10 units to the right and 12 units up
(b)



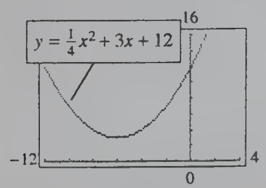
21. vertex $(1, -8)$; zeros $(-3, 0)$, $(5, 0)$



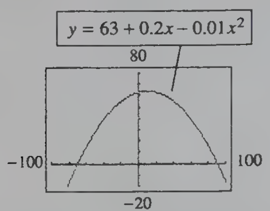
23. vertex $(-\frac{5}{2}, \frac{25}{4})$; zeros $(-5, 0)$, $(0, 0)$



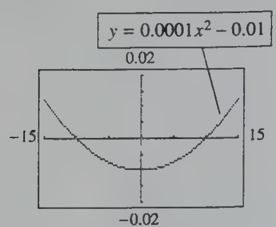
25. vertex $(-6, 3)$; no real zeros



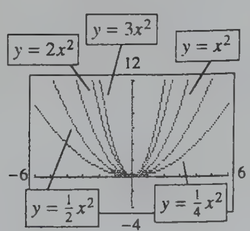
27. vertex $(10, 64)$; zeros $(90, 0)$, $(-70, 0)$

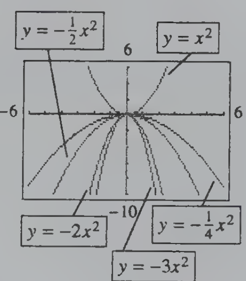


29. vertex $(0, -0.01)$; zeros $(10, 0)$, $(-10, 0)$



31. **(a)**



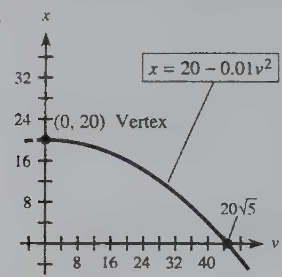


(b) When $a > 0$, the graph of $y = ax^2$ has the same basic shape as $y = x^2$. When $a < 0$, the graph is turned upside down.

(c) When $0 < a < 1$, the graph of $y = ax^2$ opens more gradually than $y = x^2$. When $a > 1$, the opening is narrower than for $y = x^2$.

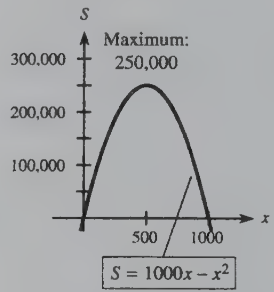
33. **(a)** 80 units **(b)** \$540

35.

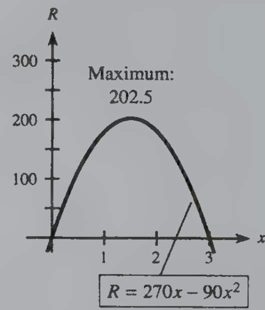


37. 400

39. Dosage = 500



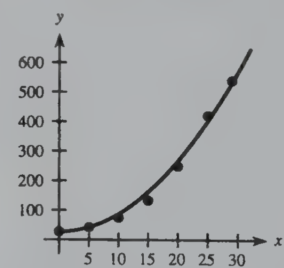
41. Intensity = 1.5



43. Equation **(a)** (384.62, 202.31) **(b)** (54, 46)
Projectile **(a)** goes higher.

45. **(a)** quadratic
(b) $a > 0$ because the graph opens upward.

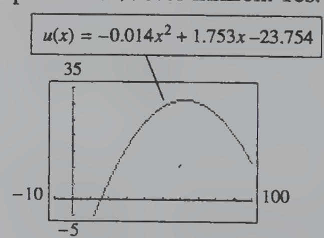
47. **(a)**



x	y
0	27.1
5	41.6
10	74.4
15	132.9
20	249.1
25	420.1
29	539.9

(b) The shape appears to be quadratic.
(c) $y = 0.6x^2 + 27.1$ fits fairly well.
(d) The model predicts 567.1.
(e) Yes, \$666.2 million is significantly larger than the predicted \$567.1 million. Yes.

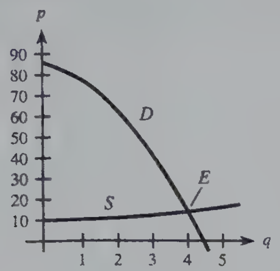
49.



51. **(a)** 2010, 1916
(b) 2011

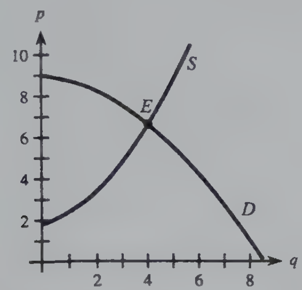
Exercise 2.3 (page 173)

1. (a) and (b)



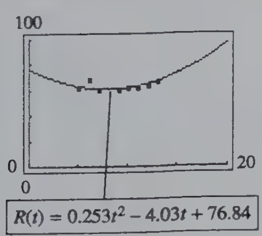
(c) See E on graph. (d) $q = 4, p = \$14$

3. (a) and (b)



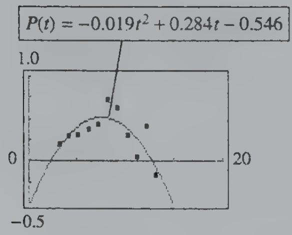
(c) See E on graph. (d) $q = 4, p = \$6.60$

5. $q = 10, p = \$196$ 7. $p = \$27.08, q = 216\frac{2}{3}$
9. $p = \$40, q = 30$ 11. $q = 90, p = \$50$
13. $q = 70, p = \$62$ 15. $x = 40$ units, $x = 50$ units
17. $x = 50, x = 300$
19. $x = 15$ units; reject $x = 100$ 21. \$41,173.61
23. \$87.50 25. $x = 55$, $P(55) = 2025$
27. (a) $P(x) = -x^2 + 350x - 15{,}000$; max is \$15,625
(b) no (c) x-values agree
29. (a) $x = 28$ units, $x = 1000$ units (b) \$651,041.67
(c) $P(x) = -x^2 + 1028x - 28{,}000$; max is \$236,196
(d) \$941.60
31. (a) $t \approx 8$; 1988 $R = \$60.792$ billion
(b) The data show a smaller revenue, $R = \$60.53$ billion, in 1987
(c)



(d) Except for 1986 ($t = 6$), the model fits the data quite well.

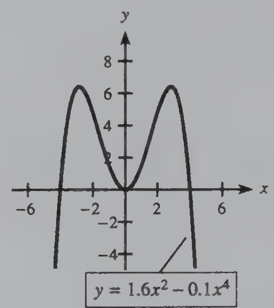
33. (a) $P(t) = -0.019t^2 + 0.284t - 0.546$ (b) 1987
(c)



(d) The model projects decreasing profits, and except for 1992, the data support this.
(e) Management would be interested in increasing revenues or reducing costs (or both) to improve profits.

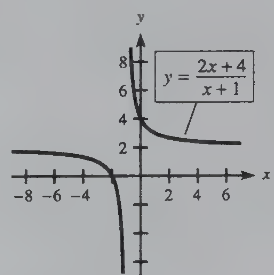
Exercise 2.4 (page 185)

1. k 3. l 5. b 7. f 9. g
11. j 13. i 15. 3rd 17. 4th 19. j
21. g 23. a 25. f 27. d
29. (a) 8/3 (b) 9.9 (c) −999.999 (d) no
31. (a) 64 (b) 1 (c) 1000 (d) 0.027
33. (a) 2 (b) 4 (c) 0 (d) 2
35. (a)



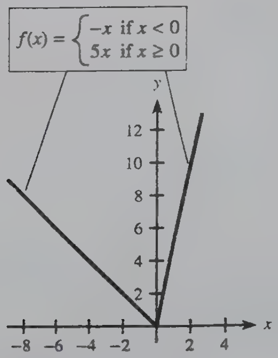
(b) polynomial (c) no asymptotes
(d) turning points at $x = 0$ and approximately $x = -2.8$ and $x = 2.8$

37. (a)



(b) rational
(c) vertical: $x = -1$, horizontal: $y = 2$
(d) no turning points

39. (a)



(b) piecewise
(c) no asymptotes
(d) turning point at $x = 0$

41. (a) 6800; 11,200 (b) $0 < x < 27$

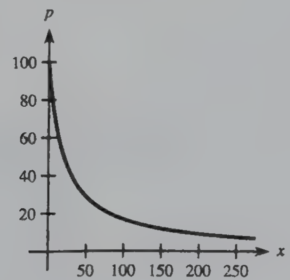
43. (a) $0 \leq p < 100$
(b) \$5972.73; to remove 45% of the particulate pollution would cost \$5972.73.
(c) \$65,700; to remove 90% of the particulate pollution would cost \$65,700.
(d) \$722,700; to remove 99% of the particulate pollution would cost \$722,700.
(e) \$1,817,700; to remove 99.6% of the particulate pollution would cost \$1,817,700.

45. (a) $A(2) = 96$; $A(30) = 600$
(b) $0 < x < 50$

47. (a) $C(5) = \$8.06$ (b) $C(6) = \$19.87$
(c) $C(3000) = \$227.65$

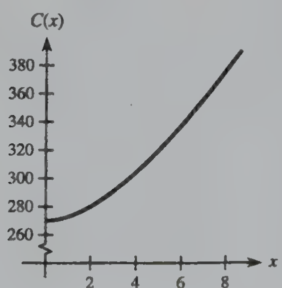
49. (a) $P(2.3) = 0.77$
The first-class postage for mailing 2.3 oz is 77¢.
(b) $p(1) = 0.33$ $p(1.01) = 0.55$

51. $p = \dfrac{200}{2 + 0.1x}$



53. $C(x) = 30(x - 1) + \dfrac{3000}{x + 10}$

(a)

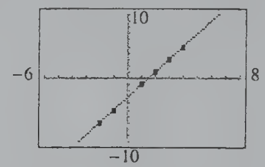


(b) Any turning point would indicate the minimum or the maximum cost. In this case, $x = 0$ gives a minimum.
(c) The y-intercept is the fixed cost of production.

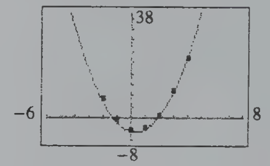
Exercise 2.5 (page 197)

1. linear 3. quadratic 5. quadratic 7. quadratic

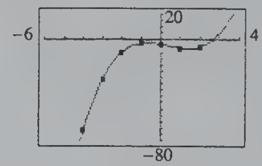
9. $y = 2x - 3$



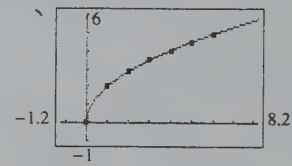
11. $y = 2x^2 - 1.5x - 4$



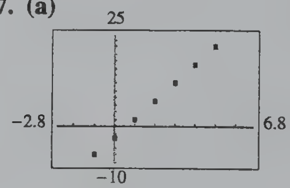
13. $y = x^3 - x^2 - 3x - 4$



15. $y = 2x^{0.5}$

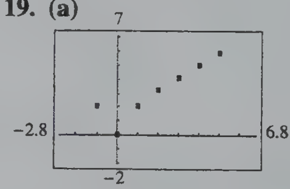


17. (a)



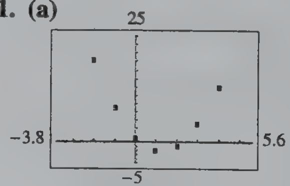
(b) linear
(c) $y = 5x - 3$

19. (a)



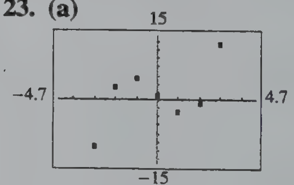
(b) quadratic
(c) $y = 0.09595x^2 + 0.4656x + 1.4758$

21. (a)



(b) quadratic
(c) $y = 2x^2 - 5x + 1$

23. (a)



(b) cubic
(c) $y = x^3 - 5x + 1$

25. (a) $y = 0.6305x + 27.0386$ (b) 0.6305
(c) For each additional dollar earned by men, \$0.63 will be earned by women.

27. (a) quadratic (b) $0.2912x^2 - 5.1350x + 84.4689$
(c) $x = 8.8$, during 1988; minimum is \$61.83 billion
(d) Predicted data are higher, but close to actual data.

29. (a) $-804.6429x^2 + 103{,}590.5286x + 320{,}812.9143$
(b) 1997 (c) 1994

31. (a) $y = -3.628x^2 + 652.1699x - 29{,}042.0332$
(b) $x = 98.39$, in 1998

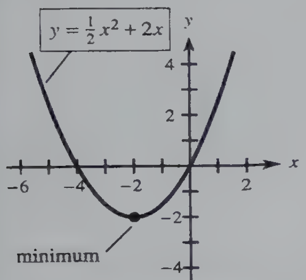
33. $y = 0.03743x^3 - 9.6275x^2 + 821.0664x - 23{,}169.1694$

35. (a) $y = 0.19x^2 - 16.59x + 1038.29$
(b) $x = 43.7$, 1944 (c) 1945

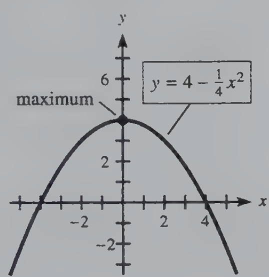
37. (a) $T(x) = 0.33x + 24.33$, $A(x) = 0.13x + 29.73$
(b) $x = 117.5$, during 2017
(c) $x = 108.1$, during 2008

Chapter 2 Review Exercises (page 203)

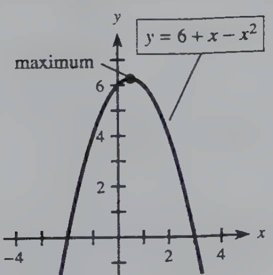
1. $x = 0, x = -\frac{5}{3}$ **2.** $x = 0, x = \frac{4}{3}$
3. $x = -2, x = -3$
4. $x = (-5 + \sqrt{47})/2, x = (-5 - \sqrt{47})/2$
5. no real solutions **6.** $x = \frac{\sqrt{3}}{2}, x = -\frac{\sqrt{3}}{2}$
7. $\frac{5}{7}, -\frac{4}{3}$ **8.** $(-1 + \sqrt{2})/4, (-1 - \sqrt{2})/4$
9. 7/2, 100 **10.** 13/5, 90
11. no real solutions **12.** $z = -9, z = 3$
13. $x = 8, x = -2$ **14.** $x = 3, x = -1$
15. $x = (-a \pm \sqrt{a^2 - 4b})/2$
16. $r = (2a \pm \sqrt{4a^2 + x^3c})/x$
17. 1.64, −7051.64 **18.** 0.41, −2.38
19. vertex (−2, −2); zeros (0, 0), (−4, 0)



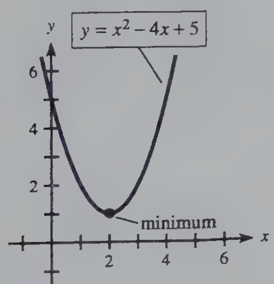
20. vertex (0, 4); zeros (4, 0), (−4, 0)



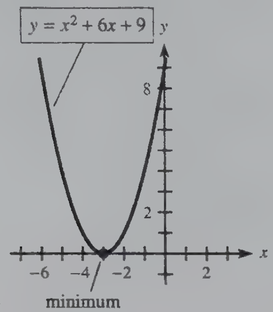
21. vertex $(\frac{1}{2}, \frac{25}{4})$; zeros (−2, 0), (3, 0)



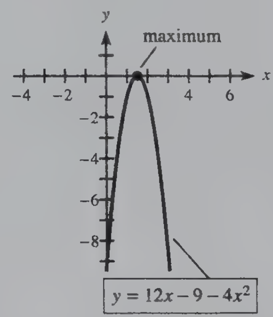
22. vertex (2, 1); no real zeros



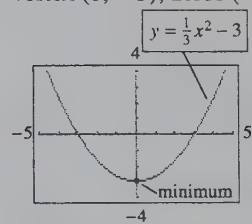
23. vertex (−3, 0); zero (−3, 0)



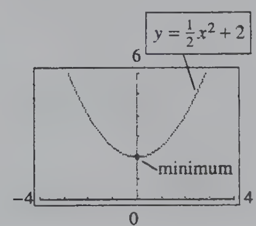
24. vertex $(\frac{3}{2}, 0)$; zero $(\frac{3}{2}, 0)$



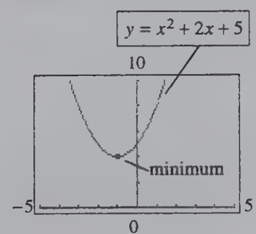
25. vertex (0, −3); zeros (−3, 0), (3, 0)



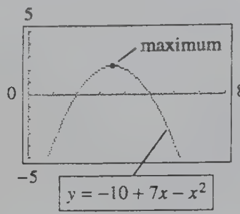
26. vertex (0, 2); no real zeros



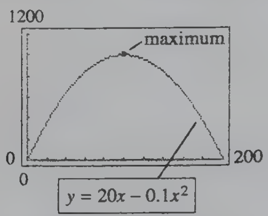
27. vertex (−1, 4); no real zeros



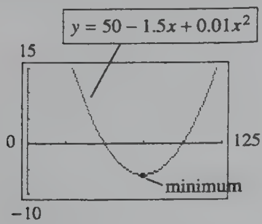
28. vertex $(\frac{7}{2}, \frac{9}{4})$; zeros $(2, 0)$, $(5, 0)$



29. vertex $(100, 1000)$; zeros $(0, 0)$, $(200, 0)$



30. vertex $(75, -6.25)$, zeros $(50, 0)$, $(100, 0)$



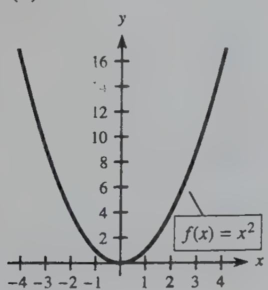
31. **(a)** $(1, -4\frac{1}{2})$ **(b)** $x = -2, x = 4$ **(c)** B

32. **(a)** $(0, 49)$ **(b)** $x = -7, x = 7$ **(c)** D

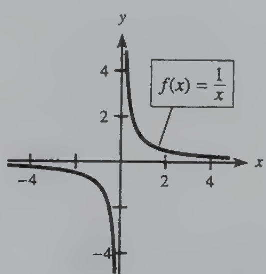
33. **(a)** $(7, 25)$ approximately, actual is $(7, 24\frac{1}{2})$
(b) $x = 0, x = 14$ **(c)** A

34. **(a)** $(-1, 9)$ **(b)** $x = -4, x = 2$ **(c)** C

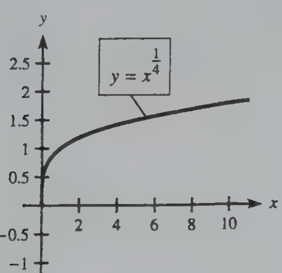
35. **(a)**



(b)



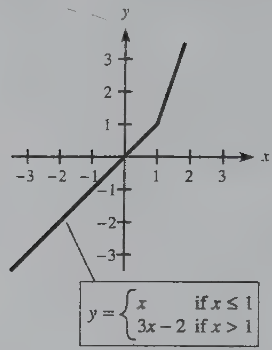
(c)



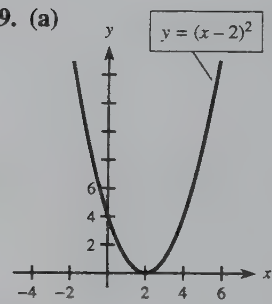
36. **(a)** 0 **(b)** 10,000 **(c)** −25 **(d)** 0.1

37. **(a)** −2 **(b)** 0 **(c)** 1 **(d)** 4

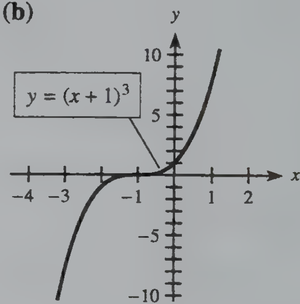
38.



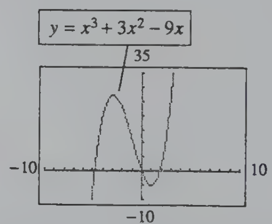
39. **(a)**



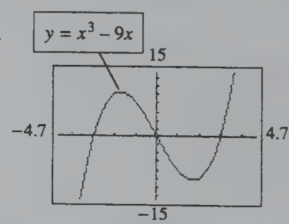
(b)



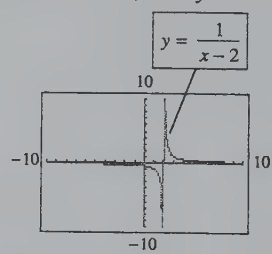
40. Turns: $(1, -5)$, $(-3, 27)$



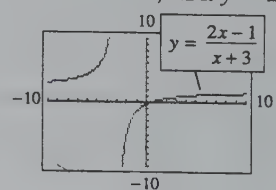
41. Turns: $(1.7, -10.4)$, $(-1.7, 10.4)$



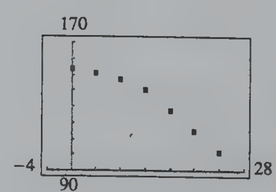
42. VA: $x = 2$; HA: $y = 0$



43. VA: $x = -3$; HA: $y = 2$

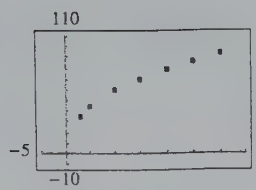


44. **(a)**



(b) $y = -2.1786x + 159.8571$
(c) $y = -0.0818x^2 - 0.2143x + 153.3095$

45. **(a)**



(b) $y = 2.1413x + 34.3913$
(c) $y = 22.2766x^{0.4259}$

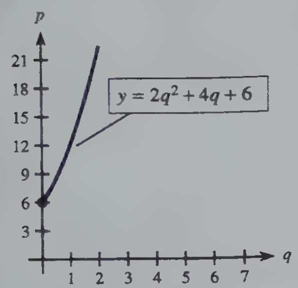
46. **(a)** $t = -1.65$ $t = 3.65$ **(b)** Just $t = 3.65$
(c) At 3.65 seconds

47. $x = 20, x = 800$

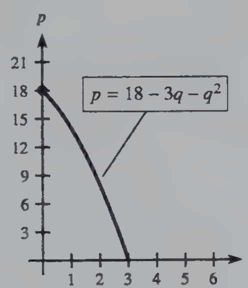
48. **(a)** $x = 76.9$, in 1977; $x = 108$, in 2008
(b) $x = 92.5$, in 1992; 9.15%

49. **(a)** $x = 200$ **(b)** $A = 30{,}000$ square feet

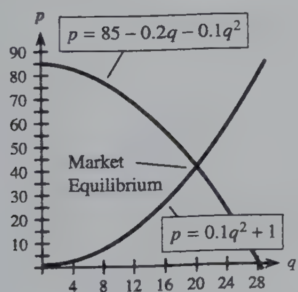
50.



51.



52. **(a)**



(b) $p = 41, q = 20$

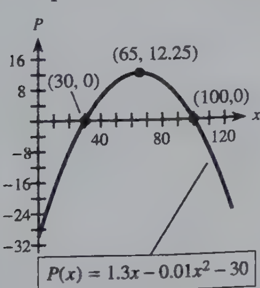
53. $p = 400, q = 10$ **54.** $p = 10, q = 20$

55. $x = 46 + 2\sqrt{89}, x = 46 - 2\sqrt{89}$

56. (15, 1275), (60, 2400)

57. max revenue = \$2500; max profit = \$506.25

58. max profit = 12.25; break-even $x = 100, x = 30$

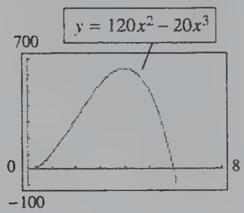


59. $x = 50, P(50) = 640$

60. **(a)** $C = 15{,}000 + 140x + 0.04x^2$; $R = 300x - 0.06x^2$
(b) 100, 1500 **(c)** 2500
(d) $P = 160x - 15{,}000 - 0.1x^2$; max at 800
(e) at 2500: $P = -240{,}000$; at 800: $P = 49{,}000$

61. **(a)** power **(b)** 5,599,885
(c) 20.481364; this model predicts 2,048,136 cases in 1995.

62. **(a)**

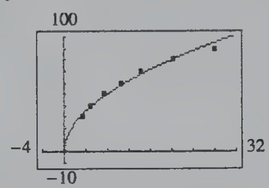


(b) $0 \le x \le 6$

63. **(a)** rational **(b)** $0 \le p < 100$
(c) 0; it costs \$0 to remove no pollution.
(d) \$475,200

64. **(a)** $x = 12$; $C(12) = \$18.68$
(b) $x = 825$; $C(825) = \$909.70$

65. **(a)** $y = 17.3969x^{0.5094}$
(b)

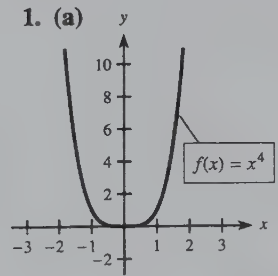


(c) 39.5 mph
(d) 18.3 seconds

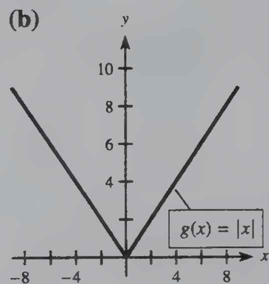
66. **(a)** $y = 1.8155x^2 - 11.1607x + 29.1845$
(b) Use $x = 11$; 126 million cubic yards

Chapter 2 Test *(page 206)*

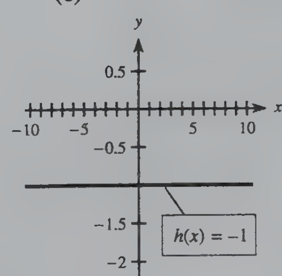
1. **(a)**



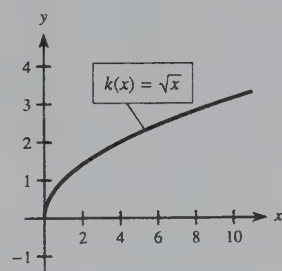
(b)



(c)



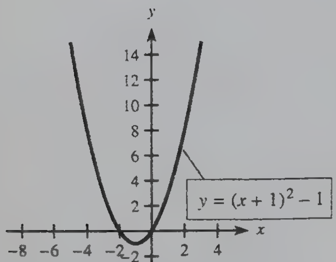
(d)



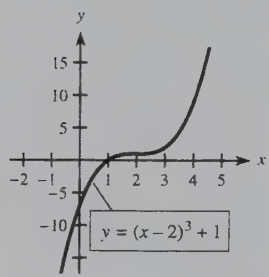
2. *b; a*

3.

4. (a)



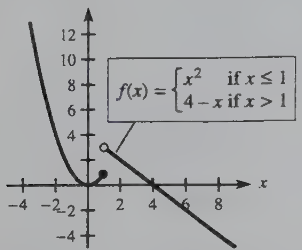
(b)



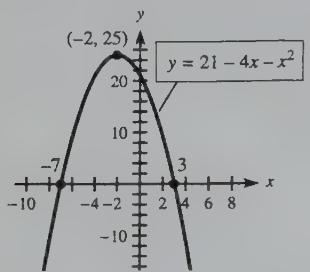
5. *b*; is cubic; $f(1) < 0$

6. (a) -10 (b) $-15\frac{1}{2}$ (c) -7

7.



8. vertex $(-2, 25)$; zeros $-7, 3$



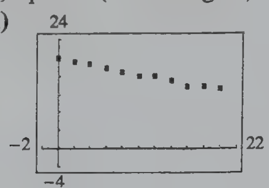
9. $x = 2, x = 1/3$

10. $x = \dfrac{-3 + 3\sqrt{3}}{2}, x = \dfrac{-3 - 3\sqrt{3}}{2}$

11. $x = 2/3$ 12. c

13. (a) quartic (fourth-degree) (b) cubic

14. (a)



Model: $y = -0.3577x + 19.9227$

(b) 5.6

(c) At $x = 55.7$

15. $q = 300, p = \$80$

16. (a) $P(x) = -x^2 + 250x - 15{,}000$

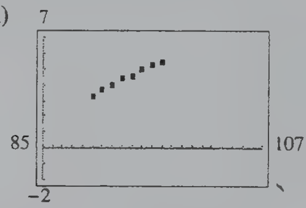
(b) 125 units, \$625

(c) 100 units, 150 units

17. (a) $f(15) = -31$ means that when the air temperature is 0°F and the wind speed is 15 mph, the air temperature feels like -31°F.

(b) -55°F

18. (a)



(b) $y = 0.3131x - 24.8619$

(c) No; the slope is positive.

(d) The predicted debt continues to increase.

19. (a) quadratic

(b) $y = -0.01607x^2 + 3.31845x - 165.27798$

(c) 5.59 in 1998; 5.75 in 1999

(d) $x = 103.2$ or after 2003

Exercise 3.1 *(page 222)*

1. 3 3. $\begin{bmatrix} -1 & -2 & -3 \\ 1 & 0 & -1 \\ -2 & 3 & 4 \end{bmatrix}$ 5. *A, C, D, F, Z*

7. *A, F,* and *Z* are 3×3; *C* and *D* are 2×2

9. 1 11. $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix}$ 13. no

15. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 17. $\begin{bmatrix} 9 & 5 \\ 4 & 7 \end{bmatrix}$ 19. $\begin{bmatrix} 0 & -2 & -1 \\ 4 & 2 & 0 \\ 2 & 3 & 7 \end{bmatrix}$

21. $\begin{bmatrix} 2 & 3 & 6 \\ 3 & 4 & 1 \\ 6 & 1 & 6 \end{bmatrix}$ 23. impossible 25. $\begin{bmatrix} 3 & 1 & 7 \\ 4 & 2 & 2 \\ 5 & 4 & 3 \end{bmatrix}$

27. $x = 3, y = 2, z = 3, w = 4$

29. $x = 4, y = 1, w = 3, z = 3$

31. $x = 2, y = 2, z = -3$

33. (a) $A = \begin{bmatrix} 55 & 74 & 14 & 7 & 65 \\ 9 & 16 & 19 & 5 & 40 \end{bmatrix}$

$B = \begin{bmatrix} 252 & 178 & 65 & 8 & 11 \\ 19 & 6 & 14 & 1 & 0 \end{bmatrix}$

(b) $A + B = \begin{bmatrix} 307 & 252 & 79 & 15 & 76 \\ 28 & 22 & 33 & 6 & 40 \end{bmatrix}$

(c) $\begin{bmatrix} 197 & 104 & 51 & 1 & -54 \\ 10 & -10 & -5 & -4 & -40 \end{bmatrix}$

more species in U.S.

35. $\begin{bmatrix} 11{,}041.7 & 8978.4 & 6461 \\ 8739.8 & 9159.6 & 6877.3 \\ 9798.1 & 9086.7 & 6448.4 \\ 9696.6 & 8926.7 & 6109.5 \end{bmatrix}$

37. **(a)** $\begin{bmatrix} 825 & 580 & 1560 \\ 810 & 650 & 350 \end{bmatrix}$

(b) $\begin{bmatrix} -75 & 20 & -140 \\ 10 & -50 & 50 \end{bmatrix}$

39. $A = \begin{array}{cc} \text{M} & \text{F} \\ \end{array}\begin{bmatrix} 54.4 & 55.6 \\ 62.1 & 66.6 \\ 67.4 & 74.1 \\ 70.7 & 78.1 \end{bmatrix}$ $B = \begin{array}{cc} \text{M} & \text{F} \\ \end{array}\begin{bmatrix} 45.5 & 45.2 \\ 51.5 & 54.9 \\ 61.1 & 67.4 \\ 65.3 & 73.6 \end{bmatrix}$

$A - B = C = \begin{bmatrix} 8.9 & 10.4 \\ 10.6 & 11.7 \\ 6.3 & 6.7 \\ 5.4 & 4.5 \end{bmatrix}$

41. **(a)** $\begin{array}{c|cc} & \text{wt} & \text{l} \\ \hline \text{I} & 140 & 5.5 \\ \text{II} & 151 & 5.7 \\ \text{III} & 141 & 5.5 \end{array}$ **(b)** $\begin{array}{c|cc} & \text{wt} & \text{l} \\ \hline \text{I} & 250 & 12.5 \\ \text{II} & 215 & 11.8 \\ \text{III} & 190 & 9.8 \end{array}$

43. **(a)** $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ **(b)** $B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

(c) person 2

45. **(a)** $\begin{bmatrix} 80 & 75 \\ 58 & 106 \end{bmatrix}$ **(b)** $\begin{bmatrix} 176 & 127 \\ 139 & 143 \end{bmatrix}$

(c) $\begin{bmatrix} 10 & 4 \\ 7 & 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} -10 & 19 \\ -7 & 20 \end{bmatrix}$ Shortage, taken from inventory.

47. **(a)** 3, 4, 5, 6 **(b)** 1

49. Worker 1: 0.9625 Worker 2: 0.9375
Worker 3: 0.9125 Worker 4: 0.8875
Worker 5: 0.85 Worker 6: 0.875
Worker 7: 0.90 Worker 8: 0.925
Worker 9: 0.95

Worker 5 is least efficient; at center 5

Exercise 3.2 (page 237)

1. [32] **3.** [11 17] **5.** $\begin{bmatrix} 3 & 0 & 6 \\ 9 & 6 & 3 \\ 12 & 0 & 9 \end{bmatrix}$

7. $\begin{bmatrix} 28 & 16 \\ 10 & 18 \end{bmatrix}$ **9.** impossible **11.** $\begin{bmatrix} 29 & 25 \\ 10 & 12 \end{bmatrix}$

13. $\begin{bmatrix} 22 & 16 \\ 20 & 19 \end{bmatrix}$ **15.** $\begin{bmatrix} 8 & -2 & 2 & 4 \\ 13 & -5 & 12 & -11 \end{bmatrix}$

17. $\begin{bmatrix} 7 & 5 & 3 & 2 \\ 14 & 9 & 11 & 3 \\ 13 & 10 & 12 & 3 \end{bmatrix}$ **19.** impossible

21. $\begin{bmatrix} 13 & 9 & 3 & 4 \\ 9 & 7 & 16 & 1 \end{bmatrix}$ **23.** $\begin{bmatrix} 9 & 7 & 16 \\ 5 & 17 & 20 \end{bmatrix}$

25. $\begin{bmatrix} 9 & 0 & 8 \\ 13 & 4 & 11 \\ 16 & 0 & 17 \end{bmatrix}$ **27.** $\begin{bmatrix} 161 & 126 \\ 42 & 35 \end{bmatrix}$ **29.** no

31. no **33.** $\begin{bmatrix} -55 & 88 & 0 \\ -42 & 67 & 0 \\ 28 & -44 & 1 \end{bmatrix}$ **35.** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

37. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **39.** A **41.** Z

43. no (see Problem 35)

45. $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

47. $\begin{bmatrix} 2-2+2 \\ 6-4-4 \\ 4+0-2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$; is solution

49. $\begin{bmatrix} 1+2+2 \\ 4+0+1 \\ 2+2+1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$; is solution

51. **(a)** $AB = \begin{bmatrix} 11 & 4 & 20 \\ -4 & -2 & -11 \\ 7 & 2 & 9 \end{bmatrix}$ $BA = \begin{bmatrix} 8 & -2 & 3 \\ 25 & -4 & 12 \\ 7 & 8 & 14 \end{bmatrix}$

(b) no

53. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (Some entries may appear as decimal approximations of 0.)

55. **(a)** $\begin{bmatrix} 292 & 275 \\ 451 & 403 \\ 550 & 453 \\ 582 & 464 \\ 514 & 403 \\ 389 & 353 \end{bmatrix}$ **(b)** $\begin{bmatrix} 350.40 & 330 \\ 541.20 & 483.60 \\ 660 & 543.60 \\ 698.40 & 556.80 \\ 616.80 & 483.60 \\ 466.80 & 423.60 \end{bmatrix}$

57. **(a)** $\begin{bmatrix} 23.10 & 42.00 & 105.00 & 5.25 \\ 21.00 & 42.00 & 21.00 & 0.00 \\ 29.40 & 73.50 & 47.25 & 0.00 \\ 15.75 & 73.50 & 21.00 & 10.50 \\ 21.00 & 0.00 & 105.00 & 5.25 \end{bmatrix}$

(b) $\begin{bmatrix} 24.20 & 44.00 & 110.00 & 5.50 \\ 22.00 & 44.00 & 22.00 & 0.00 \\ 30.80 & 77.00 & 49.50 & 0.00 \\ 16.50 & 77.00 & 22.00 & 11.00 \\ 22.00 & 0.00 & 110.00 & 5.50 \end{bmatrix}$

59. $\begin{bmatrix} 22{,}000 & 24{,}640 \\ 30{,}600 & 35{,}700 \end{bmatrix}$

61. **(a)** $\begin{bmatrix} 2703 & 3428 \\ 3608 & 2481 \\ 2383 & 2314 \\ 4376 & 4584 \\ 679 & 664 \\ 472 & 477 \\ 1443 & 1523 \\ 3152 & 2988 \\ 1289 & 1224 \\ 254 & 257 \\ 916 & 949 \\ 1455 & 1413 \end{bmatrix}$ $\begin{bmatrix} 96.1 & 97.9 & 48.1 & 79.4 & 65.4 & 98.4 & 99.0 & 77.3 & 65.8 & 82.8 & 89.5 & 76.9 \\ 109.0 & 105.0 & 63.3 & 91.1 & 79.2 & 96.4 & 121.0 & 87.5 & 85.4 & 91.3 & 95.3 & 76.4 \end{bmatrix}$

(b) 12×12 **(c)** the diagonal entries

63. 1300 teens, 1520 single, 1620 married

65. 172, 208, 268, 327, 101, 123, 268, 327, 216, 263, 162, 195, 176, 215, 343, 417

67. $\begin{bmatrix} 2137 & 84 & 41.5 & 128.5 & 158.5 & 317 & 738 \\ 1285.5 & 64 & 30.5 & 115 & 136 & 229 & 590 \\ 969.5 & 46.5 & 24 & 98 & 139 & 224 & 476.5 \\ 852.5 & 45 & 23 & 97.5 & 142.5 & 236.5 & 463 \\ 809 & 44.5 & 22.5 & 99 & 141.5 & 240.5 & 455.5 \end{bmatrix}$

69. **(a)** $\begin{bmatrix} 0.665 & 8.075 & 9.690 & 1.045 & 5.320 & 3.420 \\ 0.475 & 0.190 & 5.795 & 1.235 & 0.190 & 0.950 \\ 2.090 & 0.380 & 8.360 & 1.140 & 1.140 & 4.560 \\ 239.210 & 60.230 & 77.520 & 33.440 & 51.585 & 136.990 \\ 28.500 & 0.950 & 0.950 & 0.950 & 0.950 & 0.950 \\ 749.455 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0.7560 & 9.1800 & 11.0160 & 1.1880 & 6.0480 & 3.8880 \\ 0.5400 & 0.2160 & 6.5880 & 1.4040 & 0.2160 & 1.0800 \\ 2.3760 & 0.4320 & 9.5040 & 1.2960 & 1.2960 & 5.1840 \\ 271.9440 & 68.4720 & 88.1280 & 38.0160 & 58.6440 & 155.7360 \\ 32.4000 & 1.0800 & 1.0800 & 1.0800 & 1.0800 & 1.0800 \\ 852.0121 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

71. $\begin{bmatrix} 0.900 & 0.855 & 0.855 & 0.855 & 0.855 & 0.765 & 0.765 & 0.765 & 0.765 \\ 0.765 & 0.900 & 0.855 & 0.855 & 0.855 & 0.855 & 0.765 & 0.765 & 0.765 \\ 0.765 & 0.765 & 0.900 & 0.855 & 0.855 & 0.855 & 0.855 & 0.765 & 0.765 \\ 0.765 & 0.765 & 0.765 & 0.900 & 0.855 & 0.855 & 0.855 & 0.855 & 0.765 \\ 0.765 & 0.765 & 0.765 & 0.765 & 0.900 & 0.855 & 0.855 & 0.855 & 0.855 \\ 0.855 & 0.765 & 0.765 & 0.765 & 0.765 & 0.900 & 0.855 & 0.855 & 0.855 \\ 0.855 & 0.855 & 0.765 & 0.765 & 0.765 & 0.765 & 0.900 & 0.855 & 0.855 \\ 0.855 & 0.855 & 0.855 & 0.765 & 0.765 & 0.765 & 0.765 & 0.900 & 0.855 \\ 0.855 & 0.855 & 0.855 & 0.855 & 0.765 & 0.765 & 0.765 & 0.765 & 0.900 \end{bmatrix}$

Exercise 3.3 (page 254)

1. $\left[\begin{array}{ccc|c} -1 & 2 & 1 & 7 \\ 0 & 7 & 5 & 21 \\ 4 & 2 & 2 & 1 \end{array}\right]$ **3.** $\left[\begin{array}{ccc|c} 3 & 2 & 4 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & -2 & -4 & 0 \end{array}\right]$

5. $\left[\begin{array}{ccc|c} 1 & -3 & 4 & 2 \\ 2 & 0 & 2 & 1 \\ 1 & 2 & 1 & 1 \end{array}\right]$ **7.** $x = 2, y = 1/2, z = -5$

9. $x = 18, y = 10, z = 0$ **11.** $x = -5, y = 2, z = 1$

13. $x = 4, y = 1, z = -2$ **15.** $x = \frac{1}{3}, y = \frac{5}{3}$

17. $x = 15, y = -13, z = 2$

19. $x = 15, y = 0, z = 2$ **21.** $x = 1, y = -1, z = 1$

23. $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0$

25. $x = 1, y = 3, z = 1, w = 0$

27. no solution **29.** $x = (11 + 2z)/3, y = (-1 - z)/3$

31. $x = 0, y = -z$ **33.** $x = 3z - 2, y = 3 - 5z$

35. $x = 1 - z, y = \frac{1}{2}z$ 37. $x = 2z - 2, y = 1 + z$
39. $x = \frac{9}{2}, y = -\frac{1}{2}, z = -1$ 41. $x_1 = \frac{23}{38}x_3, x_2 = \frac{12}{19}x_3$
43. $x = 7/5, y = -3/5, z = w$ 45. no solution
47. $x_1 = 0.5 - 2x_4, x_2 = 3.5 + 5x_4, x_3 = -2.5 - 3x_4$
49. $x = (b_2c_1 - b_1c_2)/(a_1b_2 - a_2b_1)$
51. Beef: 2 cups; sirloin: 8 cups
53. AB: 2 oz, SFF: 2 oz, NMG: 1 oz
55. 2 of Portfolio I, 2 of Portfolio II
57. $\frac{3}{8}$ pound of red meat, 6 slices of bread, 4 glasses of milk
59. \$ 13,500 @ 15% and \$ 10,000 @ 16%
61. Type I = 3 Type IV, Type II = 1000 − 2(Type IV), Type III = 500 − Type IV, Type IV = any amount (less than 500 bags)
63. Oil = $6138 - 3.3C - 0.6R$,
Bank = $7.5C + R - 12{,}787.50$; R = any amount (less than 10,230); C = any amount (less than 1705)
65. Bacteria III = any amount (between 1800 and 2300)
Bacteria I = 6900 − 3 (Bacteria III)
Bacteria II = $\frac{1}{2}$ (Bacteria III) − 900
67. There are three possibilities:
(1) 4 of I and 2 of II
(2) 5 of I, 1 of II, and 1 of III
(3) 6 of I and 2 of III

Exercise 3.4 (page 269)

1. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 3. yes 5. $\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

7. $\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$ 9. no inverse 11. $\begin{bmatrix} \frac{5}{2} & -1 \\ -2 & 1 \end{bmatrix}$

13. $\begin{bmatrix} -\frac{1}{10} & \frac{7}{10} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$ 15. $\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

17. $\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{7}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{5}{3} \end{bmatrix}$ 19. no inverse
21. no inverse

23. $\begin{bmatrix} 2 & 2 & 0 & 2 & 2 \\ 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 2 & 0 & 2 & 2 & 1 \\ 1 & 0 & 0 & 0 & 2 \end{bmatrix}$ 25. $\begin{bmatrix} 13 \\ 5 \end{bmatrix}$ 27. $\begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}$

29. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ 31. $x = 2, y = 1$

33. $x = 1, y = 2$ 35. $x = 1, y = 1, z = 1$
37. $x = 1, y = 1, z = 1$ 39. $x = 1, y = 3, z = 2$
41. $x_1 = 5.6, x_2 = 5.4, x_3 = 3.25, x_4 = 6.1, x_5 = 0.4$
43. −2 45. 14 47. −5 49. −19 51. yes
53. no 55. Hang on 57. Answers in back

59. $x_0 = 2400, y_0 = 1200$
61. A = 5.5 mg and B = 8.8 mg for Patient I;
A = 10 mg and B = 16 mg for Patient II
63. \$68,000 at 18%, \$77,600 at 10%
65. 7 cc of 20%, 3 cc of 5%
67. (a) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (b) 108

Exercise 3.5 (page 282)

1. (a) 15 (b) 4 3. 8 5. 40
7. most: raw materials; least: fuels
9. raw materials, manufacturing, service
11. 5 units of manufacturing; 10 units of ag; 105 units of utilities
13. farm products = 200; machinery = 40
15. agricultural products = 100; oil = 700
17. utilities = 200; manufacturing = 400
19. mining = 310; manufacturing = 530
21. electronic components = 1200; computers = 320
23. fishing = 100; oil = 1250
25. development = \$21,000; promotional = \$12,000
27. engineering = \$15,000; computer = \$13,000
29. agricultural goods = 400; manufactured goods = 500; fuels = 400
31. electronics = 1240; steel = 1260; autos = 720
33. products = $\frac{7}{17}$ households; machinery = $\frac{1}{17}$ household
35. government = $\frac{10}{19}$ households;
industry = $\frac{11}{19}$ households
37. manufacturing = 3 households;
utilities = 3 households

39. $\begin{bmatrix} 24 \\ 96 \\ 24 \\ 120 \\ 492 \\ 3456 \end{bmatrix}$ 3456 bolts, 492 braces, 120 sheets

41. $\begin{bmatrix} 10 \\ 10 \\ 20 \\ 56 \\ 20 \\ 26 \\ 300 \end{bmatrix}$ 56 2 × 4's, 20 braces, 26 clamps, 300 nails

Chapter 3 Review Exercises (page 287)

1. 4 2. 0 3. A, B 4. none

5. D, F, G, I 6. $\begin{bmatrix} -2 & 5 & 11 & -8 \\ -4 & 0 & 0 & -4 \\ 2 & 2 & -1 & -9 \end{bmatrix}$

7. Zero matrix 8. Order

9. $\begin{bmatrix} 6 & -1 & -9 & 3 \\ 10 & 3 & -1 & 4 \\ -2 & -2 & -2 & 14 \end{bmatrix}$ 10. $\begin{bmatrix} 3 & -3 \\ 4 & -1 \\ 2 & -6 \\ 1 & -2 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix}$ 12. $\begin{bmatrix} 12 & -6 \\ 15 & 0 \\ 18 & 0 \\ 3 & 9 \end{bmatrix}$ 13. $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

14. $\begin{bmatrix} 2 & -12 \\ -8 & -22 \end{bmatrix}$ 15. $\begin{bmatrix} 9 & 20 \\ 4 & 5 \end{bmatrix}$ 16. $\begin{bmatrix} 5 & 16 \\ 6 & 15 \end{bmatrix}$

17. $\begin{bmatrix} 2 & 37 & 61 & -55 \\ -2 & 9 & -3 & -20 \\ 10 & 10 & -14 & -30 \end{bmatrix}$ 18. $\begin{bmatrix} 43 & -23 \\ 33 & -12 \\ -13 & 15 \end{bmatrix}$

19. $\begin{bmatrix} 10 & 16 \\ 15 & 25 \\ 18 & 30 \\ 6 & 11 \end{bmatrix}$ 20. $\begin{bmatrix} 17 & 73 \\ 7 & 28 \end{bmatrix}$ 21. $\begin{bmatrix} 3 & 7 \\ 23 & 42 \end{bmatrix}$

22. F 23. F 24. $\begin{bmatrix} -19 & 12 \\ -8 & 5 \end{bmatrix}$

25. F 26. $(1, 2, 1)$

27. $x = 22, y = 9$ 28. $x = -3, y = 3, z = 4$

29. $x = -\frac{3}{2}, y = 7, z = -\frac{11}{2}$ 30. no solution

31. $x = 2 - 2z; y = -1 - 2z$

32. $x_1 = 1, x_2 = 11, x_3 = -4, x_4 = -5$

33. yes 34. $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{5}{2} & \frac{7}{4} \end{bmatrix}$ 35. $\begin{bmatrix} -1 & -2 & 8 \\ 1 & 2 & -7 \\ 1 & 1 & -4 \end{bmatrix}$

36. $\begin{bmatrix} 2 & 1 & -2 \\ 7 & 5 & -8 \\ -13 & -9 & 15 \end{bmatrix}$ 37. $\begin{bmatrix} -33 \\ 30 \\ 19 \end{bmatrix}$ 38. $\begin{bmatrix} 4 \\ 5 \\ -13 \end{bmatrix}$

39. $A^{-1} = \begin{bmatrix} -41 & 32 & 5 \\ 17 & -13 & -2 \\ -9 & 7 & 1 \end{bmatrix}; x = 4, y = -2, z = 2$

40. No 41. $\begin{bmatrix} 250 & 140 \\ 480 & 700 \end{bmatrix}$ 42. $\begin{bmatrix} 1030 & 800 \\ 700 & 1200 \end{bmatrix}$

43. **(a)** higher in June **(b)** higher in July

44. $\begin{array}{cc} \text{men} & \text{women} \\ \begin{bmatrix} 865 & 885 \\ 210 & 270 \end{bmatrix} & \begin{array}{l} \text{Robes} \\ \text{Hoods} \end{array} \end{array}$ 45. $\begin{bmatrix} 1750 \\ 480 \end{bmatrix} \begin{array}{l} \text{Robes} \\ \text{Hoods} \end{array}$

46. **(a)** $\begin{bmatrix} 13{,}500 & 12{,}400 \\ 10{,}500 & 10{,}600 \end{bmatrix}$
(b) Dept. A buys from Kink; Dept. B buys from Ace

47. **(a)** [0.20 0.30 0.50] **(b)** $\begin{bmatrix} 0.013469 \\ 0.013543 \\ 0.006504 \end{bmatrix}$

(c) $[0.20\ 0.30\ 0.50] \begin{bmatrix} 0.013469 \\ 0.013543 \\ 0.006504 \end{bmatrix} = 0.20(0.013469) +$
$0.30(0.013543) + 0.50(0.006504) = 0.0100087$

(d) The historical return of the portfolio, 0.0100087, is the estimated expected monthly return of the portfolio. This is roughly 1% per month.

48. 400 fast food, 700 computer, 200 pharmaceutical

49. $A = 2C, b = 2000 - 4C$

50. 3 passenger, 4 transport, 4 jumbo

51. $S = 6000; A = 800$ 52. $S = 1000; C = 500$

53. $X = \begin{bmatrix} 366 \\ 322 \\ 402 \end{bmatrix}$, approximately

54. $G = \frac{64}{93}H; A = \frac{59}{93}H; M = \frac{40}{93}H$

Chapter 3 Test (page 290)

1. $\begin{bmatrix} 3 & 1 & 5 \\ 1 & 3 & 6 \end{bmatrix}$ 2. $\begin{bmatrix} -1 & 2 & 2 \\ 1 & -1 & 6 \end{bmatrix}$

3. $\begin{bmatrix} -12 & -16 & -155 \\ 5 & 12 & 87 \end{bmatrix}$ 4. $\begin{bmatrix} 23 & 6 \\ 182 & 45 \\ 21 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 0 & -7 \\ 26 & 1 \end{bmatrix}$ 6. $\begin{bmatrix} -43 & -46 & -207 \\ 39 & 30 & -77 \\ 17 & 5 & -216 \end{bmatrix}$

7. $\begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$ 8. $\begin{bmatrix} -3 & 2 & 2 \\ 1 & 0 & -1 \\ 1/2 & -1/2 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 5 \\ 14 \\ 15 \end{bmatrix}$ 10. $x = -0.5, y = 0.5, z = 2.5$

11. $x = 4 - 1.8z, y = 0.2z$ 12. no solution

13. $x = 2, y = 2, z = 0, w = -2$

14. $x = 6w - 0.5, y = 0.5 - w, z = 2.5 - 3w$

15. **(a)** $\begin{bmatrix} .08 & .22 & .12 \\ .10 & .08 & .19 \\ .05 & .07 & .09 \\ .10 & .26 & .15 \\ .12 & .04 & .24 \end{bmatrix}$

(b) 0.08, 0.22, 0.12 consumed by carnivores 1, 2, 3
(c) Plant 5 by 1, Plant 4 by 2, Plant 5 by 3

16. **(a)** [1000 4000 2000 1000]
(b) [45,000 55,000 90,000 70,000]

(c) $\begin{bmatrix} 5 \\ 3 \\ 4 \\ 4 \end{bmatrix}$ **(d)** [$1,030,000] **(e)** $\begin{array}{cc} & \$ \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 65 \\ 145 \\ 125 \\ 135 \end{bmatrix} \end{array}$

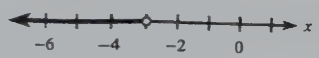
17. Growth, 2000; blue-chip, 400, utility, 400
18. Ag: 315, M: 245
19. Profit = Households
Non-profit = $\frac{2}{3}$ Households
20.

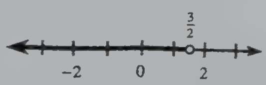
$$\begin{array}{c} \\ \text{Ag} \\ \text{M} \\ \text{F} \\ \text{S} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Ag} & \text{M} & \text{F} & \text{S} \end{array} \\ \begin{bmatrix} 0.2 & 0.1 & 0.1 & 0.1 \\ 0.3 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.3 & 0.3 \\ 0.1 & 0.4 & 0.2 & 0.2 \end{bmatrix} \end{array}$$

21. Ag: 5000, M: 8000, S: 7000
22. Ag: $\frac{520}{699}$ Hshds, S: $\frac{236}{233}$ Hshds, F: $\frac{159}{233}$ Hshds

Exercise 4.1 (page 302)

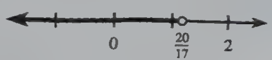
1. $x < -4$ 3. $x < 2$ 5. $x < -4$
7. $x \leq -1$ 9. $x \geq 1.949$
11. $x < 3$ 

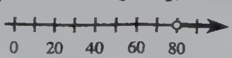
13. $x < -3$ 

15. $x < \frac{3}{2}$ 

17. $x < -6$ 

19. $x < 2$ 

21. $x < \frac{20}{17}$ 

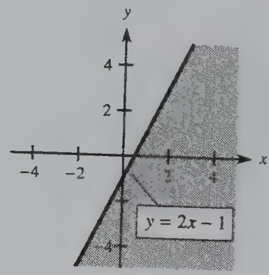
23. $-\frac{1}{2} < x \leq 3$ 25. $1 < x < 4$
27. $x > 4$ 29. $-50 \leq x < -22$
31. $(1, 3]$, half-open
33. $(2, 10)$, open 35. $[-3, 2]$, closed
37. $(-4, 3)$, open 39. $[4, 6]$, closed
41. $x > 80$ 

43. $695 + 5.75x \leq 900$; 35 or fewer
45. (a) $0 \leq I \leq 25{,}350$
$25{,}351 \leq I \leq 61{,}400$
$61{,}401 \leq I \leq 128{,}100$
$128{,}101 \leq I \leq 278{,}450$
$I \geq 278{,}451$
(b) $0 \leq T \leq 3802.50$
$3802.50 < T \leq 13{,}896.50$
$13{,}896.50 < T \leq 34{,}573.50$
$34{,}573.50 < T \leq 88{,}699.50$
$T > 88{,}699.50$
47. (a) $0.479 \leq h \leq 1$; $h = 1$ means 100% humidity
(b) $0 \leq h \leq 0.237$
49. $C \geq 37$

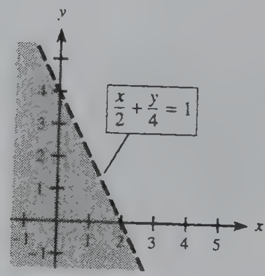
51. (a) (1) $x_4 \leq 1100$, (2) $x_4 \leq 1000$,
(3) $x_4 \leq 900$, (4) $x_4 \geq 0$
(b) $0 \leq x_4 \leq 900$
(c) $x_1 \in [100, 1000]$, $x_2 \in [200, 1100]$,
$x_3 \in [300, 1200]$

Exercise 4.2 (page 310)

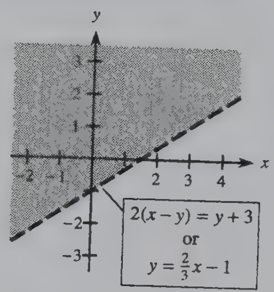
1.



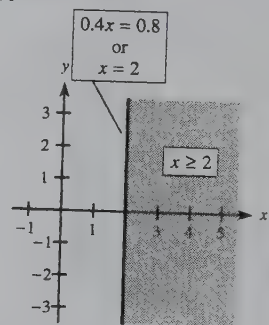
3.



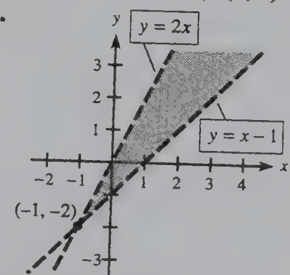
5.



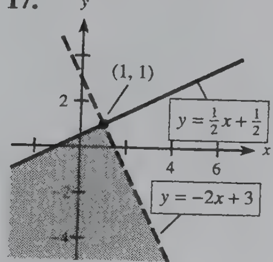
7.



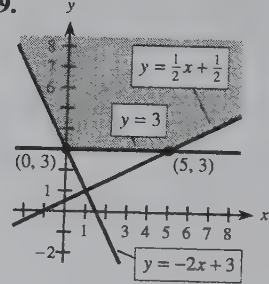
9. (0, 0), (20, 10), (0, 15), (25, 0)
11. (0, 0), (7, 0), (5, 5), (2, 6), (0, 4)
13. (0, 5), (1, 2), (3, 1), (6, 0)
15.



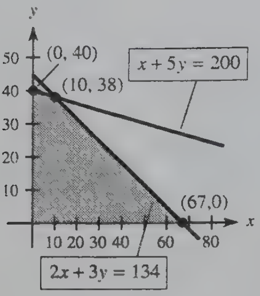
17.



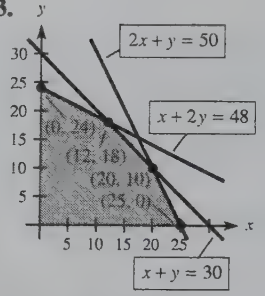
19.



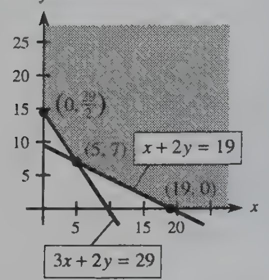
21.



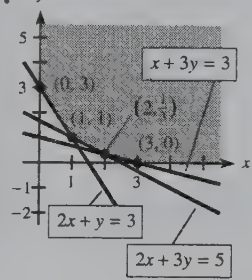
23.



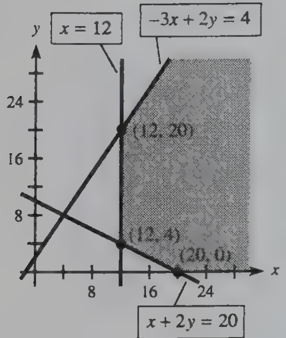
25.



27.

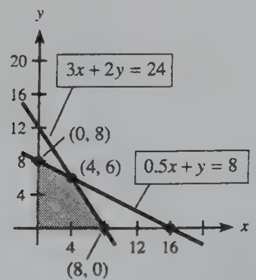


29.



31. (a) Let x = deluxe model
and y = economy model
$3x + 2y \le 24$
$\frac{1}{2}x + y \le 8$
$x \ge 0, y \ge 0$

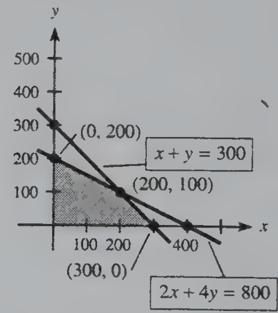
(b)



Corners: (0, 0), (8, 0), (4, 6), (0, 8)

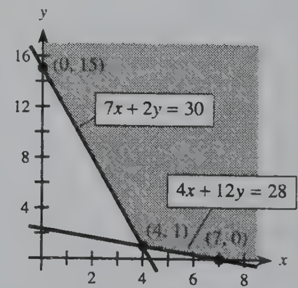
33. Let x = cord-type trimmer
and y = cordless trimmer.
(a) Constraints are
$x + y \le 300$
$2x + 4y \le 800$
$x \ge 0, y \ge 0$

(b)



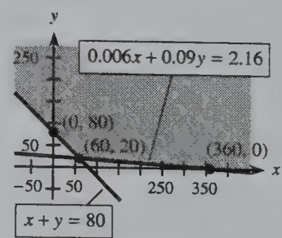
35. (a) $7x + 2y \ge 30$
$4x + 12y \ge 28$
$x \ge 0, y \ge 0$

(b)



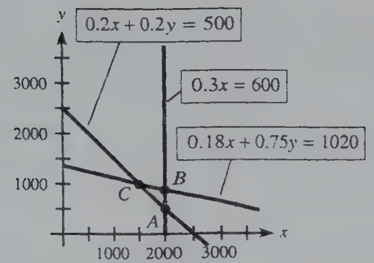
37. Let x = minutes of radio
and y = minutes of television
(a) Constraints are
$x + y \ge 80$
$0.006x + 0.09y \ge 2.16$
$x \ge 0, y \ge 0$

(b)



39. (a) x = pounds of regular
y = pounds of all beef
$0.18x + 0.75y \le 1020$
$0.2x + 0.2y \ge 500$
$0.3x \qquad \le 600$

(b)



$A = (2000, 500)$
$B = (2000, 880)$
$C = (1500, 1000)$

Exercise 4.3 (page 322)

1. (12, 0) 3. max = 20 at (4, 4); min = 0 at (0, 0)
5. max = 68 at (12, 4); min = 14 at (2, 2)
7. no max; min = 10 at (2, 1)
9. (0, 0), (0, 20), (10, 18), (15, 10), (20, 0); max = 66 at (10, 18)
11. (0, 60), (10, 30), (20, 20), (70, 0); min = 90 at (10, 30)
13. max = 382 at $x = 10$, $y = 38$
15. max = 80 at $x = 20$, $y = 10$
17. min = 115 at $x = 5$, $y = 7$
19. min = 36 at $x = 3$, $y = 0$
21. max = 6 at (0, 2)
23. max = 22 at (2, 4)
25. max = 30 on line between (0, 5) and (3, 4)
27. min = 32 at (2, 3) 29. min = 9 at (2, 3)
31. min = 75 at (15, 15)
33. max = 10 at (2, 4)
35. min = 3100 at (40, 60)
37. max = \$132 at (4, 6)
39. max = \$10,000 at 200 cord-type, 100 cordless
41. 250 fish: 150 bass and 100 trout
43. R = \$366,000 with 6 satellite and 17 full-service branches
45. radio = 60, TV = 20, C = \$16,000
47. 30 days for Factory 1 and 20 days for Factory 2; cost = \$700,000
49. 60 days for Location I and 70 days for Location II; cost = \$86,000
51. reg = 2000 lb; beef = 880 lb; profit = \$1328
53. From Pittsburgh: 20 to Blairsville, 40 to Youngstown
From Erie: 15 to Blairsville, 0 to Youngstown
Minimum cost = \$1550

Exercise 4.4 (page 342)

1. $3x + 5y + s_1 = 15$, $3x + 6y + s_2 = 20$

3. $\left[\begin{array}{ccccc|c} 2 & 5 & 1 & 0 & 0 & 30 \\ 1 & 5 & 0 & 1 & 0 & 25 \\ \hline -2 & -4 & 0 & 0 & 1 & 0 \end{array}\right]$

5. $\left[\begin{array}{ccccc|c} 1 & 5 & 1 & 0 & 0 & 200 \\ 2 & 3 & 0 & 1 & 0 & 134 \\ \hline -4 & -9 & 0 & 0 & 1 & 0 \end{array}\right]$

7. $\left[\begin{array}{ccccccc|c} 2 & 7 & 9 & 1 & 0 & 0 & 0 & 100 \\ 6 & 5 & 1 & 0 & 1 & 0 & 0 & 145 \\ 1 & 2 & 7 & 0 & 0 & 1 & 0 & 90 \\ \hline -2 & -5 & -2 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$

9. $\left[\begin{array}{ccccc|c} 2 & 4 & 1 & 0 & 0 & 24 \\ 1 & 1 & 0 & 1 & 0 & 5 \\ \hline -4 & -11 & 0 & 0 & 1 & 0 \end{array}\right]$

11. $\left[\begin{array}{cccccc|c} 10 & 27 & 1 & 0 & 0 & 0 & 200 \\ 4 & 51 & 0 & 1 & 0 & 0 & 400 \\ 15 & 27 & 0 & 0 & 1 & 0 & 350 \\ \hline -6 & -7 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$

13. $\left[\begin{array}{ccccc|c} 2 & 0 & 1 & -\frac{3}{4} & 0 & 12 \\ 3 & 1 & 0 & \frac{1}{3} & 0 & 15 \\ \hline -4 & 0 & 0 & 3 & 1 & 15 \end{array}\right]$

15. Solution is complete

17. $\left[\begin{array}{ccccccc|c} 4 & 4 & 1 & 0 & 0 & 2 & 0 & 12 \\ 2 & 4 & 0 & 1 & 0 & -1 & 0 & 4 \\ -3 & -11 & 0 & 0 & 1 & -1 & 0 & 6 \\ \hline -3 & -3 & 0 & 0 & 0 & 4 & 1 & 150 \end{array}\right]$

Either circled number may act as the next pivot, but only one of them.

19. No solution is possible. 21. $x = 11, y = 9; f = 20$
23. $x = 0, y = 14, z = 11; f = 525$
25. $x = 50, y = 10, z = 0, f = 100$. Multiple solutions are possible.
Next pivot is circled.

$\left[\begin{array}{cccccc|c} 1 & 0 & 3 & 0 & 6 & 0 & 50 \\ 0 & 0 & 4 & 1 & -4 & 0 & 6 \\ 0 & 1 & -2 & 0 & 2 & 0 & 10 \\ \hline 0 & 0 & 9 & 0 & 0 & 1 & 100 \end{array}\right]$

27. $x = 0, y = 5, f = 50$ 29. $x = 4, y = 3, f = 17$
31. $x = 2, y = 5, f = 17$ 33. $x = 4, y = 3, f = 11$
35. $x = 0, y = 2, z = 5, f = 40$
37. $x = 15, y = 15, z = 25, f = 780$
39. $x = 6, y = 2, z = 26, f = 206$
41. $x = 8, y = 16, f = 32$ 43. no solution
45. $x = 0, y = 50$ or $x = 40, y = 40, f = 600$

47. (a) $\left[\begin{array}{rrrrr|r} 1 & 1 & 1 & 0 & 0 & 60 \\ 1 & 3 & 0 & 1 & 0 & 120 \\ \hline -40 & -60 & 0 & 0 & 1 & 0 \end{array}\right]$

(b) Max profit is \$3000 with 30 ink jet and 30 laser printers

49. (a) $\left[\begin{array}{rrrrr|r} 4 & 3 & 1 & 0 & 0 & 50 \\ 3 & 5 & 0 & 1 & 0 & 43 \\ \hline -300 & -300 & 0 & 0 & 1 & 0 \end{array}\right]$

(b) Max profit is \$3900 with 11 axles and 2 wheels

51. 500 tomatoes, 1800 peaches; $P = 4100$

53. 21 newspaper, 13 radio

55. Medium 1 = 10, Medium 2 = 10, Medium 3 = 12

57. \$1650 profit with 46A, 20B, 6C

59. 8000 Regular, 0 Special, and 1000 Kitchen Magic; \$32,000

Exercise 4.5 (page 353)

1. (a) $\left[\begin{array}{rr|r} 5 & 2 & 10 \\ 1 & 2 & 6 \\ \hline 3 & 1 & g \end{array}\right]$ transpose $= \left[\begin{array}{rr|r} 5 & 1 & 3 \\ 2 & 2 & 1 \\ \hline 10 & 6 & g \end{array}\right]$

(b) Maximize $f = 10x + 6y$ subject to $5x + y \leq 3$, $2x + 2y \leq 1$, $x \geq 0$, $y \geq 0$.

3. (a) $\left[\begin{array}{rr|r} 1 & 1 & 9 \\ 1 & 3 & 15 \\ \hline 5 & 2 & g \end{array}\right]$ transpose $= \left[\begin{array}{rr|r} 1 & 1 & 5 \\ 1 & 3 & 2 \\ \hline 9 & 15 & g \end{array}\right]$

(b) Maximize $f = 9x_1 + 15x_2$ subject to
$x_1 + x_2 \leq 5$
$x_1 + 3x_2 \leq 2$

5. (a) $y_1 = 8, y_2 = 2, y_3 = 0$; min: $g = 252$
(b) $x_1 = 5, x_2 = 0, x_3 = 9$; max $f = 252$

7. $g = 5$ at $y_1 = 0, y_2 = 5$; $f = 5$ at $x_1 = 1/2, x_2 = 0$

9. $y_1 = 0, y_2 = 9$; $g = 18$ (min); $x_1 = 2, x_2 = 0$; $f = 18$ (max)

11. Maximize $f = 11x_1 + 12x_2 + 6x_3$
subject to $4x_1 + 3x_2 + 3x_3 \leq 3$
$x_1 + 2x_2 + x_3 \leq 1$
Primal: $y_1 = 2, y_2 = 3$; $g = 9$ (min)
Dual: $x_1 = 3/5, x_2 = 1/5, x_3 = 0$; $f = 9$ (max)

13. Maximize $f = x_1 + 3x_2 + x_3$
subject to $x_1 + 4x_2 \leq 12$
$3x_1 + 6x_2 + 4x_3 \leq 48$
$x_2 + x_3 \leq 8$
$y_1 = \frac{2}{5}, y_2 = \frac{1}{5}, y_3 = \frac{1}{5}$; $g = 16$ (min)
Dual: $x_1 = 4, x_2 = 2, x_3 = 6$; $f = 16$ (max)

15. min = 32 at $x = 16, y = 0$

17. min = 28 at $x = 2, y = 0, z = 1$

19. min = 480 at $y_1 = 0, y_2 = 0, y_3 = 16$

21. (a) Minimize $g = 120y_1 + 50y_2$
subject to $3y_1 + y_2 \geq 40$
$2y_1 + y_2 \geq 20$

(b) Primal: $x_1 = 40, x_2 = 0$; $f = 1600$ (max)
Dual: $y_1 = 40/3, y_2 = 0$; $g = 1600$ (min)

23. Line 1 for 4 hours, Line 2 for 1 hour; \$1200

25. $A = 12$ weeks, $B = 0$ weeks, $C = 0$ weeks; cost = \$12,000

27. Factory 1: 50 days, Factory 2: 0 days. Minimum cost \$500,000.

29. 105 minutes on radio, nothing on TV. Minimum cost \$10,500.

31. 16 oz of Food I, 0 oz of Food II, 0 oz of Food III, Minimum cost = \$16.

Exercise 4.6 (page 363)

1. $-3x + y \leq -5$ 3. $-6x - y \leq -40$

5. $\left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 0 & 0 & 6 \\ -4 & -2 & 0 & 1 & 0 & -12 \\ \hline -4 & -5 & 0 & 0 & 1 & 0 \end{array}\right]$

7. $\left[\begin{array}{rrrrr|r} 6 & 4 & 1 & 0 & 0 & 24 \\ -5 & -2 & 0 & 1 & 0 & -16 \\ \hline -2 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$

9. (a) Maximize $f = 2x + 3y$ subject to
$7x + 4y \leq 28$
$3x - y \leq -2$
$x \geq 0, y \geq 0$

(b) $\left[\begin{array}{rrrrr|r} 7 & 4 & 1 & 0 & 0 & 28 \\ 3 & -1 & 0 & 1 & 0 & -2 \\ \hline -2 & -3 & 0 & 0 & 1 & 0 \end{array}\right]$

11. (a) Maximize $-g = -3x - 8y$ subject to
$4x - 5y \leq 50$
$x + y \leq 80$
$x - 2y \leq -4$
$x \geq 0, y \geq 0$

(b) $\left[\begin{array}{rrrrrr|r} 4 & -5 & 1 & 0 & 0 & 0 & 50 \\ 1 & 1 & 0 & 1 & 0 & 0 & 80 \\ 1 & -2 & 0 & 0 & 1 & 0 & -4 \\ \hline 3 & 8 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$

13. $x = 6, y = 8, z = 12$; $f = 120$

15. $x = 10, y = 17$; $f = 57$ 17. $x = 5, y = 7$; $f = 31$

19. $x = 4, y = 8$; $f = 16$ 21. $x = 12, y = 8$; $f = 76$

23. $x = 5, y = 15$; $f = 45$

25. $x = 10, y = 20$; $f = 120$

27. $x = 20, y = 10, z = 0$; $f = 40$

29. $x = 5, y = 0, z = 3$; $f = 22$

31. $x = 0, y = 44, z = 98$; $f = 34$

33. $x = 70, y = 0, z = 40$; $f = 2100$

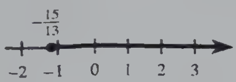
35. Produce 200 of each at Monaca; produce 300 commercial components and 550 domestic furnaces at Hamburg; profit = \$355,250

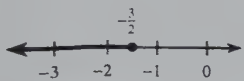
37. Produce 200 of each at Monaca; produce 300 commercial components and 550 domestic furnaces at Hamburg; cost = \$337,750
39. regular = 2000 lb; beef = 880 lb; profit = \$1328
41. I = 3 million, II = 0, III = 3 million; cost = \$180,000

Chapter 4 Review Exercises *(page 367)*

1. $x \le 3$ 

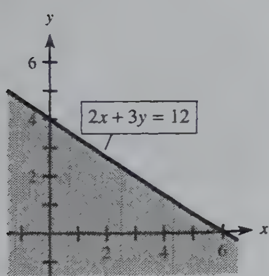
2. $x \ge -20/3$ 

3. $x \ge -15/13$ 

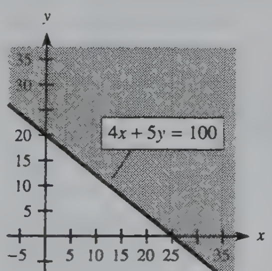
4. $x \le -3/2$ 

5. **(a)** closed; $[0, 5]$ **(b)** half-open; $[3, 7)$ **(c)** open; $(-3, 2)$
6. **(a)** $-1 < x < 16$ **(b)** $-12 < x \le 8$ **(c)** $x < -1$

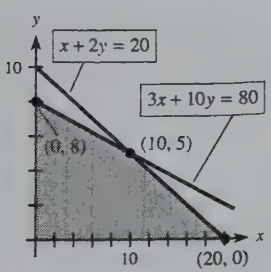
7.



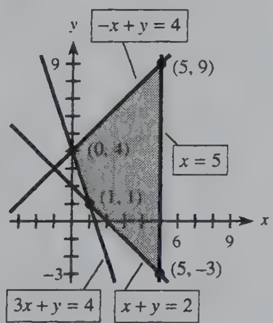
8.



9.

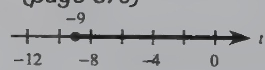


10.

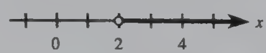


11. max = 25 at (5, 10); min = −12 at (12, 0)
12. max = 194 at (17, 23); min = 104 at (8, 14)
13. $f = 66$ at (6, 6) **14.** $f = 91$ at (9, 2)
15. $f = 43$ at (7, 9) **16.** $f = 31$ at (12, 7)
17. $g = 24$ at (3, 3) **18.** $g = 20$ at (5, 1)
19. $g = \frac{247}{3}$ at $(\frac{212}{3}, \frac{7}{3})$ **20.** $g = 80$ at (5, 45)
21. $f = 168$ at (12, 7) **22.** $f = 260$ at (60, 20)
23. $f = 360$ at (40, 30) **24.** $f = 270$ at (5, 3, 2)
25. $f = 640$ on the line between (160, 0) and (90, 70)
26. no solution **27.** $g = 32$ at $y_1 = 2, y_2 = 3$
28. $g = 20$ at $y_1 = 4, y_2 = 2$
29. $g = 7$ at $y_1 = 1, y_2 = 5$
30. $g = 16$ at $y_1 = \frac{2}{5}, y_2 = \frac{1}{5}, y_3 = \frac{1}{5}$
31. $f = 165$ at $x = 20, y = 21$
32. $f = 54$ at $x = 6, y = 5$
33. $f = 156$ at $x = 15, y = 2$
34. $f = 31$ at $x = 4, y = 5$
35. $p = \$14,750$ when 110 large and 75 small swingsets are made
36. $C = \$300,000$ when #1 operates 30 days, #2 operates 25 days
37. $P = \$320$; I = 40, II = 20
38. $P = \$420$; Jacob's ladders = 90, locomotives = 30
39. food I = 0 oz, food II = 3 oz; $C = \$0.60$ (min)
40. Cost = \$5.60; $A = 40$ lb, $B = 0$ lb
41. Cost = \$8500; $A = 20$ days, $B = 15$ days, $C = 0$ days
42. pancake mix = 8000 lb; cake mix = 3000 lb; profit = \$3550
43. Texas: 55 desks, 65 computer tables; Louisiana: 75 desks, 65 computer tables; cost = \$4245

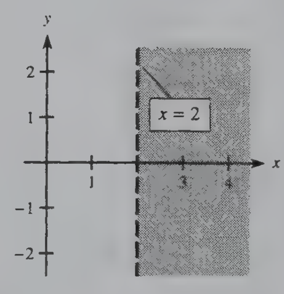
Chapter 4 Test *(page 370)*

1. $t \ge -9$ 

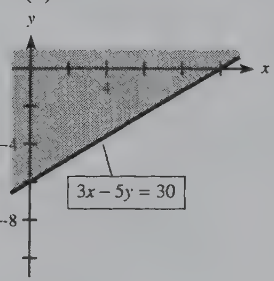
2. $-1 < x \le 4$

3. **(a)** 

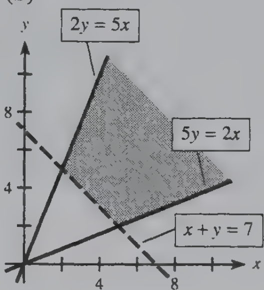
(b)



4. **(a)**



(b)



5. Max = 120 at (0, 24) **6.** Min = 136 at (28, 52)

7. Max = 6300 at $x = 90$, $y = 0$

8. (a) C; $\left[\begin{array}{ccccccc|c} 1 & 2 & 0 & 1 & 0 & -3/2 & 0 & 40 \\ 0 & 1 & 0 & -2 & 1 & 1/2 & 0 & 15 \\ 0 & 3 & 1 & -1 & 0 & 1/4 & 0 & 60 \\ \hline 0 & 0 & 0 & 4 & 0 & 6 & 1 & 220 \end{array}\right]$

$-2R_2 + \text{ to } R_1 \quad -3R_2 + \text{ to } R_3$

(b) A; Pivot column is column 3, but new pivot is undefined.

9. Maximize $f = 100x_1 + 120x_2$
subject to $3x_1 + 4x_2 \le 2$
$5x_1 + 6x_2 \le 3$
$x_1 + 3x_2 \le 5$
$x_1 \ge 0, x_2 \ge 0$

10. Maximize $-g = -7x - 3y$
subject to $x - 4y \le -4$
$x - y \le 5$
$2x + 3y \le 30$

11. Max: $x_1 = 17, x_2 = 15, x_3 = 0; f = 658$ (max)
Min: $y_1 = 4, y_2 = 18, y_3 = 0; g = 658$ (min)

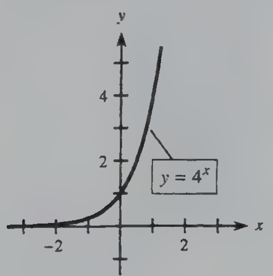
12. Maximize $P = 7x + 6y$ $\quad x$ = barrels of lager;
subject to $3x + 2y \le 1200$ $\quad y$ = barrels of ale
$2x + 2y \le 1000$
$P = \$3200$ (max) at $x = 200$, $y = 300$

13. x = number of day calls, y = number of evening calls
Minimize $C = 3x + 4y$
subject to $0.3x + 0.3y \ge 150$
$0.1x + 0.3y \ge 120$
$C = \$1850$ (min) at $x = 150$, $y = 350$

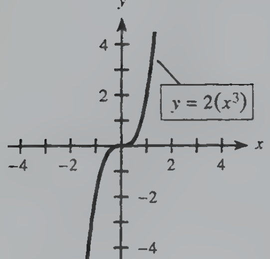
Exercise 5.1 *(page 389)*

1. 3.162278 **3.** 0.01296525 **5.** 1.44225

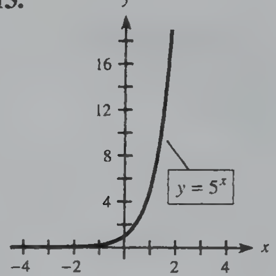
7. 7.3891 **9.**



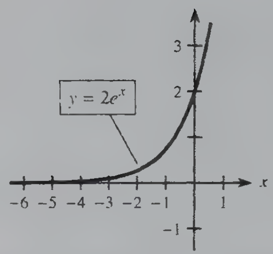
11.



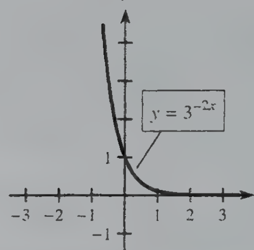
13.



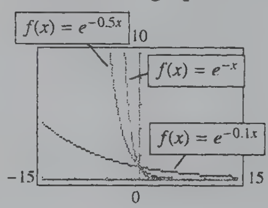
15.

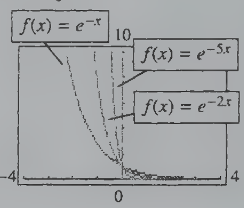


17.

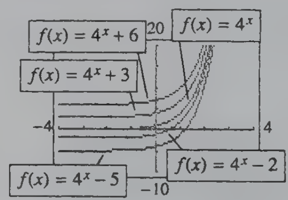


19. All graphs have the same basic shape. For larger positive k, the graphs fall more sharply. For positive k nearer 0, the graphs fall more slowly.

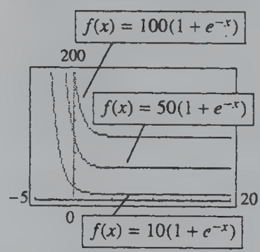




21. $y = f(x) + C$ is the same graph as $y = f(x)$ but shifted C units on the y-axis.

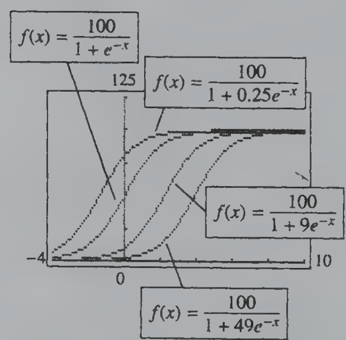


23. (a)



(b) As c changes, the y-intercept and the asymptote change.

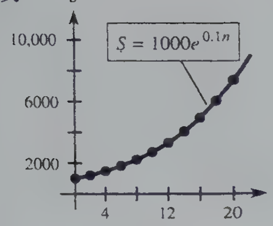
25. (a)



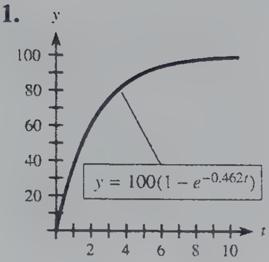
(b) Different c values change the y-intercept and how the graph approaches the asymptote.

27. \$1884.54

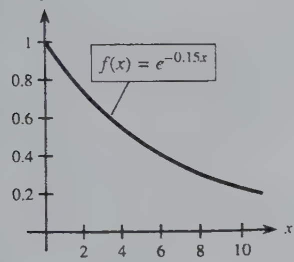
29.



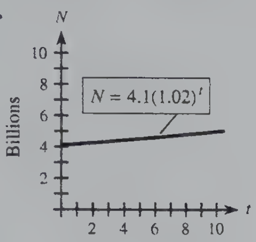
31.



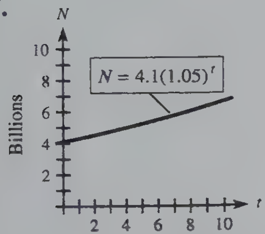
33.



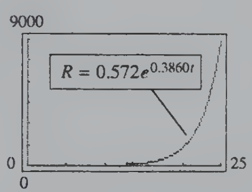
35.



37.



39.



41. The linear model fails in 2007, giving a negative number of processors.

43. **(a)** $y = 6.869(1.0851)^x$
(b) \$52.92

45. **(a)** $y = 2.366(1.04457)^x$ **(b)** 230.39
(c) $x = 101.76$, 2001

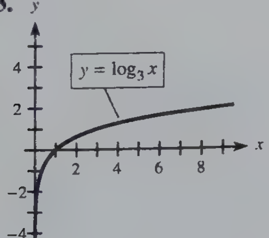
47. $y = 0.1018(1.06443)^x$

Exercise 5.2 *(page 402)*

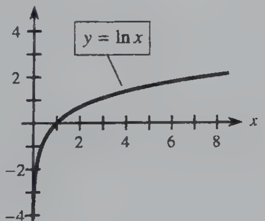
1. $2^4 = 16$ **3.** $4^{1/2} = 2$ **5.** $x = 8$

7. $x = \frac{1}{2}$ **9.** $\log_2 32 = 5$ **11.** $\log_4(\frac{1}{4}) = -1$

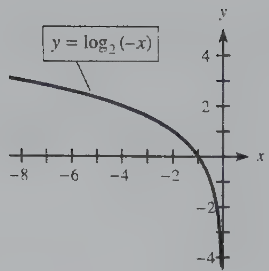
13.



15.



17.



19. **(a)** 3 **(b)** -1

21. x **23.** 3

25. **(a)** 4.9 **(b)** 0.4 **(c)** 12.4 **(d)** 0.9

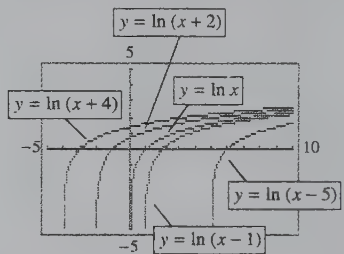
27. $\log x - \log(x + 1)$ **29.** $\log_7 x + \frac{1}{3}\log_7(x + 4)$

31. $\ln(x/y)$ **33.** $\log_5[x^{1/2}(x + 1)]$

35. equivalent; Properties V and III

37. not equivalent; $\log(\sqrt[3]{8/5})$

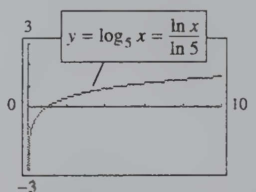
39. **(a)**



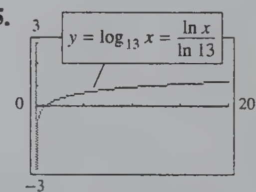
(b) For each c, the domain is $x > c$ and the vertical asymptote is at $x = c$.
(c) Each x-intercept is at $x = c + 1$.
(d) The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted c units on the x-axis.

41. **(a)** 4.0875 **(b)** -0.1544

43.



45.

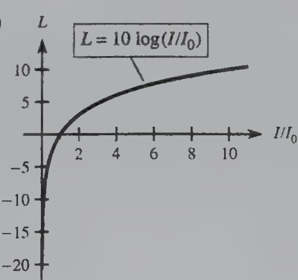


47. If $\log_a M = u$ and $\log_a N = v$, then $a^u = M$ and $a^v = N$. Therefore, $\log_a(M/N) = \log_a(a^u/a^v) = \log_a(a^{u-v}) = u - v = \log_a M - \log_a N$.

49. $y = 4.2885e^{-0.05751x}$ **51.** 2.5 times as severe

53. 14 times as severe **55.** 40

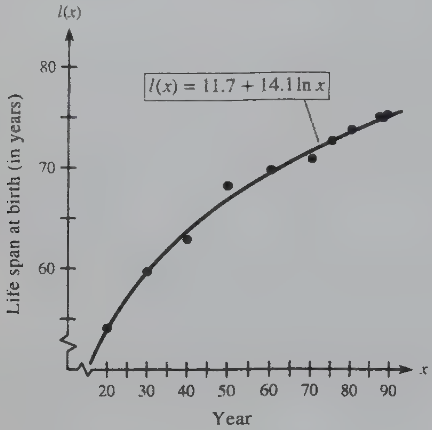
57. $L = 10\log(I/I_0)$



59. 0.1 and 1×10^{-14}

61. $\text{pH} = \log \dfrac{1}{[\text{H}^+]} = \log 1 - \log[\text{H}^+] = -\log[\text{H}^+]$

63.

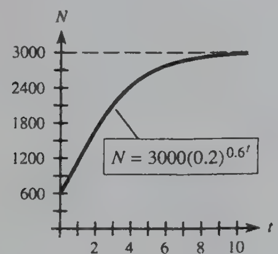


Very similar.

65. (a) $y = -22.88 + 320.88 \ln x$ (b) 1010

Exercise 5.3 *(page 415)*

1. (a) 2038 (b) 4.9 months
3. (a) 69% (b) 37,204 years (approximately)
5. 24.5 years 7. 128,402
9. (a) 600 (b) 2119 (c) 3000
   (d)

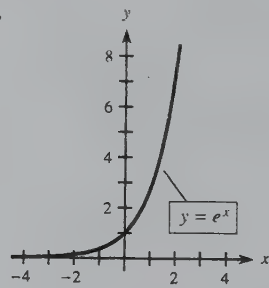


11. (a) 10 (b) 2.5 years
13. (a) 37 (b) 1.5 hours 15. (a) 52 (b) tenth
17. (a) \$4.98 (b) 8 19. \$502 21. \$420.09
23. \$2706.71 25. (a) \$10,100.31 (b) 6.03 years
27. (a) \$5469.03
    (b) 7 years, 9 months (approximately)
29. (a) 2% (b) 20 months ($x = 19.6$)
31. (a) 44.7 years (approximately)
    (b) The intent of such a plan would be to reduce future increases in health care expenditures. A new model might not be exponential, or, if it were, it would be one that rose more gently.
33. (a) 18.3 yrs (approx.) (b) \$42,340.00
35. (a) $\log_{1.005}(2) = t$
    (b) 139 months (approximately)
37. (a) 0.23 km³ (b) 5.9 years
39. (a) 0 lb (b) 29.95 lb (c) 0.21 min
41. (a) $x = 16$; 0.2% (b) 10.56 min (c) 0.06%

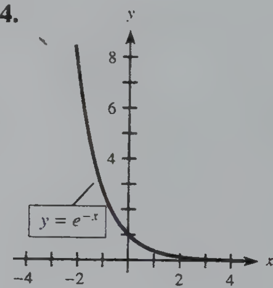
Chapter 5 Review Exercises *(page 419)*

1. (a) $\log_2 y = x$ (b) $\log_3 2x = y$
2. (a) $7^{-2} = \frac{1}{49}$ (b) $4^{-1} = x$

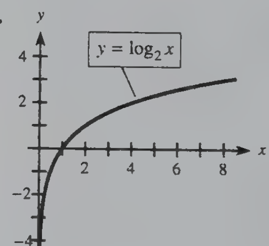
3.



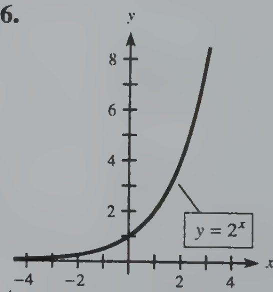
4.



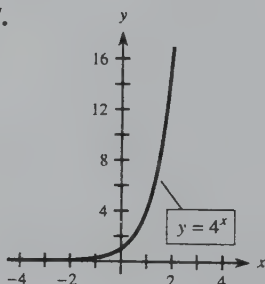
5.



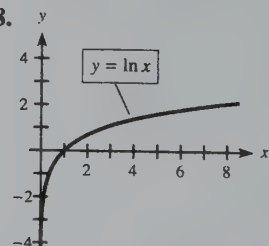
6.



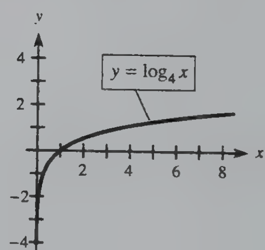
7.



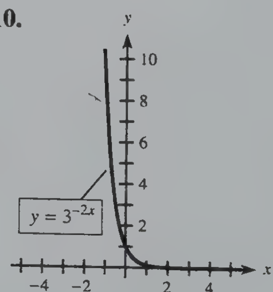
8.



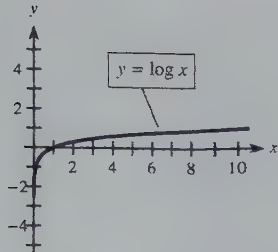
9.



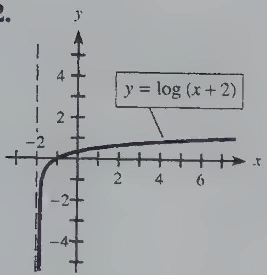
10.



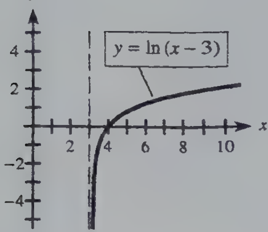
11.



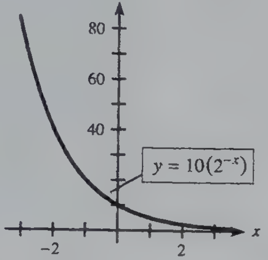
12.



13.

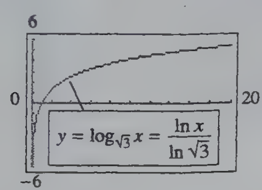


14.

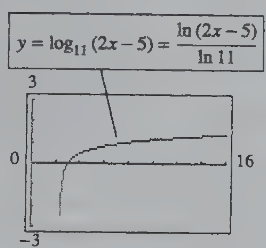


15. 0 **16.** 2 **17.** $\frac{1}{2}$ **18.** -1 **19.** 8
20. 1 **21.** 5 **22.** 3.15 **23.** -2.7
24. 0.6 **25.** 5.1 **26.** 15.6 **27.** $\log y + \log z$
28. $\frac{1}{2}\ln(x+1) - \frac{1}{2}\ln x$ **29.** no
30. -2 **31.** 5 **32.** 1 **33.** 0
34. 3.4939 **35.** -1.5845

36.

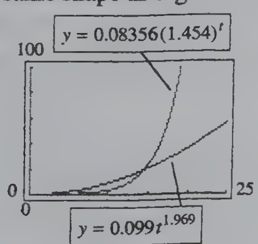


37.



38. Growth exponential because the general outline has the same shape as a growth exponential.

39. (a)



(b) The function $y = 0.099t^{1.969}$

40. Exponential because the general shape is similar to the graph of a growth exponential.

41. (a) $y = 604.9211\,(1.082)^x$, $x = 0$ in 1900
(b) 2,374,440

42. (a) $y = 18.1607 + 5.1865 \ln(x)$
(b) 38.4% **(c)** 2017 ($x = 67$)

43. (a) -3.9 **(b)** $0.14B_0$ **(c)** $0.004B_0$ **(d)** yes

44. (a) 3000 **(b)** 8603 **(c)** 10,000

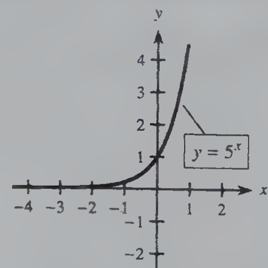
45. (a) 27,441 **(b)** 12 weeks

46. 1366 **47.** 5.8 years

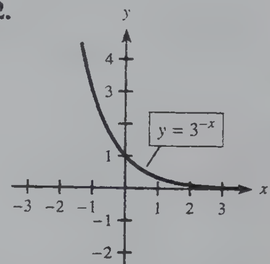
48. (a) \$5532.77 **(b)** 5.13 years

Chapter 5 Test (page 422)

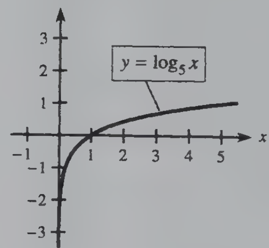
1.



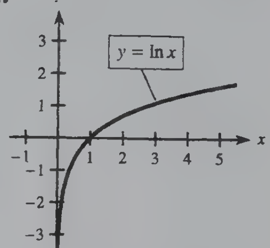
2.



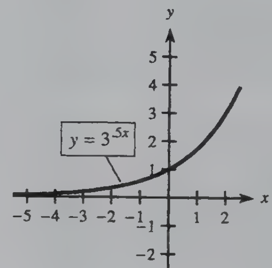
3.



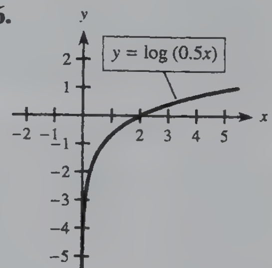
4.



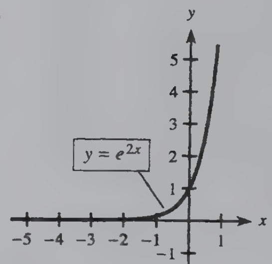
5.



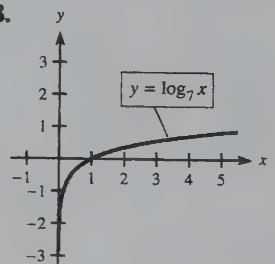
6.



7.



8.



9. 54.59815 **10.** 0.122456 **11.** 1.38629
12. 3.04452 **13.** $x = 17^{3.1} \approx 6522.163$
14. $\log_3(27) = 2x$, so $x = \frac{3}{2}$ **15.** 3
16. x^4 **17.** 3 **18.** x^2
19. $\ln(M) + \ln(N)$ **20.** $\ln(x^3 - 1) - \ln(x + 2)$
21. $\dfrac{\ln(x^3+1)}{\ln 4} \approx 0.721 \ln(x^3 + 1)$
22. A decay exponential

23. A growth exponential 24. During 2103
25. (a) $y = 2227.8035\,(3.0489)^x$
(b) Model predicts attendance for 2001 will be about 471,468,000 (to the nearest thousand). It seems unlikely that this many people would attend one rally.
26. $y = 0.06626\,(1.126724)^x$
27. $y = 0.080073\,(1.12401)^x$; The models are quite similar.

Exercise 6.1 (page 437)

1. (a) \$9600 (b) \$19,600
3. (a) \$30 (b) \$1030
5. \$864 7. \$3850 9. 13%
11. (a) 6.5% (b) 5.65% 13. \$1631.07
15. \$12,000 17. 10 years 19. pay on time
21. (a) \$2120 (b) \$2068.29 (nearest cent)
23. 3, 6, 9, 12, 15, 18, 21, 24, 27, 30
25. $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}$ 27. $-\frac{1}{4}, \frac{1}{8}, -\frac{1}{12}, \frac{1}{16}, \frac{-1}{20}, \frac{1}{24}$
29. $-\frac{1}{3}, \frac{1}{5}, \frac{-1}{7}, \frac{1}{9}, \frac{-1}{11}, \frac{1}{13}$ 31. $-1, -\frac{1}{4}, -\frac{1}{15}, 0; a_{10} = \frac{1}{20}$
33. (a) $d = 3, a_1 = 2$ (b) 11, 14, 17
35. (a) $d = \frac{3}{2}, a_1 = 3$ (b) $\frac{15}{2}, 9, \frac{21}{2}$ 37. 105
39. −35 41. 203 43. 2185 45. 1907.5
47. 2550 49. −15,862.5 51. 21, 34, 55
53. \$2400 55. the job starting at \$20,000
57. (a) \$3000 (b) \$4500 (c) Plan II, by \$1500
(d) \$10,000 (e) \$13,500 (f) Plan II, by \$3500
(g) Plan II

Exercise 6.2 (page 452)

(Minor differences may occur because of rounding.)
1. (a) 6% (b) 6 (c) 3% = 0.03 (d) 12
3. (a) 8% (b) 7 (c) 2% = 0.02 (d) 28
5. (a) 9% (b) 5 (c) $(\frac{9}{12})\% = .0075$ (d) 60
7. \$24,846.78 9. \$3086.45 11. \$4755.03
13. \$6844.36 15. \$6661.46 17. \$5583.95
19. \$7309.98 21. \$502.47
23. (a) \$12,245.64 (b) \$11,080.32
(c) A $\frac{1}{2}$% increase in the interest rate reduces the amount required by \$1165.32.
25. \$50.26 more at 8% 27. 8.67% 29. 7.55%
31. 6.18%
33. 8% compounded monthly, 8% compounded quarterly, 8% compounded annually
35. The higher graph is for continuous compounding because its yield (its effective annual rate) is higher.
37. 26.8% 39. 3 years 41. 4% 43. \$3996.02
45. (a) \$63,128.75 (b) \$14,263.10
47. 5.12 years (approximately) 49. \$13,916.24
51.

	A	B	C
1		Future Value	(Yearly)
2	End of Year	Quarterly	Monthly
3	0	\$5000.00	\$5000.00
4	1	\$5322.52	\$5324.26
5	2	\$5665.84	\$5669.54
6	3	\$6031.31	\$6037.22
7	4	\$6420.36	\$6428.74
8	5	\$6834.50	\$6845.65
9	6	\$7275.35	\$7289.60
10	7	\$7744.64	\$7762.34
11	8	\$8244.20	\$8265.74
12	9	\$8775.99	\$8801.79
13	10	\$9342.07	\$9372.59

(a) From quarterly and monthly spreadsheets: after $6\frac{1}{2}$ years (26 quarters or 78 months)
(b) See the spreadsheet.
53. (a) 24, 48, 96 (b) 24, 16, $\frac{32}{3}$
55. 40,960 57. $4 \cdot (\frac{3}{2})^{15}$ 59. $\frac{6(1 - 3^{17})}{-2}$
61. $\frac{8}{3}[1 + (\frac{1}{2})^{21}]$ 63. $\frac{3^{35} - 1}{2}$ 65. $18[1 - (\frac{2}{3})^{18}]$
67. \$350,580 (approx.) 69. 24.4 million (approx.)
71. 35 years 73. 40.5 ft 75. \$4096
77. 320,000 79. \$7,231,366 81. \$1801.14
83. 15,625 85. 305,175,780

Exercise 6.3 (page 464)

1. $3\frac{1}{2}\% = 0.035$ for 22 periods
3. $\frac{1}{4}\% = 0.0025$ for 60 periods
5. (a) The higher graph is \$1120 per year.
(b) \$8772.71
7. \$7328.22 9. \$1072.97 11. \$77,313.47
13. \$4774.55 15. \$4372.20 17. \$226.10
19. \$1083.40 21. \$265.25 23. \$741.47
25. A sinking fund is a savings plan, so the 10% rate (a) is better.
27. \$53,677.40 29. \$4651.61 31. \$1180.78
33. \$4152.32 35. \$26,517.13 37. \$3787.92
39. \$235.16

41. The spreadsheet shows the amount at the end of each of the first 12 months and the last 12 months. Amount after 10 years is shown.

	A	B	C
1		Future Value	
2	End of Month	Ordinary Ann.	Annuity Due
3	0	0	100
4	1	$100.00	$100.60
5	2	$200.60	$201.80
6	3	$301.80	$303.61
7	4	$403.61	$406.04
8	5	$506.04	$509.07
9	6	$609.07	$612.73
10	7	$712.73	$717.00
11	8	$817.00	$821.91
12	9	$921.91	$927.44
13	10	$1027.44	$1033.60
14	11	$1133.60	$1140.40
15	12	$1240.40	$1247.85
	⋮	⋮	⋮
112	109	$15324.39	$15416.34
113	110	$15516.34	$15609.44
114	111	$15709.44	$15803.70
115	112	$15903.70	$15999.12
116	113	$16099.12	$16195.71
117	114	$16295.71	$16393.49
118	115	$16493.49	$16592.45
119	116	$16692.45	$16792.60
120	117	$16892.60	$16993.96
121	118	$17093.96	$17196.52
122	119	$17296.52	$17400.30
123	120	$17500.30	$17605.30

(b) $12,000

(c) Annuity due. Each payment for an annuity due earns 1 month's interest more than that for an ordinary annuity.

Exercise 6.4 (page 475)

1. $976.32 **3.** $69,913.77 **5.** $4595.46
7. $5541.23 **9.** $27,590.62 **11.** $2,128,391

13. **(a)** $69,552.35
(b) $1045.23 per annual payment; $10,452.30 over 10 years

15. **(a)** The higher graph corresponds to 8%.
(b) $1500 (approximately)
(c) With an interest rate of 10%, a present value of about $9000 is needed to purchase an annuity of $1000 for 25 years. If the interest rate is 8%, about $10,500 is needed.

17. **(a)** $30,078.99 **(b)** $16,900
(c) $607.02 **(d)** $36,421.20

19. Ordinary annuity—payments at the end of each period
Annuity due—payments at the beginning of each period

21. $69,632.02 **23.** $445,962.23 **25.** $316,803.61
27. $2145.59 **29.** $146,235.06 **31.** $22,663.74
33. $7957.86 **35.** $74,993.20 **37.** $19,922.97
39. $1317.98 **41.** $85,804.29

43. **(a)** The spreadsheet shows the payments for first 12 months and the last 12 months. Full payments for $13\frac{1}{2}$ years.

	A	B	C	D
1	End of Month	Acct. Value	Payment	New Balance
2	0	$100000.00	$0.00	$100000.00
3	1	$100650.00	$1000.00	$99650.00
4	2	$100297.73	$1000.00	$99297.73
5	3	$99943.16	$1000.00	$98943.16
6	4	$99586.29	$1000.00	$98586.29
7	5	$99227.10	$1000.00	$98227.10
8	6	$98865.58	$1000.00	$97865.58
9	7	$98501.70	$1000.00	$97501.70
10	8	$98135.47	$1000.00	$97135.47
11	9	$97766.85	$1000.00	$96766.85
12	10	$97395.83	$1000.00	$96395.83
13	11	$97022.40	$1000.00	$96022.40
14	12	$96646.55	$1000.00	$95646.55
	⋮	⋮	⋮	⋮
154	152	$10684.71	$1000.00	$9684.71
155	153	$9747.66	$1000.00	$8747.66
156	154	$8804.52	$1000.00	$7804.52
157	155	$7855.25	$1000.00	$6855.25
158	156	$6899.81	$1000.00	$5899.81
159	157	$5938.16	$1000.00	$4938.16
160	158	$4970.25	$1000.00	$3970.25
161	159	$3996.06	$1000.00	$2996.06
162	160	$3015.53	$1000.00	$2015.53
163	161	$2028.64	$1000.00	$1028.64
164	162	$1035.32	$1000.00	$35.32
165	163	$35.55	$35.55	$0.00

43. (b) The spreadsheet shows the payments for the first 12 months and the last 12 months. Full payments for almost 4 years.

	A	B	C	D
1	End of Month	Acct. Value	Payment	New Balance
2	0	$100000.00	$0.00	$100000.00
3	1	$100650.00	$2500.00	$98150.00
4	2	$98787.98	$2500.00	$96287.98
5	3	$96913.85	$2500.00	$94413.85
6	4	$95027.54	$2500.00	$92527.54
7	5	$93128.97	$2500.00	$90628.97
8	6	$91218.05	$2500.00	$88718.05
9	7	$89294.72	$2500.00	$86794.72
10	8	$87358.89	$2500.00	$84858.89
11	9	$85410.47	$2500.00	$82910.47
12	10	$83449.39	$2500.00	$80949.39
13	11	$81475.56	$2500.00	$78975.56
14	12	$79488.90	$2500.00	$76988.90
	⋮	⋮	⋮	⋮
38	36	$27734.95	$2500.00	$25234.95
39	37	$25398.98	$2500.00	$22898.98
40	38	$23047.83	$2500.00	$20547.83
41	39	$20681.39	$2500.00	$18181.39
42	40	$18299.57	$2500.00	$15799.57
43	41	$15902.26	$2500.00	$13402.26
44	42	$13489.38	$2500.00	$10989.38
45	43	$11060.81	$2500.00	$8560.81
46	44	$8616.45	$2500.00	$6116.45
47	45	$6156.21	$2500.00	$3656.21
48	46	$3679.98	$2500.00	$1179.98
49	47	$1187.65	$1187.65	$0.00

Exercise 6.5 *(page 485)*

1. (a) The 10-year loan because the loan must be paid more quickly.
   (b) The 25-year loan because the loan is paid more slowly.

3. $2504.56 5. $1288.29

7.

Period	*Payment*	*Interest*
1	$39,505.50	$9000.00
2	39,505.50	6254.51
3	39,505.43	3261.92
	118,516.43	18,516.43

Period	*Balance Reduction*	*Unpaid Balance*
		$100,000.00
1	$30,505.50	69,494.50
2	33,250.99	36,243.51
3	36,243.51	0.00
	100,000.00	

9.

Period	*Payment*	*Interest*
1	$5380.54	$600.00
2	5380.54	456.58
3	5380.54	308.87
4	5380.54	156.71
	21,522.16	1522.16

Period	*Balance Reduction*	*Unpaid Balance*
		$20,000.00
1	$4780.54	15,219.46
2	4923.96	10,295.50
3	5071.67	5,223.83
4	5223.83	0.00
	20,000.00	

11. $8852.05 13. $5785.83

15. (a) $4359.23 (b) $87,184.60 (c) $37,184.60

17. (a) $1237.78 (b) $9902.24 (c) $1902.24

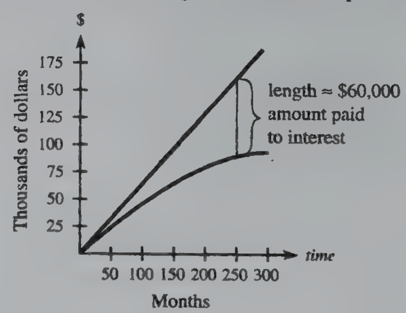
19. $2967.75 21. $62,473.28

23. (a) $1,239,676.52 (b) $1,270,768.38

25. (a) $89,120.53 (b) $6451.45

27. The line is the total amount paid ($644.30 per month × the number of months). The other curve is the total amount paid toward the principal.

29.



31.

	Payment	*Total Interest*
(a) 8%	$366.19	$2577.12
8.5%	$369.72	$2746.56
(b) 6.75%	$518.88	$106,796.80
7.25%	$545.74	$116,466.40

(c) The duration of the loan seems to have the greatest effect. It greatly influences payment size (for a $15,000 loan vs. one for $80,000), and it also affects total interest paid.

33.

	Payment	*Points*	*Total paid*
(a) (i)	$738.99	—	$221,697
(ii)	$722.81	$1000	$217,843
(iii)	$706.78	$2000	$214,034

(b) The 7% loan with 2 points.

35. The spreadsheet shows the amortization schedule for the first 12 and the last 12 payments.

	A	B	C	D	E
1	Period	Payment	Interest	Bal. Reduction	Unpaid Bal.
2	0				$16700.00
3	1	$409.27	$114.12	$295.15	$16404.85
4	2	$409.27	$112.10	$297.17	$16107.68
5	3	$409.27	$110.07	$299.20	$15808.48
6	4	$409.27	$108.02	$301.25	$15507.23
7	5	$409.27	$105.97	$303.30	$15203.93
8	6	$409.27	$103.89	$305.38	$14898.55
9	7	$409.27	$101.81	$307.46	$14591.09
10	8	$409.27	$99.71	$309.56	$14281.52
11	9	$409.27	$97.59	$311.68	$13969.84
12	10	$409.27	$95.46	$313.81	$13656.03
13	11	$409.27	$93.32	$315.95	$13340.08
14	12	$409.27	$91.16	$318.11	$13021.97
	⋮	⋮	⋮	⋮	⋮
39	37	$409.27	$32.11	$377.16	$4322.49
40	38	$409.27	$29.54	$379.73	$3942.75
41	39	$409.27	$26.94	$382.33	$3560.43
42	40	$409.27	$24.33	$384.94	$3175.49
43	41	$409.27	$21.70	$387.57	$2787.91
44	42	$409.27	$19.05	$390.22	$2397.70
45	43	$409.27	$16.38	$392.89	$2004.81
46	44	$409.27	$13.70	$395.57	$1609.24
47	45	$409.27	$11.00	$398.27	$1210.97
48	46	$409.27	$8.27	$401.00	$809.97
49	47	$409.27	$5.53	$403.74	$406.24
50	48	$409.02	$2.78	$406.24	$0.00

37. The spreadsheet shows the amortization schedule for the first 12 payments and the last 12. Paying an extra $15 takes 57 full payments (rather than 60) and a final payment of $44.07.

	A	B	C	D	E
1	Period	Payment	Interest	Bal. Reduction	Unpaid Bal.
2	0				$18000.00
3	1	$383.43	$126.00	$257.43	$17742.57
4	2	$383.43	$124.20	$259.23	$17483.34
5	3	$383.43	$122.38	$261.05	$17222.29
6	4	$383.43	$120.56	$262.87	$16959.42
7	5	$383.43	$118.72	$264.71	$16694.70
8	6	$383.43	$116.86	$266.57	$16428.14
9	7	$383.43	$115.00	$268.43	$16159.70
10	8	$383.43	$113.12	$270.31	$15889.39
11	9	$383.43	$111.23	$272.20	$15617.19
12	10	$383.43	$109.32	$274.11	$15343.08
13	11	$383.43	$107.40	$276.03	$15067.05
14	12	$383.43	$105.47	$277.96	$14789.09
	⋮	⋮	⋮	⋮	⋮
48	46	$383.43	$31.07	$352.36	$4086.36
49	47	$383.43	$28.60	$354.83	$3731.53
50	48	$383.43	$26.12	$357.31	$3374.22
51	49	$383.43	$23.62	$359.81	$3014.41
52	50	$383.43	$21.10	$362.33	$2652.09
53	51	$383.43	$18.56	$364.87	$2287.22
54	52	$383.43	$16.01	$367.42	$1919.80
55	53	$383.43	$13.44	$369.99	$1549.81
56	54	$383.43	$10.85	$372.58	$1177.23
57	55	$383.43	$8.24	$375.19	$802.04
58	56	$383.43	$5.61	$377.82	$424.22
59	57	$383.43	$2.97	$380.46	$43.76
60	58	$44.07	$0.31	$43.76	$0.00

Chapter 6 Review Exercises *(page 488)*

1. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$
2. Arithmetic: (a) and (c) **(a)** $d = -5$ **(c)** $d = \frac{1}{6}$
3. 235 **4.** 109 **5.** 315
6. Geometric: (a) and (b) **(a)** $r = 8$ **(b)** $r = \frac{-3}{4}$
7. 8 **8.** $2{,}391{,}484\frac{4}{9}$ **9.** $10,880 **10.** $6\frac{2}{3}\%$
11. $2941.18 **12.** $4650
13. the $20,000 job ($245,000 vs $234,000)
14. **(a)** 40 **(b)** $2\% = 0.02$
15. **(a)** $S = P(1 + i)^n$ **(b)** $S = Pe^{rt}$
16. (b) monthly **17.** $372.79 **18.** $1601.03
19. $14,510.26 **20.** $1616.07 **21.** $3,466.64
22. $21,299.21 **23.** 14.5 years **24.** 34.3 months
25. **(a)** 16.32% **(b)** 14.22% **26.** 13.29%
27. **(a)** 7.40% **(b)** 7.47%
28. 2^{63} **29.** $2^{32} - 1$ **30.** $29,428.47
31. $1863.93 **32.** $213.81 **33.** $6069.44
34. $31,194.18 **35.** $10,841.24 **36.** $130,079.36
37. $12,007.09 **38.** $32,834.69
39. **(a)** $11.828 million **(b)** $161.5 million
40. $1726.85 **41.** $5390.77 **42.** $12,162.06
43. $88.85
44. **(a)** $592.76 **(b)** $177,828 **(c)** $85,828
45. $3443.61 **46.** $34,597.40
47.

Payment Number	*Payment Amount*	*Interest*	*Balance Reduction*	*Unpaid Balance*
57	$699.22	$594.01	$105.21	$94,936.99
58	$699.22	$593.36	$105.86	$94,831.13

Chapter 6 Test (page 491)

1. 25.3 years (approximately) 2. \$840.75
3. 6.82% 4. \$158,524.90 5. 33.53%
6. (a) \$698.00 (b) \$112,400 7. \$2625
8. \$7999.41 9. 8.73% 10. \$119.912.92
11. \$40,552.00 12. \$32,488 (to the nearest dollar)
13. \$6781.17 14. (a) \$95,164.21 (b) \$1300.14
15. \$1688.02 16. (a) \$279,841.35 (b) \$13,124.75
17. \$23,381.82 18. \$29,716.47
19. (a) The difference between successive terms is always −5.5.
(b) 23.8
(c) 8226.3
20. 1000 mg (correct to three decimal places)

Exercise 7.1 (page 505)

1. $\frac{1}{4}$ 3. 1 5. (a) $\frac{2}{5}$ (b) 0 (c) 1
7. (a) $\frac{3}{10}$ (b) $\frac{1}{2}$ (c) $\frac{1}{5}$ (d) $\frac{3}{5}$ (e) $\frac{7}{10}$
9. (a) $\frac{1}{13}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$
11. {HH, HT, TH, TT}; (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$
13. (a) $\frac{1}{12}$ (b) $\frac{1}{12}$ (c) $\frac{1}{36}$ 15. (a) $\frac{1}{2}$ (b) $\frac{5}{12}$
17. (a) 431/1200
(b) If fair, Pr(6) = $\frac{1}{6}$; 431/1200 not close to $\frac{1}{6}$, so not a fair die
19. (a) 2:3 (b) 3:2 21. (a) 3:8 (b) 8:3
23. (a) $\frac{1}{21}$ (b) $\frac{20}{21}$ 25. (a) $\frac{57}{80}$ (b) $\frac{23}{80}$
27. (a) 1/3601 (b) 100/3601 (c) 3500/3601
29. S = {A+, A−, B+, B−, AB+, AB−, O+, O−}
31. .46 33. (a) .04 (b) .96
35. (a) .627 (b) .373 37. .03 39. .75
41. $\frac{1}{3}$ 43. $\frac{1}{3}$
45. .22; yes, .39 is much higher than .22
47. $\frac{1}{4}$ 49. $\frac{3}{8}$
51. (a) no (b) {BB, BG, GB, GG} (c) $\frac{1}{2}$
53. $\frac{1}{5}$ 55. $\frac{3}{125}$
57. Pr(A) = 0.000019554 or about 1.9 accidents per 100,000
Pr(B) = 0.000035919 or about 3.6 accidents per 100,000
Pr(C) = 0.000037679 or about 3.8 accidents per 100,000
Intersection C is the most dangerous.
59. (a) 557/1200 (b) 11/120
61. (a) boy: 1/5; girl: 4/5
(b) boy: .4946; girl: .5054 (c) part b

Exercise 7.2 (page 515)

1. $\frac{1}{2}$ 3. $\frac{1}{6}$ 5. $\frac{2}{5}$ 7. (a) $\frac{1}{7}$ (b) $\frac{5}{7}$
9. $\frac{3}{4}$ 11. $\frac{2}{3}$ 13. $\frac{10}{17}$ 15. $\frac{2}{3}$
17. (a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{8}{9}$ (d) $\frac{1}{9}$
19. .54 21. (a) 362/425 (b) $\frac{66}{85}$
23. $\frac{17}{50}$ 25. (a) $\frac{8}{15}$ (b) $\frac{7}{15}$
27. (a) $\frac{11}{12}$ (b) $\frac{5}{6}$ 29. (a) $\frac{1}{2}$ (b) $\frac{7}{8}$ (c) $\frac{3}{4}$
31. .56 33. .965 35. (a) .72 (b) .84 (c) $\frac{61}{100}$
37. $\frac{51}{65}$ 39. .13

Exercise 7.3 (page 526)

1. (a) $\frac{1}{2}$ (b) $\frac{1}{13}$ 3. (a) $\frac{1}{3}$ (b) $\frac{1}{3}$ 5. $\frac{4}{7}$
7. (a) $\frac{2}{3}$ (b) $\frac{4}{9}$ (c) $\frac{3}{5}$ 9. (a) $\frac{1}{4}$ (b) $\frac{1}{2}$
11. $\frac{1}{36}$ 13. (a) $\frac{1}{8}$ (b) $\frac{7}{8}$ 15. (a) $\frac{3}{50}$ (b) $\frac{1}{15}$
17. (a) $\frac{4}{25}$ (b) $\frac{9}{25}$ (c) $\frac{6}{25}$ (d) 0 19. $\frac{5}{68}$
21. (a) $\frac{1}{5}$ (b) $\frac{3}{5}$ (c) 0 23. (a) $\frac{1}{17}$ (b) 13/204
25. (a) $\frac{13}{17}$ (b) $\frac{4}{17}$ (c) $\frac{8}{51}$ 27. $\frac{31}{52}$ 29. $\frac{25}{96}$
31. $\frac{43}{50}$ 33. $\frac{65}{87}$ 35. 35/435 = $\frac{7}{87}$ 37. $\frac{1}{10}$
39. 1/144,000,000 41. .004292 43. .06
45. .045 47. $(.95)^5 = .774$ 49. .06
51. (a) .366 (b) .634 53. (a) .4565 (b) .5435
55. (a) $(\frac{1}{3})^3(\frac{1}{5})^4 = 1/16{,}875$
(b) $(\frac{2}{3})^3(\frac{4}{5})^4 = 2048/16{,}875$ (c) 14,827/16,875
57. 4/11; 4:7
59. (a) 364/365 (b) $\frac{1}{365}$ 61. (a) .59 (b) .41

Exercise 7.4 (page 538)

1. (a) $\frac{4}{9}$ (b) $\frac{5}{9}$ 3. $\frac{2}{5}$ 5. $\frac{1}{325}$
7. (a) $\frac{2}{21}$ (b) $\frac{4}{21}$ (c) $\frac{23}{35}$
9. (a) $\frac{1}{30}$ (b) $\frac{1}{2}$ (c) $\frac{5}{6}$ 11. $\frac{3}{5}$
13. (a) $\frac{6}{25}$ (b) $\frac{9}{25}$ (c) $\frac{12}{25}$ (d) $\frac{19}{25}$
15. $\frac{2}{3}$ 17. $\frac{2}{3}$ 19. 0.3095
21. (a) 81/10,000 (b) 1323/5000
23. (a) $\frac{6}{35}$ (b) $\frac{6}{35}$ (c) $\frac{12}{35}$
25. (a) $\frac{4}{7}$ (b) $\frac{5}{14}$ (c) $\frac{7}{10}$ (d) $\frac{16}{25}$
27. $\frac{17}{45}$ 29. 0.079 31. (a) 49/100 (b) $\frac{12}{49}$

Exercise 7.5 (page 546)

1. 360 3. 151,200 5. 60 7. 1
9. (a) $6 \cdot 5 \cdot 4 \cdot 3 = 360$ (b) $6^4 = 1296$
11. 1 13. $n + 1$ 15. 16 17. 15
19. 4950 21. 1 23. 1 25. 1
27. ${}_{13}C_{10} = 286$; ${}_{12}C_9 + {}_{12}C_{10} = 220 + 66 = 286$
29. 10 31. 604,800 33. 120 35. 24 37. 64
39. 720 41. $2^{10} = 1024$ 43. 10! = 3,628,800
45. 252 47. 30,045,015 49. 792 51. 210
53. 2,891,999,880 55. 3,700,000

Exercise 7.6 (page 551)

1. $\frac{1}{120}$ 3. (a) 120 (b) $\frac{1}{120}$ 5. .639
7. (a) 1/10,000 (b) 1/5040 9. $1/10^6$
11. (a) $(\frac{1}{5})^{10} = 1/9{,}765{,}625$ (b) $(\frac{4}{5})^{10} = .107$
(c) .893
13. 1/10! = 1/3,628,800

15. (a) $\frac{1}{22}$ (b) $\frac{6}{11}$ (c) $\frac{9}{22}$ 17. .098

19. $\frac{{}_{90}C_{28} \cdot {}_{10}C_2}{{}_{100}C_{30}}$ 21. (a) .119 (b) .0476 (c) .476

23. .0238 25. (a) .721 (b) .262 (c) .279

27. (a) $\frac{1}{3}$ (b) $\frac{1}{6}$ 29. $\frac{{}_{20}C_{10}}{{}_{80}C_{10}} = .00000011$

31. (a) .033 (b) .633 33. (a) .0005 (b) .002

35. .00748

Exercise 7.7 (page 560)

1. can 3. cannot, sum $\neq 1$ 5. cannot, not square
7. can 9. [.248 .752] 11. [.228 .236 .536]
13. $[\frac{1}{4}\ \frac{3}{4}]$ 15. $[\frac{1}{4}\ \frac{1}{4}\ \frac{1}{2}]$
17. [.5 .4 .1]; [.44 .43 .13]; [.431 .43 .139]; [.4292 .4291 .1417]
19. $\begin{array}{c c} & \begin{array}{cc} R & N \end{array} \\ \begin{array}{c} R \\ N \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \end{array}$ 21. 0.45
23. $\begin{array}{c c} & \begin{array}{ccc} A & F & V \end{array} \\ \begin{array}{c} A \\ F \\ V \end{array} & \begin{bmatrix} 0 & .7 & .3 \\ .6 & 0 & .4 \\ .8 & .2 & 0 \end{bmatrix} \end{array}$
25. [.3928 .37 .2372]
27. [46/113 38/113 29/113]
29. $\begin{array}{c c} & \begin{array}{cc} r & u \end{array} \\ \begin{array}{c} r \\ u \end{array} & \begin{bmatrix} .7 & .3 \\ .1 & .9 \end{bmatrix} \end{array}$; [1/4 3/4]
31. $[\frac{1}{14}\ \frac{3}{14}\ \frac{5}{7}]$ 33. $[\frac{4}{7}\ \frac{2}{7}\ \frac{1}{7}]$
35. [49/100 42/100 9/100]

Chapter 7 Review Exercises (page 563)

1. (a) $\frac{5}{9}$ (b) $\frac{1}{3}$ (c) $\frac{2}{9}$
2. (a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{6}$ (d) $\frac{2}{3}$
3. (a) 3:4 (b) 4:3 4. (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$
5. (a) $\frac{3}{8}$ (b) $\frac{1}{8}$ (c) $\frac{3}{8}$ 6. $\frac{2}{13}$ 7. 16/169
8. $\frac{3}{4}$ 9. $\frac{2}{13}$ 10. $\frac{7}{13}$ 11. (a) $\frac{2}{9}$ (b) $\frac{2}{3}$ (c) $\frac{7}{9}$
12. $\frac{2}{7}$ 13. $\frac{1}{2}$ 14. $\frac{8}{15}$ 15. (a) $\frac{3}{14}$ (b) $\frac{4}{7}$ (c) $\frac{3}{8}$
16. 30 17. 35 18. 26^3 19. $\frac{5}{8}$ 20. $\frac{29}{50}$
21. $\frac{5}{56}$ 22. $\frac{33}{56}$ 23. $\frac{15}{22}$ 24. $4! = 24$
25. ${}_8P_4 = 1680$ 26. ${}_{12}C_4 = 495$ 27. ${}_8C_4 = 70$
28. (a) ${}_{12}C_2 = 66$ (b) ${}_{12}C_3 = 220$ 29. 62,193,780
30. If her assumption about blood groups is accurate, there would be $4 \cdot 2 \cdot 4 \cdot 8 = 256$, not 288, unique groups.
31. (a) 63/2000 (b) $\frac{60}{63}$ 32. 39/116
33. $\frac{1}{24}$ 34. $\frac{3}{1250}$ 35. $\frac{3}{500}$
36. (a) .3398 (b) .1975 37. $\frac{1}{10}$
38. (a) $({}_{10}C_5)({}_2C_1)/{}_{12}C_6$
(b) $\frac{({}_{10}C_5)({}_2C_1) + ({}_{10}C_4)({}_2C_2)}{{}_{12}C_6}$

39. [.135 .51 .355], [.09675 .3305 .57275], [.0640875 .288275 .6476375]
40. [12/265 68/265 37/53]

Chapter 7 Test (page 565)

1. (a) $\frac{4}{7}$ (b) $\frac{3}{7}$ 2. (a) $\frac{2}{7}$ (b) $\frac{5}{7}$
3. (a) 0 (b) 1 4. $\frac{1}{7}$ 5. $\frac{1}{7}$
6. (a) $\frac{2}{7}$ (b) $\frac{4}{7}$ 7. $\frac{2}{7}$ 8. $\frac{3}{7}$ 9. $\frac{2}{3}$
10. 1/17,576 11. .2389 12. (a) $\frac{1}{5}$ (b) $\frac{1}{20}$
13. (a) $\frac{3}{95}$ (b) $\frac{6}{19}$ (c) $\frac{21}{38}$ (d) 0
14. 1/5,245,786
15. (a) 2,118,760 (b) 1/2,118,760
16. .064 17. (a) .633 (b) .962 18. .229
19. (a) $\frac{1}{5}$ (b) $\frac{1}{14}$ (c) $\frac{13}{14}$ 20. $\frac{3}{14}$
21. (a) 2^{10} (b) $\frac{1}{2^{10}}$ (c) $\frac{1}{3}$ (d) Change the code.
22. (a) $A = \begin{bmatrix} .80 & .20 \\ .07 & .93 \end{bmatrix}$ (b) [.25566 .74434]
(c) $\frac{7}{27}$; 25.9% of market

Exercise 8.1 (page 575)

1. .0595 3. (a) $\frac{1}{64}$ (b) $\frac{5}{16}$ (c) $\frac{15}{64}$
5. .0284 7. (a) .2304 (b) .0102 (c) .3174
9. .0585 11. .2759 13. (a) .375 (b) .0625
15. (a) .1157 (b) .4823
17. (a) $\frac{27}{64}$ (b) $\frac{27}{128}$ (c) $\frac{81}{256}$
19. (a) .0729 (b) .5905 (c) .9914
21. .2457 23. .0007
25. (a) .1323 (b) .0308
27. (a) .9044 (b) .0914 (c) .0043
29. (a) .8683 (b) .2099 31. .740

Exercise 8.2 (page 583)

1. no; $\Pr(x) \not\geq 0$ 3. yes 5. yes
7. no; $\Sigma \Pr(x) > 1$ 9. $\frac{15}{8}$ 11. 5 13. $\frac{8}{3}$ 15. 2
17. $\mu = \frac{13}{8}$, $\sigma^2 = 1.48$, $\sigma = 1.22$
19. $\mu = 4.2$, $\sigma^2 = 9.56$, $\sigma = 3.09$
21. $\mu = \frac{13}{3}$, $\sigma^2 = 2.22$, $\sigma = 1.49$
23. 3 25. 2 27. 1.85
29. TV, 37,500; P.A., 35,300 31. 2
33. 0 35. −\$0.39 37. Expect to lose \$2 each time.
39. If he buys 0, his profit is 0. If the buys 100, his expected profit is \$3(100)(.25) + \$3(50)(.20) + \$3(10)(.55) + \$(−1) 50(.20) + \$(−1)90(.55) = \$62. If he buys 200, his expected profit is \$3(180)(.25) + \$3(50)(.20) + \$3(10)(.55) + (−\$1)(20)(.25) + (−\$1)(150)(.20) + (−\$1)(190)(.55) = \$42. He should buy 100.
41. TV: 37,500; newspapers: 39,200; newspapers
43. \$10

Exercise 8.3 (page 590)

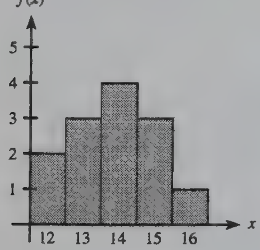
1. (a)

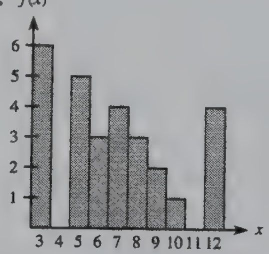
x	$\Pr(x)$
0	125/216
1	25/72
2	5/72
3	1/216

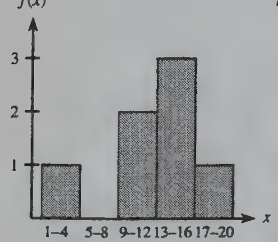
(b) $3(\frac{1}{6}) = \frac{1}{2}$ (c) $\sqrt{3(\frac{1}{6})(\frac{5}{6})} = (\frac{1}{6})\sqrt{15}$

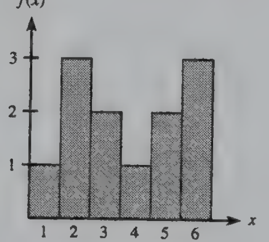
3. (a) 42 (b) 3.55 5. 2, 1.29
7. 4 9. 2 11. 2 13. 15
15. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$
17. $x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$
19. (a) $100(.10) = 10$ (b) $\sqrt{100(.10)(90)} = 3$
21. (a) 60,000 (b) $\sqrt{24{,}000} = 154.919$
23. 59,690 25. (a) 4 (b) 1.79
27. 2, 1.41 29. 300

Exercise 8.4 (page 601)

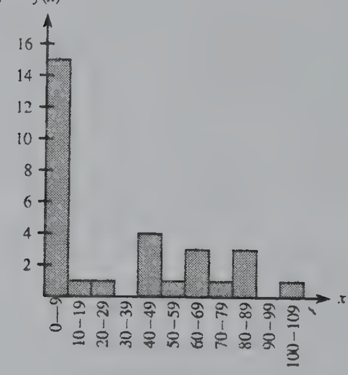
1.


3.


5.


7.


9. 3 11. 13 13. 2 15. 1
17. mode = 2, median = 4.5, mean = 6
19. mode = 17, median = 18.5, mean = 23.5
21. mode = 5.3, median = 5.3, mean = 5.32
23. 12.21, 14.5, 14.5 25. 9 27. 14
29. 4, 8.5714, 2.9277 31. 14, 4.6667, 2.1602
33. 2.73, 1.35 35. 6.75, 2.96

37.


Total Executions in U.S. 1950–79

39. (a) \$30,000 (b) \$18,000 (c) \$16,000
41. The mean will give the highest measure.
43. The median will give the most representative average.
45. 33,000 47. 3.32, 0.677
49. (a) 6.32% (b) 8.93% (c) 7.625% (d) 2.69%
51.

Gold Prod.	1249.1	thousand oz
Gold Sales	1233.9	thousand oz
Revenue	423	\$/oz
Aver. price	355.7	\$/oz
Oper. cost	179.7	\$/oz
Net income	138.3	\$/oz

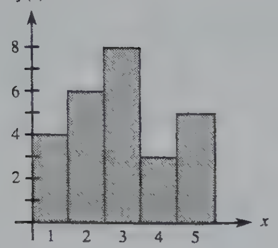
53. $\bar{x} = 159.8$, $s = 17.123$

Exercise 8.5 (page 612)

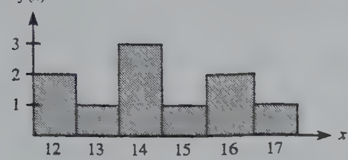
1. 0.4641 3. 0.4641 5. 0.9153 7. 0.1070
9. 0.0166 11. 0.0227 13. 0.8849 15. 0.1915
17. 0.3944 19. 0.5381 21. 0.2957 23. 0.3446
25. 0.7258 27. (a) 0.3413 (b) 0.3944
29. 0.9876
31. (a) 0.4192 (b) 0.0227 (c) 0.0581
(d) 0.8965
33. (a) 0.0668 (b) 0.3085 (c) 0.3830
35. (a) 0.0475 (b) 0.2033 (c) 0.5934
37. (a) 0.0227 (b) 0.1587 (c) 0.8186

Chapter 8 Review Exercises (page 614)

1. $\frac{11}{27}$ 2. (a) $(99{,}999/100{,}000)^{99{,}999} \approx 0.37$
(b) $1 - (99{,}999/100{,}000)^{100{,}000} \approx 0.63$
3. .2048 4. 2.4 5. yes 6. no; $\Sigma \Pr(x) \neq 1$
7. yes 8. no; $\Pr(x) \not\geq 0$ 9. 2
10. (a) 4.125 (b) 2.7344 (c) 1.654
11. (a) $\frac{37}{12}$ (b) .9097 (c) .9538
12. $\mu = 4$, $\sigma = (2\sqrt{3})/3$ 13. (a) 1 (b) $(\frac{4}{5})^4$
14. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

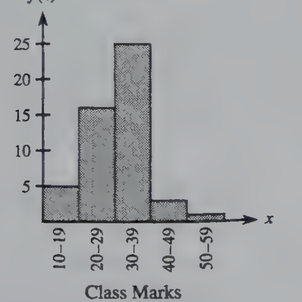
15. 

16. 3
17. $\frac{77}{26} = 2.96$
18. 3

19. 

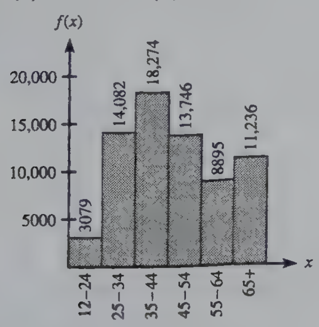
20. 14
21. 14
22. 14.3

23. $\bar{x} = 3.86$; $s^2 = 6.81$; $s = 2.61$
24. $\bar{x} = 2$; $s^2 = 2.44$; $s = 1.56$
25. .9165 26. .1498 27. .1039 28. .3413
29. .6826 30. .1360 31. .297 32. .16308
33. \$−.50 34. 455 35. 3 36. \$18.00

37. 

38. 30.3%
39. 8.35%
40. (a) .4773
(b) .1360
(c) .0227
41. 15%

Chapter 8 Test (page 616)

1. (a) $\frac{40}{243}$ (b) $\frac{51}{243}$ 2. 4
3. $\mu = 4$, $\sigma^2 = \frac{8}{3}$, $\sigma = \frac{2}{3}\sqrt{6}$ 4. 5.1
5. $\mu = 16.7$, $\sigma^2 = 26.61$, $\sigma = 5.16$
6. $\mu = 21.57$, median = 21, mode = 21
7. (a) .4706 (b) .8413 (c) .0669
8. (a) .3891 (b) .5418 (c) .1210
9. 

10. $\bar{x} = 46.48$, $s = 15.40$

11. 34.47
12. 1980, 6.4 yr; 1994, 8.1 yr; Higher prices, better quality
13. (a) .00003 (b) 30 14. 2 (1.8) 15. 5 (5.4)
16. 0 (.054) with correct use; 1 (.76) with typical use
17. (a) .0158 (b) .0901 (c) .5383

Exercise 9.1 (page 637)

1. (a) 1 (b) 1 3. (a) −8 (b) −8
5. (a) 10 (b) does not exist 7. (a) 0 (b) −6
9. (a) does not exist ($+\infty$) (b) does not exist ($+\infty$)
(c) does not exist ($+\infty$) (d) does not exist
11. (a) 3 (b) −6 (c) does not exist (d) −6

13.

x	$f(x)$
1.9	4.1579
1.99	4.0151
1.999	4.0015
↓	↓
2	4
↑	↑
2.001	3.9985
2.01	3.9851
2.1	3.8571

$\lim_{x\to 2} f(x) = 4$

15.

x	$f(x)$
0.9	−2.9
0.99	−2.99
0.999	−2.999
↓	↓
1	−3
↑	↑
1.001	−3.001
1.01	−3.01
1.1	−3.1

$\lim_{x\to 1} f(x) = -3$

17.

x	$f(x)$
0.9	3.5
0.99	3.95
0.999	3.995
↓	↘ 4
1	Different
↑	↗ 5
1.001	4.995999
1.01	4.9599
1.1	4.59

$\lim_{x\to 1} f(x)$ does not exist

19. −1 21. −4 23. −2 25. 6
27. 3/4 29. 0 31. does not exist
33. −3 35. does not exist 37. does not exist
39. $3x^2$ 41. −2 43. $\frac{1}{30}$ 45. does not exist
47. 5/7 49. −4 51. 9

53.

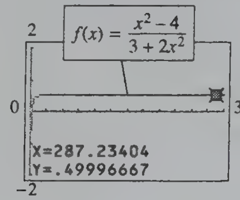
a	$(1 + a)^{1/a}$
0.1	2.5937
0.01	2.7048
0.001	2.7169
0.0001	2.7181
0.00001	2.71827
↓	↓
0	≈2.718

55. (a) 2 (b) 6 (c) −8 (d) $-\frac{1}{2}$

57. \$150,000
59. (a) \$32 (thousands) (b) \$55.04 (thousands)
61. (a) \$2800 (b) \$700 (c) \$560
63. (a) 1.52 units/hr (b) 0.85 units/hr (c) lunch
65. (a) 0; $p \to 100^-$ means the water approaches not being treated (containing 100% or all of its impurities); the associated costs of nontreatment approach zero.
(b) ∞ (c) No, because $C(0)$ is undefiined.
67. (a) \$3697.50 (b) \$3697.50 (c) \$3697.50
69. \$1091.70
71. 7780 (approximately) This corresponds to the Dow Jones opening average on October 5, 1998.
73. (a) -1.449 (approximately)
(b) This predicts the percentage of U.S. workers in farm occupations as the year approaches 2000.
(c) The model is inaccurate because the percentage must be 0 or positive; it cannot be negative.

Exercise 9.2 (page 651)

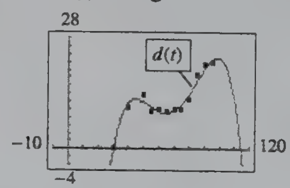
1. (a) continuous
(b) discontinuous; $f(1)$ does not exist
(c) discontinuous; $\lim_{x\to 3} f(x)$ does not exist
(d) discontinuous; $f(0)$ does not exist and $\lim_{x\to 0} f(x)$ does not exist
3. continuous 5. continuous
7. discontinuous; $f(-3)$ does not exist
9. discontinuous; $f(-1)$ and $\lim_{x\to -1} f(x)$ do not exist
11. continuous
13. discontinuous; $\lim_{x\to 1} f(x)$ does not exist
15. continuous
17. discontinuity at $x = -2$; $g(-2)$ and $\lim_{x\to -2} g(x)$ do not exist
19. continuous 21. continuous
23. discontinuity at $x = -1$; $f(-1)$ does not exist
25. discontinuity at $x = 3$; $\lim_{x\to 3} f(x)$ does not exist
27. vertical asymptote: $x = -2$; $\lim_{x\to +\infty} f(x) = 0$; $\lim_{x\to -\infty} f(x) = 0$
29. vertical asymptotes: $x = -2, x = 3$; $\lim_{x\to +\infty} f(x) = 2$; $\lim_{x\to -\infty} f(x) = 2$
31. 0 33. 1 35. $\frac{5}{3}$ 37. does not exist ($+\infty$)
39. (a)



$\lim_{x\to +\infty} f(x) = 0.5$

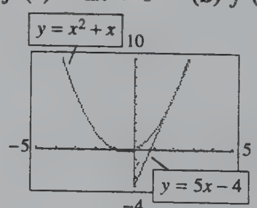
(b) The table indicates $\lim_{x\to +\infty} f(x) = 0.5$.

41. (a) $x = -1000$ (b) $y = 1000$
(c) These values are so large that experimenting with windows to discover asymptotes may never locate them.
43. (a) no, not at $p = -8$ (b) yes (c) yes
(d) $p > 0$
45. (a) yes, $q = -1$ (b) yes
47. (a) R/i (b) \$10,000 49. yes, $0 \le p \le 100$
51. 100%; No, for p to approach 100% (as a limit) requires spending to increase without bound, which is impossible.
53. $R(x)$ is discontinuous at $x = 41{,}200$; $x = 99{,}600$; $x = 151{,}750$; and $x = 271{,}050$.
55. (a) \$79.40 (b) $\lim_{x\to 100} C(x) = 19.40$; $\lim_{x\to 500} C(x) = 49.40$ (c) yes
57. (a) Fourth degree is better.
$d(t) = -0.00001543t^4 + 0.004274t^3 - 0.4215t^2 + 17.58t - 251.274$
(b) 19.75% (c) $-\infty$
(d) No, $d(t)$ is negative for t-values greater than 114.
(e)

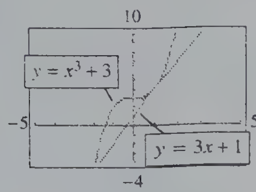


Exercise 9.3 (page 667)

1. (a) 32 (b) 32 (c) (4, 64)
3. (a) verification (b) -5 (c) -5 (d) $(-1, 3)$
5. (a) $P(1, 1), A(3, 0)$ (b) $-\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) $-\frac{1}{2}$
7. (a) $P(1, 3), A(0, 3)$ (b) 0 (c) 0 (d) 0
9. (a) $f'(x) = -6$ (b) -6; -6 (c) -6
11. (a) $f'(x) = 8x - 2$ (b) $8x - 2$; -26 (c) -26
13. (a) $p'(q) = 2q + 4$ (b) $2q + 4$; 14 (c) 14
15. (a) 89.000024 (b) 89.0072 (c) ≈ 89
17. (a) 294.000008 (b) 294.0084 (c) ≈ 294
19. -31 21. $f'(-3) = \frac{5}{2}$; $f(-3) = -9$
23. $y = 3x - 4$
25. (a) a, b, d (b) c (c) A, C, E
27. (a) A, B, C, D (b) A, D
29. (a) $f'(x) = 2x + 1$ (b) $f'(2) = 5$ (c) $y = 5x - 4$
(d)



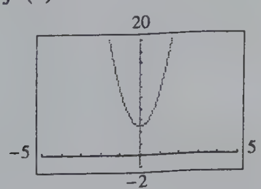
31. (a) $f'(x) = 3x^2$ (b) $f'(1) = 3$ (c) $y = 3x + 1$
(d)



33. (a) −32 ft/s (b) the ball is falling, so S is decreasing
35. (a) $\frac{-100}{3}$ (b) $\frac{-4}{3}$ 37. 68 mph
39. (a) $R'(x) = \overline{MR} = 300 - 2x$ (b) 200
(c) −100 (d) 0
(e) It changes from increasing to decreasing
41. 200
43. (a) 100; the expected profit from the sale of the 201st car is \$100.
(b) −100; the expected profit from the sale of the 301st car is a loss of \$100.
45. (a) 1.039
(b) If humidity changes by 1%, the heat index will change by about 1.039°F.
47. (a) 1.164 million infections per year (approximately)
(b) Low: 3.6 million infections per year
High: 20 million infections per year

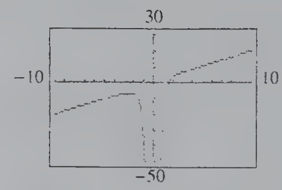
Exercise 9.4 (page 680)

1. $y' = 0$ 3. $y' = 1$ 5. $f'(x) = 6x^2 - 5x^4$
7. $y' = 24x^3 - 10x + 1$
9. $g'(x) = 90x^8 - 25x^4 + 21x^2 + 5$
11. (a) 19 (b) 19 13. (a) 0 (b) 0
15. $y' = -5x^{-6} - 8x^{-9}$
17. $y' = 11x^{8/3} - \frac{7}{2}x^{3/4} - \frac{1}{2}x^{-1/2}$
19. $f'(x) = -4x^{-9/5} - \frac{8}{3}x^{-7/3}$
21. $g'(x) = -\dfrac{12}{x^5} - \dfrac{10}{x^6} + \dfrac{5}{3\sqrt[3]{x^2}}$
23. $y = -3x + 6$ 25. $y = 3$
27. $(1, -1), (5, 31)$ 29. $(0, 9), (3, -18)$
31. (a) −1/2 (b) −0.5000 (to four decimal places)
33. (a) $f'(x) = 6x^2 + 5$
(b)



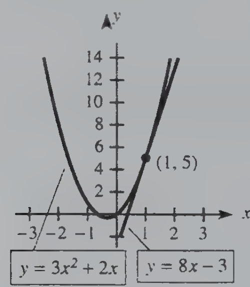
Graph of $f'(x)$ and numerical derivative of $f(x)$

35. (a) $h'(x) = \dfrac{-30}{x^4} + \dfrac{4}{\sqrt[5]{x^7}} + 2x$
(b)



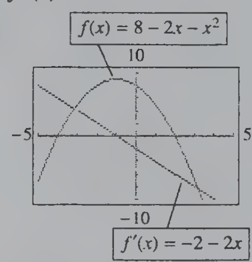
Graph of $h'(x)$ and numerical derivative of $h(x)$

37. (a) $y = 8x - 3$
(b)

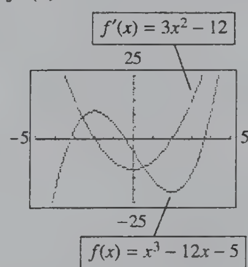


(c) x:0.7 → 1.6; y:3.0 → 7.9

39. (a) $f'(x) = -2 - 2x$
(b)



(c) $f'(x) = 0$ at $x = -1$; $f'(x) > 0$ for $x < -1$
$f'(x) < 0$ for $x > -1$
(d) $f(x)$ has max when $x = -1$
$f(x)$ rises for $x < -1$
$f(x)$ falls for $x > -1$
41. (a) $f'(x) = 3x^2 - 12$
(b)



(c) $f'(x) = 0$ at $x = -2$ and $x = 2$
$f'(x) > 0$ for $x < -2$ and $x > 2$
$f'(x) < 0$ for $-2 < x < 2$
(d) $f(x)$ has max when $x = -2$, min when $x = 2$
$f(x)$ rises when $x < -2$ and when $x > 2$
$f(x)$ falls when $-2 < x < 2$
43. (a) 40, revenue increasing
(b) −20, revenue decreasing
45. (a) 920 (b) 926

47. **(a)** -4; if the price changes to \$26, the quantity demanded will change by approximately -4 units
(b) $-\frac{1}{2}$; if the price changes to \$101, the quantity demanded will change by approximately $-\frac{1}{2}$ unit

49. **(a)** $\overline{C(x)}' = (-4000/x^2) + 0.1$ **(b)** 200

51. **(a)** $-120{,}000$
(b) If the impurities change from 1% to 2%, then the expected change in cost is $-120{,}000$ (dollars).

53. **(a)** $WC = 55.174 - 2.15s - 23.18\sqrt{s}$
(b) -4.468
(c) If the wind speed changes by + 1 mph (to 26 mph), then the wind chill will change by approximately -4.468°F.

55. **(a)** $u(x) = 0.0003836x^3 - 0.08853x^2 + 6.2132x - 105.247$
(b) -0.78% per year
(c) $u'(80) \approx -0.59\%$ per year

57. **(a)** $f(t) = 0.030847t^3 - 8.49779t^2 + 779.66274t - 23{,}820.28$
(b) $f'(t) = 0.092541t^2 - 16.99558t + 779.66274$
(c) $f'(89) \approx 0.07$; $f'(94) \approx -0.23$
(d) $f'(89) \approx 0.07$ means that from 1989 to 1990, the consumer price index rose about 0.07%. $f'(94) \approx -0.23$ means that from 1994 to 1995, the consumer price index fell about 0.23%.

Exercise 9.5 *(page 690)*

1. $y' = 3x^2 + 2x - 6$ **3.** $dp/dq = 9q^2 - 2q + 6$

5. $f'(x) = (x^{12} + 3x^4 + 4)(12x^2) + (4x^3 - 1)(12x^{11} + 12x^3)$

7. $y' = (7x^6 - 5x^4 + 2x^2 - 1)(36x^8 + 21x^6 - 10x + 3) + (4x^9 + 3x^7 - 5x^2 + 3x)(42x^5 - 20x^3 + 4x)$

9. $y' = (x^2 + x + 1)(\frac{1}{3}x^{-2/3} - x^{-1/2}) + (x^{1/3} - 2x^{1/2} + 5)(2x + 1)$

11. **(a)** 40 **(b)** 40 **13.** $y' = (-x^2 - 1)/(x^2 - 1)^2$

15. $\dfrac{dp}{dq} = (q^2 - 4q - 1)/(q - 2)^2$

17. $\dfrac{dy}{dx} = \dfrac{4x^5 - 4x^3 - 16x}{(x^4 - 2x^2 + 5)^2}$

19. $\dfrac{dz}{dx} = 2x + \dfrac{2x - x^2}{(1 - x - 2x^2)^2}$

21. $\dfrac{dp}{dq} = \dfrac{2q + 1}{\sqrt[3]{q^2}\,(1 - q)^2}$

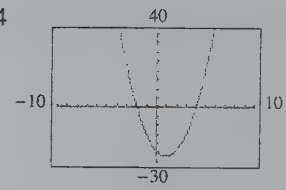
23. $y' = \dfrac{2x^3 - 6x^2 - 8}{(x - 2)^2}$

25. **(a)** $\frac{3}{5}$ **(b)** $\frac{3}{5}$ **27.** $y = 44x - 32$

29. $y = \frac{10}{3}x - \frac{10}{3}$ **31.** 104

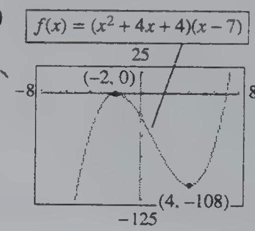
33. 1.3333 (to four decimal places)

35. **(a)** $f'(x) = 3x^2 - 6x - 24$

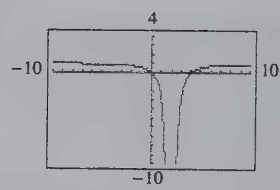


Graph of both $f'(x)$ and numerical derivative of $f(x)$

(b) Horizontal tangents where $f'(x) = 0$; at $x = -2$ and $x = 4$
(c)

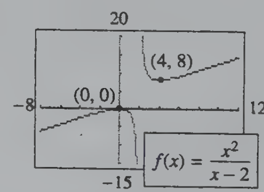


37. **(a)** $y' = \dfrac{x^2 - 4x}{(x - 2)^2}$

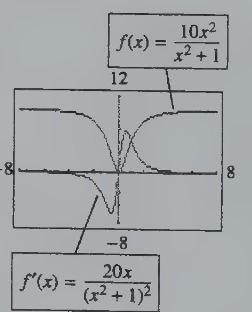


Graph of both y' and the numerical derivative of y

(b) Horizontal tangents where $y' = 0$; at $x = 0$ and $x = 4$
(c) (0, 0) (4, 8)



39. **(a)** $f'(x) = \dfrac{20x}{(x^2 + 1)^2}$ **(b)**



(c) $f' = 0$ at $x = 0$
$f' > 0$ for $x > 0$
$f' < 0$ for $x < 0$
(d) f has a min at $x = 0$
f increasing for $x > 0$
f decreasing for $x < 0$

41.
$$f'(x) = \lim_{h\to 0} \frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h}$$
$$= \lim_{h\to 0} \frac{u(x+h)\,v(x) - u(x)\,v(x+h)}{h\cdot v(x)\,v(x+h)}$$
$$= \lim_{h\to 0} \frac{u(x+h)\,v(x) - u(x)\,v(x) + u(x)v(x) - u(x)v(x+h)}{h\cdot v(x)\,v(x+h)}$$
$$= \lim_{h\to 0} \frac{v(x)\left[\dfrac{u(x+h) - u(x)}{h}\right] - u(x)\left[\dfrac{v(x+h) - v(x)}{h}\right]}{v(x)\,v(x+h)}$$
$$= \frac{v(x)\,u'(x) - u(x)\,v'(x)}{[v(x)]^2}$$

43. $C'(p) = 810{,}000/(100 - p)^2$

45. $R'(49) \approx 30.00$ The expected revenue from the sale of the next unit (the 50th) is about \$30.00.

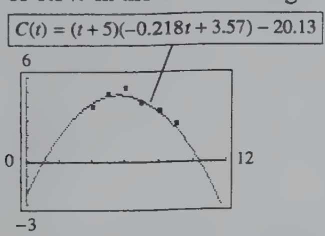
47. $R'(5) = -50$ As the group changes by 1 person (to 31), the revenue will drop by about \$50.

49. $S = 1000x - x^2$ 51. $\dfrac{dR}{dn} = \dfrac{r(1-r)}{[1+(n-1)r]^2}$

53. (a) $P'(6) \approx 0.045$ During the next (7th) month of the campaign, the proportion of voters who recognize the candidate will change by about 0.045, or 4.5%.
(b) $P'(12) \approx -0.010$ During the next (13th) month of the campaign, the proportion of voters who recognize the candidate will drop by about 0.010, or 1%.
(c) It is better for $P'(t)$ to be positive—that is, to have increasing recognition.

55. (a) $f'(20) \approx -0.79$
(b) At 0°F, if the wind speed changes by 1 mph (to 21 mph), the wind chill will change by about −0.79°F.

57. (a) $C'(5) = 0.3$
(b) From 1990 to 1991, the model predicts a change of 0.3% in the cost of living.
(c)



(d) The model fails to be valid when $C(t) < 0$, below the t-axis.

59. (a) $f'(t) =$
$$1000\left[\frac{8.88543219t^2 - 3423.843248t + 16{,}731.97758}{(1.09816t^2 - 122.183t + 21{,}472.6)^2}\right]$$
(b) $f'(70) \approx -0.5356$ $f'(170) \approx -0.2932$
(c) $f'(70)$ means that from 1870 to 1871, the model predicts a change of about −0.5356% in U.S. workers in farm occupations. $f'(170)$ means that from 1970 to 1971, the model predicts a change of about −0.2932% in U.S. workers in farm occupations.

Exercise 9.6 (page 698)

1. $6x(x^2+1)^2$ 3. $4(8x-1)(4x^2-x+8)^3$

5. $-6x/(x^2+2)^4$ 7. $-4(x+2)(x^2+4x)^{-3}$

9. $-3(2x+3)(x^2+3x+4)^{-4}$

11. $\dfrac{-3(6x^2+3)}{4(2x^3+3x+5)^{7/4}}$ 13. $(x+2)/\sqrt{x^2+4x+5}$

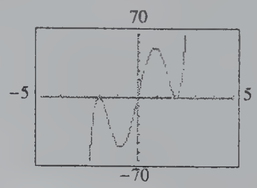
15. $(6-4x)/\sqrt{3x-x^2}$ 17. $16x(x^2-3)^4$

19. $\dfrac{15(3x+1)^4 - 3}{7}$ 21. (a) and (b) 96,768

23. (a) and (b) 2 25. $y = -3x + 4$

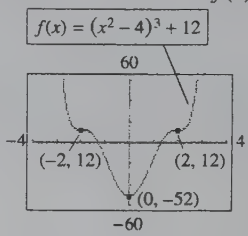
27. $9x - 5y = 2$

29. (a) $f'(x) = 6x(x^2-4)^2$
(b)

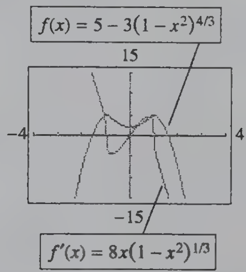


Graph of both $f'(x)$ and numerical derivative of $f(x)$

(c) $x = 0, x = 2, x = -2$
(d) $(0, -52), (2, 12), (-2, 12)$
(e) $f(x) = (x^2-4)^3 + 12$



31. (a) $f'(x) = 8x(1-x^2)^{1/3}$
(b)



(c) $f'(x) = 0$ at $x = -1, x = 0, x = 1$
$f'(x) > 0$ for $x < -1$ and $0 < x < 1$
$f'(x) < 0$ for $-1 < x < 0$ and $x > 1$
(d) $f(x)$ has a maximum at $x = -1$ and $x = 1$, a minimum at $x = 0$
$f(x)$ increasing for $x < -1$ and $0 < x < 1$
$f(x)$ decreasing for $-1 < x < 0$ and $x > 1$

33. (a) $y' = 2x^2$ (b) $y' = -2/x^4$
(c) $y' = 2(2x)^2$ (d) $y' = \dfrac{-18}{(3x)^4}$

35. 10 ft/sec 37. \$1499.85 (approximately)

39. (a) −0.114 (approximately)
(b) If the price changes by \$1, to \$22, the weekly sales volume will change by approximately −0.114 thousand unit.

41. (a) −\$3.20 per unit
(b) If the quantity demanded changes from 49 to 50 units, the change in price will be about −\$3.20.

43. $\dfrac{dy}{dx} = \left(\dfrac{8k}{5}\right)(x - x_0)^{3/5}$

45. $\dfrac{dp}{dq} = \dfrac{-100}{(2q+1)^{3/2}}$ 47. $\dfrac{dK_c}{dv} = \dfrac{8}{\sqrt{4v+1}}$

49. **(a)** \$658.75. If the interest changed from 6% to 7%, the amount of the investment would change by about \$658.75
(b) \$2156.94. If the interest rate changed from 12% to 13%, the amount of the investment would change by about \$2156.94.

51. $d'(30) \approx -0.170$ means from 1960 to 1961, the model predicts that the interest paid as a percent of federal expenditures would change by -0.17%. $d'(60) \approx 0.58$ means from 1990 to 1991, the model predicts that the interest paid as a percent of federal expenditures would change by 0.58%.

53. $p'(10) \approx -0.42$ means from 1970 to 1971, the model predicts that the number of persons below the poverty level would change by -0.42 million (a decrease). $p'(30) \approx 0.58$ means from 1990 to 1991, the model predicts that the number of persons below the poverty level would change by 0.58 million (an increase).

Exercise 9.7 *(page 706)*

1. 0 **3.** $4(-4x^{-5})$; $-16/x^5$

5. $15x^2 + 4(-x^{-2})$; $15x^2 - 4/x^2$

7. $(x^2-2)1 + (x+4)(2x)$; $3x^2 + 8x - 2$

9. $\dfrac{x^2(3x^2) - (x^3+1)(2x)}{(x^2)^2}$; $(x^3-2)/x^3$

11. $(3x^2-4)(x^3-4x)^9$

13. $\frac{5}{3}x^3[3(4x^5-5)^2(20x^4)] + (4x^5-5)^3(5x^2)$; $5x^2(4x^5-5)^2(24x^5-5)$

15. $(x-1)^2(2x) + (x^2+1)2(x-1)$; $2(x-1)(2x^2-x+1)$

17. $\dfrac{(x^2+1)\,3(x^2-4)^2(2x) - (x^2-4)^3(2x)}{(x^2+1)^2}$; $\dfrac{2x(x^2-4)^2(2x^2+7)}{(x^2+1)^2}$

19. $3[(q+1)(q^3-3)]^2[(q+1)3q^2 + (q^3-3)1]$; $3(4q^3+3q^2-3)[(q+1)(q^3-3)]^2$

21. $4[x^2(x^2+3x)]^3[x^2(2x+3) + (x^2+3x)(2x)]$; $4x^2(4x+9)[x^2(x^2+3x)]^3$

23. $4\left(\dfrac{2x-1}{x^2+x}\right)^3\left[\dfrac{(x^2+x)2-(2x-1)(2x+1)}{(x^2+x)^2}\right]$; $\dfrac{4(-2x^2+2x+1)(2x-1)^3}{(x^2+x)^5}$

25. $(8x^4+3)^2 3(x^3-4x)^2(3x^2-4) + (x^3-4x)^3 2(8x^4+3)(32x^3)$; $(8x^4+3)(x^3-4x)^2(136x^6-352x^4+27x^2-36)$

27. $\dfrac{(4-x^2)\frac{1}{3}(x^2+5)^{-2/3}(2x) - (x^2+5)^{1/3}(-2x)}{(4-x^2)^2}$; $\dfrac{2x(2x^2+19)}{3\sqrt[3]{(x^2+5)^2}\,(4-x^2)^2}$

29. $(x^2)\frac{1}{4}(4x-3)^{-3/4}(4) + (4x-3)^{1/4}(2x)$; $(9x^2-6x)/\sqrt[4]{(4x-3)^3}$

31. $(2x)\frac{1}{2}(x^3+1)^{-1/2}(3x^2) + (x^3+1)^{1/2}(2)$; $(5x^3+2)/\sqrt{x^3+1}$

33. **(a)** $F_1'(x) = 12x^3(x^4+1)^4$
(b) $F_2'(x) = \dfrac{-12x^3}{(x^4+1)^6}$
(c) $F_3'(x) = 12x^3(3x^4+1)^4$
(d) $F_4'(x) = \dfrac{-300x^3}{(5x^4+1)^6}$

35. $dP/dx = 90(3x+1)^2$

37. **(a)** \$59,900
(b) It is increasing.

39. $C'(y) = 1/\sqrt{y+1} + 0.4$

41. $dV/dx = 144 - 96x + 12x^2$

43. **(a)** -1.6 This means that from the 9th to the 10th week, sales are expected to change by -1.6 (decrease).

45. **(a)** $f'(t) =$
$$1000\left[\frac{8.885432192\,(t+20)^2 - 3423.843248\,(t+20) - 16{,}731.9776}{[1.09816\,(t+20)^2 - 122.183\,(t+20) + 21472.6]^2}\right]$$
(b) 1850: $f'(30) \approx -0.4033$
1950: $f'(130) \approx -0.3827$
(c) $f'(30)$ means that from 1850 to 1851, the model predicts a change of $-.4033\%$ in U.S. workers in farm occupations.
$f'(130)$ means that from 1950 to 1951, the model predicts a change of -0.3827% in U.S. workers in farm occupations.

Exercise 9.8 *(page 712)*

1. $24x - 30$ **3.** $60x - 2$ **5.** $6x - 2x^{-3}$

7. $6x + \frac{1}{4}x^{-3/2}$ **9.** $60x^2 - 96$

11. $1008x^6 - 720x^3$ **13.** $-6/x^4$ **15.** $\frac{3}{8}x^{-5/2}$

17. $20x^3 + \frac{1}{4}x^{-3/2}$ **19.** $\frac{3}{8}(x+1)^{-5/2}$ **21.** 0

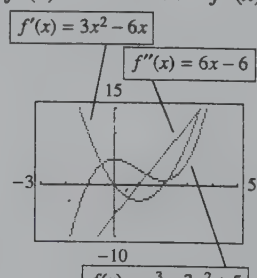
23. $-15/(16x^{7/2})$ **25.** $\frac{3}{8}(x-1)^{-5/2}$

27. $-2(x+1)^{-3}$ **29.** 26

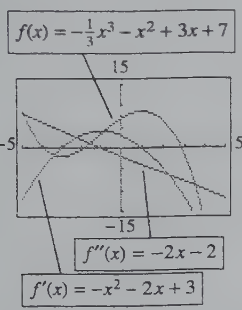
31. 16.0000 (to four decimal places)

33. 0.0004261

35. **(a)** $f'(x) = 3x^2 - 6x$ $f''(x) = 6x - 6$
(b)



(c) $f''(x) = 0$ at $x = 1$
$f''(x) > 0$ for $x > 1$
$f''(x) < 0$ for $x < 1$
(d) $f'(x)$ has a min at $x = 1$
$f'(x)$ is increasing for $x > 1$
$f'(x)$ is decreasing for $x < 1$
(e) $f''(x) < 0$ (f) $f''(x) > 0$

37. (a) $f'(x) = -x^2 - 2x + 3$ $f''(x) = -2x - 2$
(b)


(c) $f''(x) = 0$ at $x = -1$
$f''(x) > 0$ for $x < -1$
$f''(x) < 0$ for $x > -1$
(d) $f'(x)$ has a max at $x = -1$
$f'(x)$ is increasing for $x < -1$
$f'(x)$ is decreasing for $x > -1$
(e) $f''(x) < 0$ (f) $f''(x) > 0$

39. $a = 0.12$ **41.** -0.02

43. (a) $\dfrac{dR}{dm} = mc - m^2$ (b) $\dfrac{d^2R}{dm^2} = c - 2m$
(c) Second

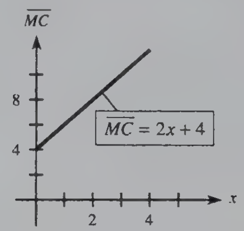
45. (a) 0.0009 (approximately)
(b) When 1 more unit is sold (beyond 25), the marginal revenue will change by about 0.0009 thousand dollars per unit, or \$0.90 per unit.

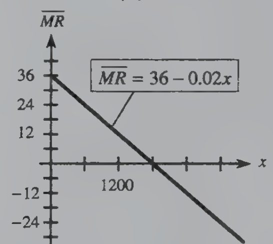
47. (a) $S' = \dfrac{-3}{(t+3)^2} + \dfrac{36}{(t+3)^3}$ (b) $S''(15) = 0$
(c) After 15 weeks, the rate of change of the rate of sales is zero because the rate of sales reaches a minimum value.

49. (a) $p'(t) = -0.005169t^2 + 0.2582t - 2.5015$
(b) 1970: $p''(10) = 0.155$
1990: $p''(30) = -0.052$
(c) $p'(10) = -0.44$ means that from 1970 to 1971, the number of people who lived below the poverty level was expected to change by about -0.44 million.
$p''(10) = 0.155$ means that from 1970 to 1971, the rate of change of the number of people who lived below the poverty level was expected to change by about 0.155 million per year.
Thus, from 1970 to 1971, the number of people who lived below the poverty level was increasing at an increasing rate.

51. (a) $f(t) = 0.030847t^3 - 0.44674t^2 + 1.48932t + 3.43357$
(b) $f'(t) = 0.092541t^2 - 0.89348t + 1.48932$
(c) 1991: $f''(4) \approx -0.153$
1995: $f''(8) \approx 0.587$
(d) $f'(8) \approx 0.264$ means that from 1995 to 1996, the expected change in the CPI was about 0.264%.
$f''(8) \approx 0.587$ means that from 1995 to 1996, the expected change in the rate of change of the CPI was about 0.587% per year.

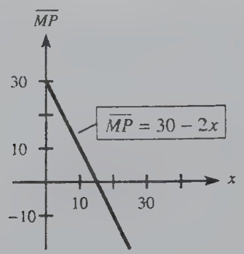
Exercise 9.9 *(page 721)*

1. $\overline{MC} = 8$ **3.** $\overline{MC} = 13 + 2x$
5. $\overline{MC} = 3x^2 - 12x + 24$ **7.** $\overline{MC} = 27 + 3x^2$
9. (a) \$10; the cost will increase by \$10. (b) \$11
11. \$46; the cost will increase by \$46.
13.


15. (a) $\overline{MR} = 4$
(b) The sale of each additional item brings in \$4 revenue at all levels of production.
17. (a) \$3500; this is revenue from the sale of 100 units.
(b) $\overline{MR} = 36 - 0.02x$
(c) \$34; Revenue will increase by \$34.
(d) Actual revenue from the sale of the 101st item is \$33.99.
19. (a) $\overline{MR} = 36 - 0.02x$ (b) $x = 1800$
(c) \$32,400


21. $\overline{MP} = 5$
23. (a) \$0 (b) $\overline{MP} = 30 - 2x$
(c) $-\$10$; profit will decrease by \$10 if 1 additional unit is sold.
(d) The sale of the 21st item results in a loss of \$11.

25. (a)



(b) 15
(c) 15
(d) \$25

27. 70 29. \$200

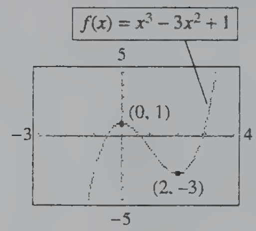
31. (a) Data: $R(7) = 60.53$ Model: $R(7) = 61.027$
$R(7)$ represents AT&T's total revenues (in billions of dollars) for 1987.
(b) $R'(t) = 0.506\,t - 4.03$
(c) $R'(12) = 2.042$
(d) $R'(12) = 2.042$ means that from 1992 to 1993, the model predicts that AT&T's revenue rose by about \$2.042 billion.

33. (a) $C'(10) = 0.252$
(b) This means that from 1990 to 1991, the model predicts that Scott Paper Company's costs and expenses will change by about \$0.252 billion. (The actual change for that period was \$0.0319 billion.)

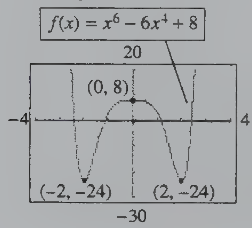
Chapter 9 Review Exercises (page 725)

1. (a) 2 (b) 2 2. (a) 0 (b) 0
3. (a) 2 (b) 1 4. (a) 2 (b) does not exist
5. (a) does not exist (b) 2
6. (a) does not exist (b) does not exist
7. 55 8. 0 9. −2 10. 6 11. $\frac{1}{2}$ 12. $\frac{1}{5}$
13. no limit 14. 0 15. 4 16. no limit
17. 3 18. no limit 19. $6x$ 20. $1 - 4x$
21. −14 22. 5 23. (a) yes (b) no
24. (a) yes (b) no 25. 2 26. no limit
27. 1 28. no 29. yes 30. yes
31. discontinuity at $x = 5$ 32. discontinuity at $x = 2$
33. continuous 34. discontinuity at $x = 1$
35. (a) $x = 0, x = 1$ (b) 0 (c) 0
36. (a) $x = -1, x = 0$ (b) $\frac{1}{2}$ (c) $\frac{1}{2}$
37. −2 38. 0 39. true 40. false
41. $f'(x) = 6x + 2$ 42. $f'(x) = 1 - 2x$
43. (a) no (b) no 44. (a) yes (b) no
45. (a) −5.9171 (to four decimal places) (b) −5.9
46. 7 47. $20x^4 - 18x^2$ 48. $8x$ 49. 3
50. $1/(2\sqrt{x})$ 51. 0 52. $-4/(3\sqrt[3]{x^4})$
53. $\dfrac{-1}{x^2} + \dfrac{1}{2\sqrt{x^3}}$ 54. $\dfrac{-3}{x^3} - \dfrac{1}{3\sqrt[3]{x^2}}$
55. $y = 15x - 18$ 56. $y = 34x - 48$

57. (a) $x = 0, x = 2$ (b) (0, 1) (2, −3)



58. (a) $x = 0, x = 2, x = -2$
(b) (0, 8) (2, −24) (−2, −24)



59. $9x^2 - 2x - 12$ 60. $15x^4 + 9x^2 + 2x$
61. $\dfrac{2(1-q)}{q^3}$ 62. $\dfrac{1-3t}{[2\sqrt{t}(3t+1)^2]}$ 63. $\dfrac{9x+2}{2\sqrt{x}}$
64. $\dfrac{5x^6 + 2x^4 + 20x^3 - 3x^2 - 4x}{(x^3+1)^2}$
65. $(9x^2 - 24x)(x^3 - 4x^2)^2$
66. $6(30x^5 + 24x^3)(5x^6 + 6x^4 + 5)^5$
67. $72x^3(2x^4 - 9)^8$ 68. $\dfrac{-(3x^2-4)}{2\sqrt{(x^3-4x)^3}}$
69. $2x(2x^4 + 5)^7(34x^4 + 5)$ 70. $\dfrac{-2(3x+1)(x+12)}{(x^2-4)^2}$
71. $36[(3x+1)(2x^3-1)]^{11}(8x^3 + 2x^2 - 1)$
72. $\dfrac{3}{(1-x)^4}$ 73. $\dfrac{(2x^2-4)}{\sqrt{x^2-4}}$ 74. $\dfrac{2x-1}{(3x-1)^{4/3}}$
75. $y'' = \frac{-1}{4}x^{-3/2} - 2$ 76. $y'' = 12x^2 - 2/x^3$
77. $\dfrac{d^5y}{dx^5} = 0$ 78. $\dfrac{d^5y}{dx^5} = -30(1-x)$
79. $\dfrac{d^3y}{dx^3} = -4/[(x^2-4)^{3/2}]$ 80. $\dfrac{d^4y}{dx^4} = \dfrac{2x(x^2-3)}{(x^2+1)^3}$
81. (a) $x'(10) = -1$ means if price changes from \$10 to \$11, the number of units demanded will change by about −1.
(b) $x'(20) = -\frac{1}{4}$ means if price changes from \$20 to \$21, the number of units demanded will change by about $-\frac{1}{4}$.
82. $\dfrac{dq}{dp} = \dfrac{-p}{\sqrt{0.02p^2 + 500}}$
83. $x'(10) = \frac{1}{6}$ means if price changes from \$10 to \$11, the number of units supplied will change by about $\frac{1}{6}$.
84. (a) $\overline{MC} = 6x + 6$ (b) 186
(c) If a 31st unit is produced, costs will change by about \$186.

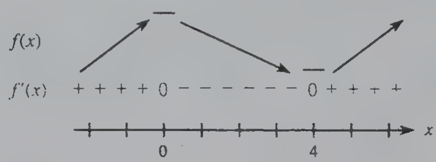
85. $C'(4) = 53$ means that a 5th unit produced would change total costs by about \$53.
86. **(a)** $\overline{MR} = 40 - 0.04x$ **(b)** $x = 1000$ units
87. $\overline{MP}(10) = 48$ means if an 11th unit is sold, profit will change by about \$48.
88. **(a)** $\overline{MR} = 80 - 0.08x$ **(b)** 72
(c) If a 101st unit is sold, revenue will change by about \$72.
89. $\dfrac{120x(x+1)}{(2x+1)^2}$ 90. $\overline{MP} = 4500 - 3x^2$
91. $\overline{MP} = 16 - 0.2x$

Chapter 9 Test (page 728)

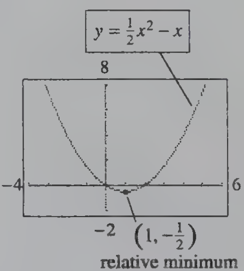
1. **(a)** $\frac{3}{4}$ **(b)** $-8/5$ **(c)** $9/8$ **(d)** does not exist
2. **(a)** $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$
(b) $f'(x)$
$$= \lim_{h\to 0} \frac{[3(x+h)^2 - (x+h) + 9] - [3x^2 - x + 9]}{h}$$
$$= \lim_{h\to 0} \frac{[3x^2 + 6xh + 3h^2 - x - h + 9] - [3x^2 - x + 9]}{h}$$
$$= \lim_{h\to 0} \frac{6xh + 3h^2 - h}{h} = \lim_{h\to 0} [6x + 3h - 1] = 6x - 1$$
3. $x = 0, x = 8$
4. **(a)** $\dfrac{99x^2 - 24x^9}{(2x^7 + 11)^2}$
(b) $(3x^5 - 2x + 3)(40x^9 + 40x^3) + (4x^{10} + 10x^4 - 17)(15x^4 - 2)$
(c) $9(10x^4 + 21x^2)(2x^5 + 7x^3 - 5)^{11}$
(d) $2(8x^2 + 5x + 18)(2x + 5)^5$
(e) $\dfrac{6}{\sqrt{x}} + \dfrac{20}{x^3}$
5. $\dfrac{d^3y}{dx^3} = 6 + 60x^{-6}$
6. **(a)** $y = -15x - 5$ **(b)** $(4, -90), (-2, 18)$
7. **(a)** 2 **(b)** does not exist **(c)** -4
8. $g(-2) = 8;\ \lim_{x\to -2^-} g(x) = 8,\ \lim_{x\to -2^+} g(x) = -8$
$\therefore \lim_{x\to -2} g(x)$ does not exist and $g(x)$ is not continuous at $x = -2$.
9. **(a)** $P(x) = 50x - 0.01x^2 - 10{,}000$
(b) $\overline{MP} = 50 - 0.02x$
(c) $\overline{MP}(1000) = 30$ means the predicted profit from the sale of the 1001st unit is approximately \$30.
10. 104
11. **(a)** -5 **(b)** -1 **(c)** 4 **(d)** does not exist
(e) 2 **(f)** $-4, 1, 3, 6$ **(g)** $-4, 3, 6$
12. **(a)** $\frac{2}{3}$ **(b)** -4 **(c)** $\frac{2}{3}$

Exercise 10.1 (page 747)

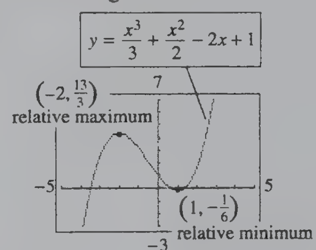
1. **(a)** $(1, 5)$ **(b)** $(4, 1)$ **(c)** $(-1, 2)$
3. **(a)** $(1, 5)$ **(b)** $(4, 1)$ **(c)** $(-1, 2)$
5. **(a)** 3, 7 **(b)** $3 < x < 7$
(c) $x < 3, x > 7$ **(d)** 7 **(e)** 3
7. $x = 0, x = 4$
9.



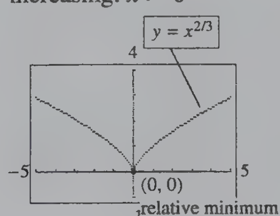
Min: $(4, -58)$; Max: $(0, 6)$
11. **(a)** $dy/dx = 3x^2 - 3$ **(b)** $x = 1, x = -1$
(c) $(1, 2), (-1, 6)$
(d) relative max at $(-1, 6)$, relative min at $(1, 2)$
13. **(a)** $dy/dx = 3x^2 + 6x + 3$ **(b)** $x = -1$
(c) $(-1, -3)$
(d) horizontal point of inflection at $(-1, -3)$
15. **(a)** $\dfrac{dy}{dx} = x - 1$ **(b)** $x = 1$ **(c)** $(1, -\frac{1}{2})$
(d) decreasing: $x < 1$ **(e)**
increasing: $x > 1$



17. **(a)** $dy/dx = x^2 + x - 2$
(b) $x = -2, x = 1$ **(c)** $(-2, \frac{13}{3}), (1, -\frac{1}{6})$
(d) increasing: $x < -2$ and $x > 1$
decreasing: $-2 < x < 1$
(e)



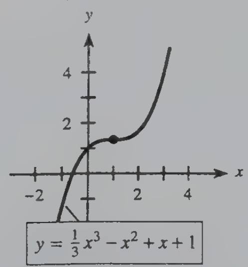
19. **(a)** $\dfrac{dy}{dx} = \dfrac{2}{3x^{1/3}}$ **(b)** $x = 0$ **(c)** $(0, 0)$
(d) decreasing: $x < 0$
increasing: $x > 0$
(e)



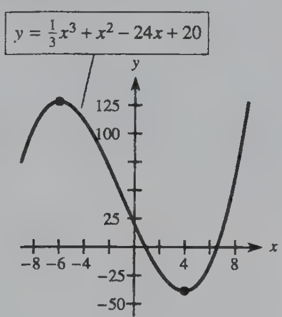
21. (a) $f'(x) = 0$ at $x = -\frac{1}{2}$
$f'(x) > 0$ for $x < -\frac{1}{2}$
$f'(x) < 0$ for $x > -\frac{1}{2}$
(b) $f'(x) = -1 - 2x$ verifies these conclusions

23. (a) $f'(x) = 0$ at $x = 0$, $x = -3$, $x = 3$
$f'(x) > 0$ for $-3 < x < 3$, $x \neq 0$
$f'(x) < 0$ for $x < -3$ and $x > 3$
(b) $f'(x) = \frac{1}{3}x^2(9 - x^2)$ verifies these conclusions.

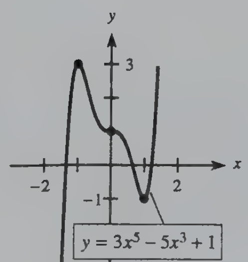
25. HPI $(1, \frac{4}{3})$
no max or min



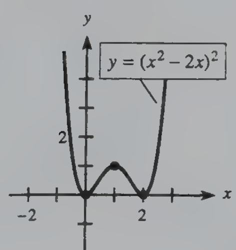
27. $(-6, 128)$ rel max;
$(4, -38\frac{2}{3})$ rel min



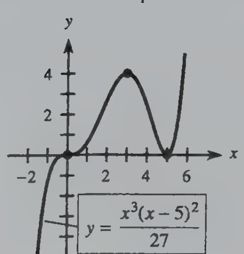
29. $(-1, 3)$ rel max;
$(1, -1)$ rel min;
HPI $(0, 1)$



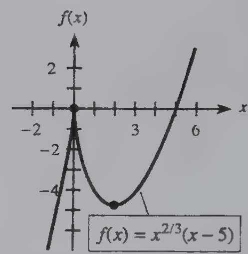
31. $(1, 1)$ rel max;
$(0, 0)$, $(2, 0)$ rel min



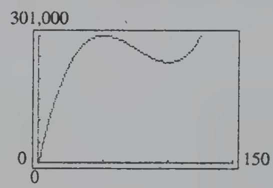
33. $(3, 4)$ rel max;
$(5, 0)$ rel min;
HPI $(0, 0)$



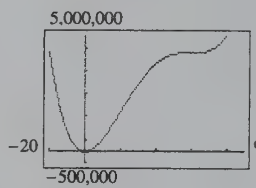
35. $(0, 0)$ rel max;
$(2, -4.8)$ rel min



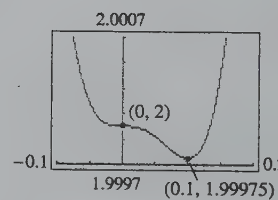
37. $(50, 300{,}500)$, $(100, 238{,}000)$
$0 \le x \le 150$, $0 \le y \le 301{,}000$



39. $(0, -40{,}000)$ $(60, 4{,}280{,}000)$
$-20 \le x \le 90$ $\quad -50{,}000 \le y \le 5{,}000{,}000$

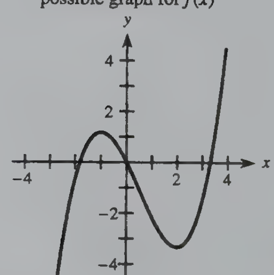


41. $f(x) = 7.5x^4 - x^3 + 2$
$(0, 2)$ HPI
$(0.1, 1.99975)$ rel min
$-0.1 \le x \le 0.2$
$1.9997 \le y \le 2.007$



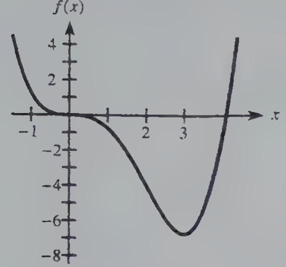
43. critical values: $x = -1$, $x = 2$
$f(x)$ increasing for $x < -1$ and $x > 2$
$f(x)$ decreasing for $-1 < x < 2$
rel max at $x = -1$; rel min at $x = 2$

possible graph for $f(x)$



45. critical values: $x = 0$, $x = 3$
$f(x)$ increasing for $x > 3$
$f(x)$ decreasing for $x < 3, x \neq 0$
rel min at $x = 3$; HPI at $x = 0$

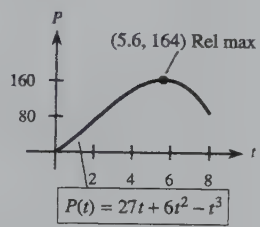
possible graph of $f(x)$



47. Graph on left is $f(x)$; on right is $f'(x)$ because $f(x)$ is increasing when $f'(x) > 0$ (i.e., above the x-axis) and $f(x)$ is decreasing when $f'(x) < 0$ (i.e., below the x-axis).

49. decreasing for $t \geq 0$

51. **(a)** $2 \pm \sqrt{13}$
(b) $2 + \sqrt{13} \approx 5.6$
(c) $0 \leq t < 2 + \sqrt{13}$



53. **(a)** $x = 5$ **(b)** decreasing for $0 < x < 5$
(c) $x > 5$

55. **(a)** at $x = 150$, increasing; at $x = 250$, changing from increasing to decreasing; at $x = 350$, decreasing
(b) increasing for $x < 250$ **(c)** 250 units

57. **(a)** $t = 6$ **(b)** 6 weeks

59. **(a)** 10 **(b)** January 1

61. **(a)** $x \approx 10.12$, during 1990
(b) $x \approx 1.76$, during 1981

63. **(a)** $y = -9.3458x^2 + 34.0863x + 235.8625$
(b) at $x = 1.82$, late in 1991; data indicate the maximum in 1992.

Exercise 10.2 (page 761)

1. **(a)** concave down **(b)** concave up

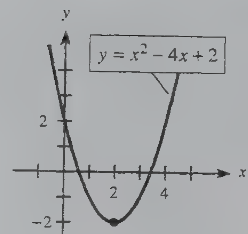
3. **(a)** concave down **(b)** concave up

5. (a, c) and (d, e) **7.** (c, d) (e, f) **9.** c, d, e

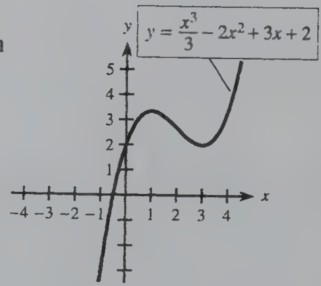
11. concave up when $x > 2$; concave down when $x < 2$; POI at $x = 2$

13. concave up when $x < -2$ and $x > 1$
concave down when $-2 < x < 1$
points of inflection at $x = -2$ and $x = 1$

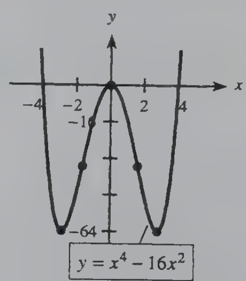
15. no points of inflection; $(2, -2)$ min



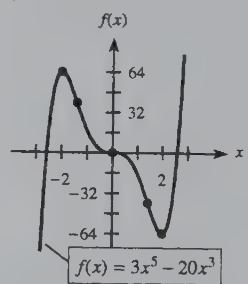
17. $(1, \frac{10}{3})$ max; $(3, 2)$ min; $(2, \frac{8}{3})$ point of inflection



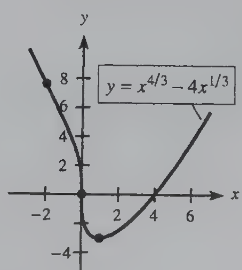
19. $(0, 0)$ rel max; $(2\sqrt{2}, -64)$, $(-2\sqrt{2}, -64)$ min; points of inflection: $(2\sqrt{6}/3, -320/9)$ and $(-2\sqrt{6}/3, -320/9)$

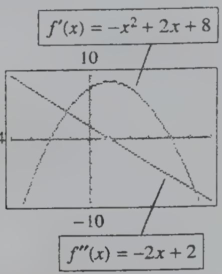


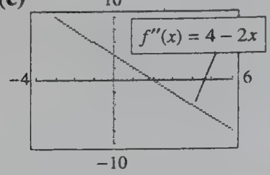
21. $(-2, 64)$ rel max; $(2, -64)$ rel min; and points of inflection: $(-\sqrt{2}, 39.6)$, $(0, 0)$ and $(\sqrt{2}, -39.6)$

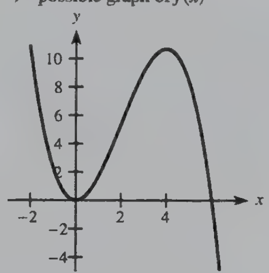


23. $(1, -3)$ min; points of inflection: $(-2, 7.6)$ and $(0, 0)$



25. (a) $f''(x) = 0$ when $x = 1$
$f''(x) > 0$ when $x < 1$
$f''(x) < 0$ when $x > 1$
(b) rel max for $f'(x)$ at $x = 1$
no rel min
(c)


27. (a) concave up when $x < 2$ concave down when $x > 2$
(b) point of inflection at $x = 2$
(c)


(d) possible graph of $f(x)$


29. (a) G (b) C (c) F (d) H (e) I
31. (a) concave up when $x < 0$
concave down when $x > 0$
point of inflection at $x = 0$
(b) concave up when $-1 < x < 1$
concave down when $x < -1$ and $x > 1$
POI at $x = -1$ and $x = 1$
(c) concave up when $x > 0$
concave down when $x < 0$
point of inflection at $x = 0$
33. (a) $P'(t)$ (b) B (c) C
35. (a) C (b) right (c) yes
37. (a) in an 8-hour shift, max when $t = 8$ (b) 4 hr
39. (a) 9 days (b) 15 days
41. $x = 9.53$, during 1989
43. (a) $y = -0.00816x^3 + 0.4635x^2 - 8.1122x + 64.7785$
(b) min at $x = 13.73$, during 1983
max at $x = 24.14$, during 1994
(c) no; data give min in 1984, max in 1991.
45. (a) $y = -0.4052x^3 + 6.0161x^2 - 17.9605x + 52$
(b) $x = 1.83$, during 1913
(c) $x = 8.07$, during 1920

Exercise 10.3 (page 773)

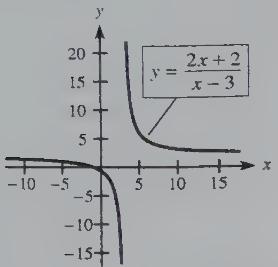
1. min -6 at $x = 2$, max $3.\overline{481}$ at $x = -2/3$
3. min -1 at $x = -2$, max 2 at $x = -1$
5. (a) $x = 1800$ units, $R = \$32{,}400$
(b) $x = 1500$ units, $R = \$31{,}500$
7. $x = 20$ units, $R = \$24{,}000$ 9. 40 people
11. $p = \$15$, $R = \$22{,}500$
13. (a) max = \$2100 (b) $x = 10$
15. $x = 5$ units, $\overline{C} = \$23$
17. $x = 10$ units, $\overline{C} = \$20$
19. 200 units ($x = 2$), $\overline{C} = \$108$ 21. $x = 5$
23. $x = 80$ units, $P = \$280{,}000$
25. $x = 10\sqrt{15} \approx 39$ units, $P \approx \$71{,}181$ (using $x = 39$)
27. $x = 1000$ units, $P = \$39{,}700$
29. $x = 600$ units, $P = \$495{,}000$
31. (a) 60 (b) \$570 (c) \$9000
33. (a) 1000 units (b) \$8066.67 (approximately)
35. 2000 units priced at \$90/unit; max profit is \$90,000/wk
37. rent = \$430
39. (a) $t \approx 12.5$ (in 1992), $R = \$5.0353$ billion
(b) the data show a max in 1990 of \$5.1686 billion
41. (a) August 1991; approximately \$3 billion
(b) August or September 1991; approximately \$5.5 billion
(c) December 1992 or January 1993; approximately \$12 billion
(d) May 1993; approximately \$9.5 billion
43. (a) About January 1980 or January 1991
(b) (January 25, 1994, 37%)
(c) No, either some citizens think crime is the most important problem or no citizens think that.
(d) Crime would be a less important issue, and the percent would drop.
(e) Not 1980, because crime was a low priority. Yes, in 1994, as seen from the high concern about crime then.

Exercise 10.4 (page 785)

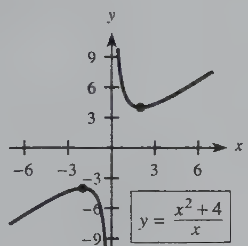
1. (a) $x_1 = \$25$ million, $x_2 = \$13.846$ million
(b) \$38.846 million
3. 400 trees 5. (a) 5 (b) 237.5 7. \$50
9. $m = c$ 11. 1 week 13. $t = 8$, $p = 45\%$
15. 240 ft 17. 300 ft × 150 ft
19. 20 ft long, $6\frac{2}{3}$ ft across (dividers run across)
21. 4 in. × 8 in. × 8 in. high 23. 30,000
25. 12,000 27. $x = 2$ 29. 3 weeks from now
31. 25 plates

Exercise 10.5 *(page 797)*

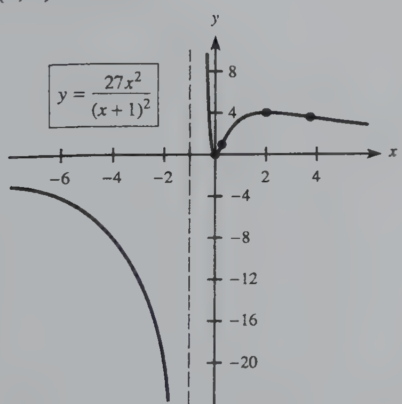
1. (a) $x = 2$ (b) 1 (c) $y = 1$ (d) 1
3. (a) $x = 2$ (b) 1 (c) $y = 1$ (d) 1
5. HA: $y = 2$; VA: $x = 3$
7. HA: $y = 0$; VA: $x = -2, x = 2$
9. HA: none; VA: none
11. HA: $y = 2$; VA: $x = 3$
no max, min, or points of inflection



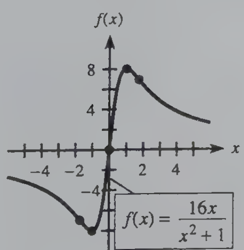
13. VA: $x = 0$;
$(-2, -4)$ rel max;
$(2, 4)$ rel min



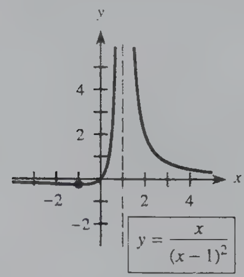
15. VA: $x = -1$; HA: $y = 0$;
$(0, 0)$ rel min; $(2, 4)$ rel max;



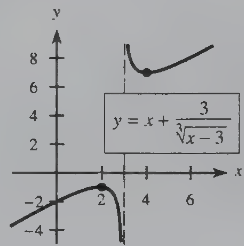
17. HA: $y = 0$; $(1, 8)$ rel max;
$(-1, -8)$ rel min;
points of inflection:
$(0, 0)$, $(-\sqrt{3}, -4\sqrt{3})$,
and $(\sqrt{3}, 4\sqrt{3})$



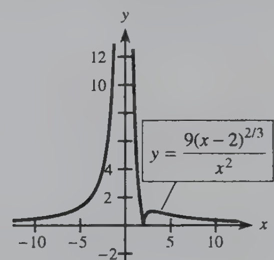
19. HA: $y = 0$; VA: $x = 1$; $(-1, -\frac{1}{4})$ rel min; point of inflection: $(-2, -\frac{2}{9})$



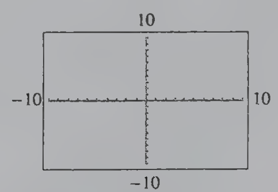
21. VA: $x = 3$;
$(2, -1)$ rel max;
$(4, 7)$ rel min



23. HA: $y = 0$; VA: $x = 0$
$(2, 0)$ rel min; $(3, 1)$ rel max
points of inflection: $(1.87, 0.66)$, $(4.13, 0.87)$

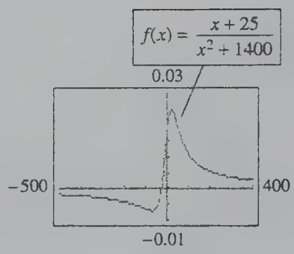


25. (a) HA: approx. $y = -2$; VA: approx. $x = 4$
(b) HA: $y = -\frac{9}{4}$; VA: $x = \frac{17}{4}$
27. (a) HA: approx. $y = 2$;
VA: approx. $x = 2.5$, $x = -2.5$
(b) HA: $y = \frac{20}{9}$; VA: $x = \frac{7}{3}$, $x = -\frac{7}{3}$
29. $f(x) = \dfrac{x + 25}{x^2 + 1400}$
(a)



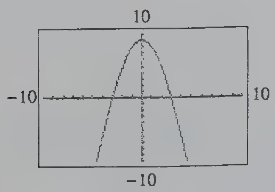
(b) HA: $y = 0$; rel min $(-70, -0.0071)$
rel max $(20, 0.025)$

(c) x: -500 to 400
y: -0.01 to 0.03



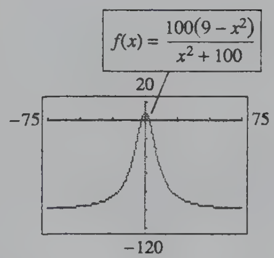
31. $f(x) = \dfrac{100(9 - x^2)}{x^2 + 100}$

(a)



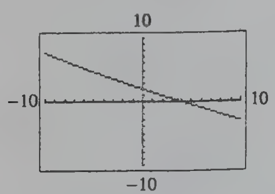
(b) HA: $y = -100$; rel max $(0, 9)$

(c) x: -75 to 75
y: -120 to 20



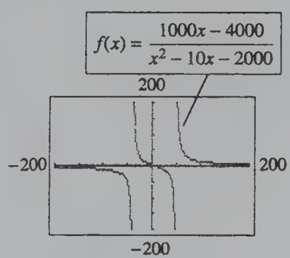
33. $f(x) = \dfrac{1000x - 4000}{x^2 - 10x - 2000}$

(a)



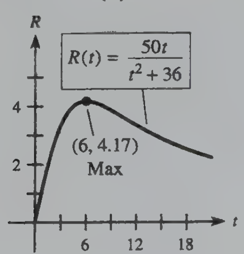
(b) HA: $y = 0$
VA: $x = -40$, $x = 50$
no max or min

(c) x: -200 to 200
y: -200 to 200



35. (a) none (b) $C \geq 0$ (c) $p = 100$ (d) no

37. (a)



(b) 6 weeks
(c) 22 weeks after its release

39. (a) yes, $x = -1$ (b) no; domain is $x \geq 5$
(c) yes, $y = -58.5731$
(d) At 0°F, as the wind speed increases, there is a limiting wind chill of about -58.6°F. This is meaningful because at high wind speeds, additional wind probably has little noticeable effect.

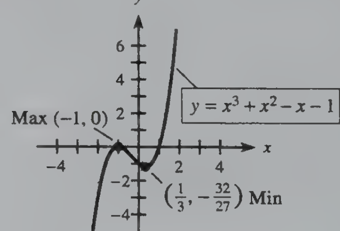
41. (a) $P = C$ (b) C (c) $P' = 0$ (d) 0

43. (a) 0
(b) As the years past 1800 increase, the percentage of workers in farm occupations approaches 0.
(c) no

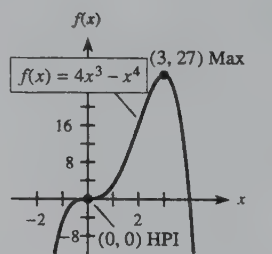
45. (a) No. Barometric pressure can drop off the scale (as shown), but it cannot decrease without bound. In fact, it must always be positive.
(b) See your library with regard to the "Storm of the Century" in March 1993.

Chapter 10 Review Exercises (page 801)

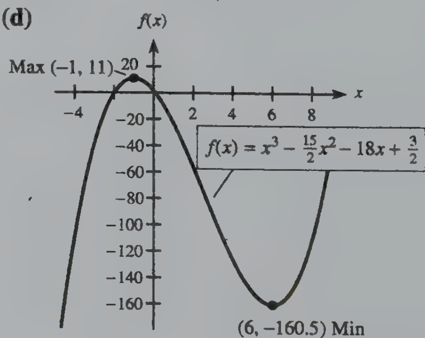
1. (0, 0) max 2. (2, −9) min 3. HPI (1, 0)
4. $(1, \frac{3}{2})$ max, $(-1, -\frac{3}{2})$ min
5. (a) $\frac{1}{3}, -1$ (b) $(-1, 0)$ rel max, $(\frac{1}{3}, -\frac{32}{27})$ rel min
(c) none (d)



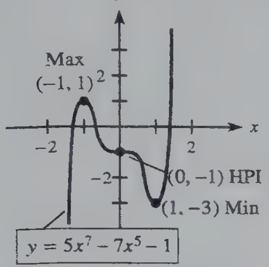
6. (a) 3, 0 (b) (3, 27) max (c) (0, 0)
(d)



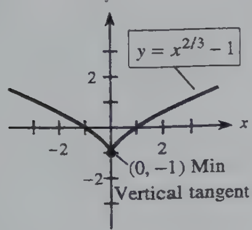
7. (a) −1, 6
(b) (−1, 11) rel max, (6, −160.5) rel min
(c) none (d)



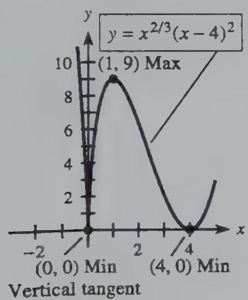
8. (a) 0, ±1 **(b)** (−1, 1) rel max, (1, −3) rel min
(c) (0, −1) **(d)**



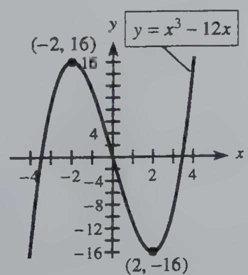
9. (a) 0 **(b)** (0, −1) min **(c)** none
(d)



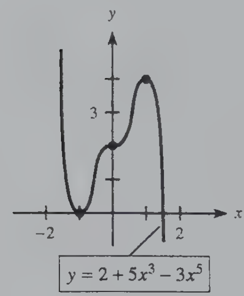
10. (a) 0, 1, 4
(b) (0, 0) rel min, (1, 9) rel max, (4, 0) rel min
(c) none **(d)**



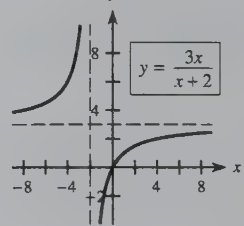
11. concave up
12. concave up when $x < -1$ and $x > 2$
concave down when $-1 < x < 2$
points of inflection at (−1, −3) and (2, −42)
13. (−1, 15) rel max; (3, −17) rel min;
point of inflection (1, −1)
14. (−2, 16) rel max; (2, −16) rel min;
point of inflection (0, 0)



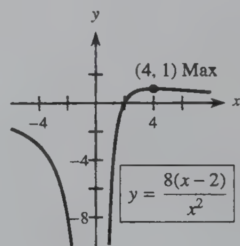
15. (1, 4) rel max; (−1, 0) rel min; points of inflection:
$\left(\frac{1}{\sqrt{2}}, 2 + \frac{7}{4\sqrt{2}}\right)$, (0, 2), and $\left(-\frac{1}{\sqrt{2}}, 2 - \frac{7}{4\sqrt{2}}\right)$



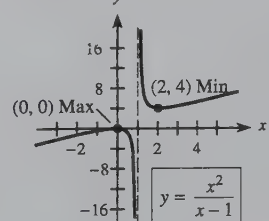
16. (a) (0, 0) absolute min; (140, 19,600) absolute max
(b) (0, 0) absolute min; (100, 18,000) absolute max
17. (a) (50, 233,333) absolute max; (0, 0) absolute min
(b) (64, 248,491) absolute max; (0, 0) absolute min
18. (a) $x = 1$ **(b)** $y = 0$ **(c)** 0 **(d)** 0
19. (a) $x = -1$ **(b)** $y = \frac{1}{2}$ **(c)** $\frac{1}{2}$ **(d)** $\frac{1}{2}$
20. HA: $y = \frac{3}{2}$; VA: $x = 2$
21. HA: $y = -1$; VA: $x = 1, x = -1$
22. (a) HA: $y = 3$; VA: $x = -2$
(b) no max or min **(c)**



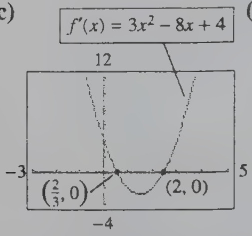
23. (a) HA: $y = 0$; VA: $x = 0$
(b) (4, 1) max **(c)**



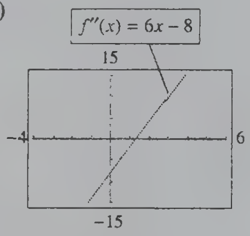
24. (a) HA: none; VA: $x = 1$
(b) (0, 0) rel max; (2, 4) rel min
(c)



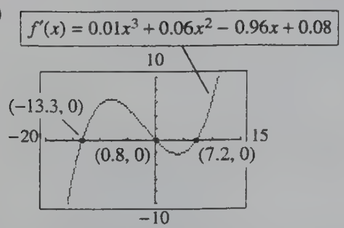
25. **(a)** $f'(x) > 0$ for $x < \frac{2}{3}$ (approximately) and $x > 2$
$f'(x) < 0$ for about $\frac{2}{3} < x < 2$
$f'(x) = 0$ about $x = \frac{2}{3}$ and $x = 2$
(b) $f''(x) > 0$ for $x > \frac{4}{3}$
$f''(x) < 0$ for $x < \frac{4}{3}$
$f''(x) = 0$ at $x = \frac{4}{3}$
(c)



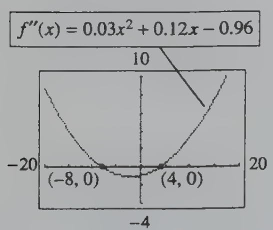
(d)



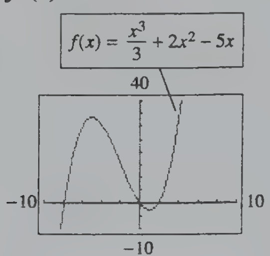
26. **(a)** $f'(x) > 0$ for about $-13 < x < 0$ and $x > 7$
$f'(x) < 0$ for about $x < -13$ and $0 < x < 7$
$f'(x) = 0$ for about $x = 0, x = -13, x = 7$
(b) $f''(x) > 0$ for about $x < -8$ and $x > 4$
$f''(x) < 0$ for about $-8 < x < 4$
$f''(x) = 0$ for about $x = -8$ and $x = 4$
(c)



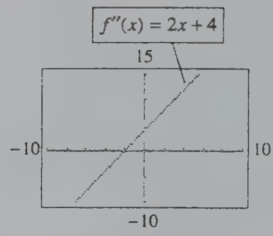
(d)



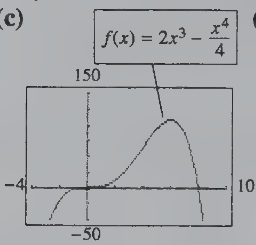
27. **(a)** $f(x)$ increasing for $x < -5$ and $x > 1$
$f(x)$ decreasing for $-5 < x < 1$
$f(x)$ has rel max at $x = -5$, rel min at $x = 1$
(b) $f''(x) > 0$ for $x > -2$ (where $f'(x)$ increases)
$f''(x) < 0$ for $x < -2$ (where $f'(x)$ decreases)
$f''(x) = 0$ for $x = -2$
(c)



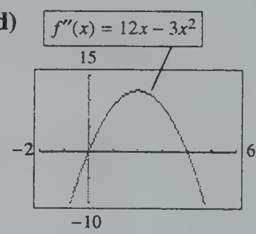
(d)



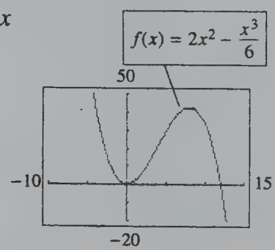
28. **(a)** $f(x)$ increasing for $x < 6, x \neq 0$
$f(x)$ decreasing for $x > 6$
$f(x)$ has max at $x = 6$, point of inflection at $x = 0$
(b) $f''(x) > 0$ for $0 < x < 4$
$f''(x) < 0$ for $x < 0$ and $x > 4$
$f''(x) = 0$ at $x = 0$ and $x = 4$
(c)



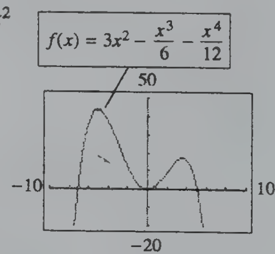
(d)



29. **(a)** $f(x)$ is concave up for $x < 4$
$f(x)$ is concave down for $x > 4$
$f(x)$ has point of inflection at $x = 4$
(b) $f''(x) = 4 - x$



30. **(a)** $f(x)$ is concave up for $-3 < x < 2$
$f(x)$ is concave down for $x < -3$ and $x > 2$
$f(x)$ has points of inflection at $x = -3$ and $x = 2$
(b) $f''(x) = 6 - x - x^2$

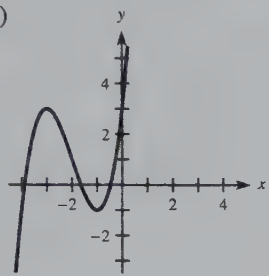


31. $x = 5$ units, $\overline{C} = \$45$
32. **(a)** $x = 1600$ units, $R = \$25{,}600$
(b) $x = 1600$ units, $R = \$25{,}600$
33. $P = \$54{,}000$ at $x = 100$ units **34.** $x = 3$ units
35. $x = 152$ units **36.** $x = 7$ units **37.** \$300
38. selling 175 sets at \$425 each

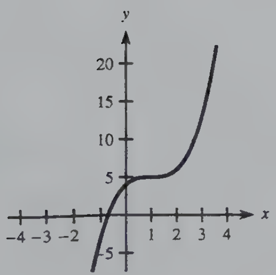
39. $93,625 at 325 units
40. (a) 150 (b) $650
41. $208,490.67 at 64 units **42.** $x = 1000$ units
43. 10:00 A.M. **44.** 325 in 1995
45. 20 mi from A, 10 mi from B **46.** 4 ft × 4 ft
47. $8\frac{3}{4}$ in. × 10 in. **48.** 500 **49.** $\frac{29}{18}$ **50.** 24,000

Chapter 10 Test *(page 805)*

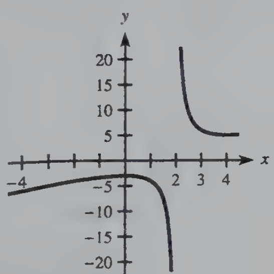
1. max $(-3, 3)$; min $(-1, -1)$



2. HPI $(1, 5)$



3. max $(0, -3)$;
min $(4, 5)$;
vert asym $x = 2$



4. $\left(-\frac{1}{\sqrt{2}}, 0\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$

5. $(0, 2)$, HPI; $\left(-\frac{1}{\sqrt{2}}, 3.237\right)$, $\left(\frac{1}{\sqrt{2}}, 0.763\right)$

6. max $(-1, 4)$; min $(1, 0)$
7. max 67 at $x = 8$; min -122 at $x = 5$
8. horiz asym $y = 200$; vert asym $x = -300$

9.

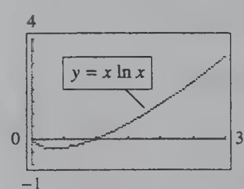
Point	f	f'	f''
A	−	+	−
B	+	−	0
C	+	0	+

10. (a) -2 (b) $x = -1$
11. local max at $(6, 10)$

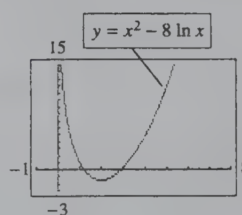
12. (a) $x = 43.7$, during 1943
(b) The ratio of priests to Catholics was highest in 1943.
13. (a) $x = 7200$ (b) $518,100 **14.** 100 units
15. $250 **16.** $\frac{10}{3}$ centimeter **17.** 28,000 units
18. (a) $y = -0.0161x^2 + 3.3185x - 165.278$
(b) $x = 103.059$; during 2003

Exercise 11.1 *(page 818)*

1. $f'(x) = 4/x$ **3.** $y' = 1/x$
5. $y' = 4/x$ **7.** $f'(x) = \dfrac{4}{4x + 9}$
9. $y' = \dfrac{4x - 1}{2x^2 - x} + 3$ **11.** $dp/dq = 2q/(q^2 + 1)$
13. (a) $y' = \dfrac{1}{x} - \dfrac{1}{x - 1}$ (b) $y' = \dfrac{-1}{x(x - 1)}$
15. (a) $y' = \dfrac{2x}{3(x^2 - 1)}$ (b) $y' = \dfrac{2x}{3(x^2 - 1)}$
17. (a) $y' = \dfrac{4}{4x - 1} - \dfrac{3}{x}$ (b) $y' = \dfrac{-8x + 3}{x(4x - 1)}$
19. (a) $y' = \dfrac{3}{x} + \dfrac{1}{2(x + 1)}$ (b) $y' = \dfrac{7x + 6}{2x(x + 1)}$
21. $\dfrac{dp}{dq} = \dfrac{(q^2 + 1)}{q(q^2 - 1)}$ **23.** $\dfrac{dy}{dt} = -\dfrac{(3t^2 - 4t - 3)}{2(1 - t)(t^2 + 3)}$
25. $\dfrac{dy}{dx} = 1 - \dfrac{1}{x}$ **27.** $y' = (1 - \ln x)/x^2$
29. $y' = 8x^3/(x^4 + 3)$ **31.** $y' = \dfrac{4(\ln x)^3}{x}$
33. $y' = \dfrac{8x^3 \ln(x^4 + 3)}{x^4 + 3}$ **35.** $y' = \dfrac{1}{x \ln 4}$
37. $y' = \dfrac{4x^3 - 12x^2}{(x^4 - 4x^3 + 1) \ln 6}$
39. rel min $(e^{-1}, -e^{-1})$



41. rel min $(2, 4 - 8 \ln 2)$



43. (a) $\overline{MC} = \dfrac{400}{2x + 1}$
(b) $\overline{MC} = \dfrac{400}{401} \approx 1.0$; the approximate cost of the 201st unit is $1.00

45. (a) $\overline{MR} = \dfrac{2500[(x+1)\ln(10x+10) - x]}{(x+1)\ln^2(10x+10)}$
(b) 309.67; at 100 units, selling 1 additional unit yields \$309.67.

47. (a) -5.23 (b) -1.89 (c) increasing

49. A/B **51.** $dR/dI = 1/(I \ln 10)$

53. (a) $y = 561.2713 + 2646.5135 \ln(x)$
(b) \$115/year

Exercise 11.2 (page 825)

1. $y' = 5e^x - 1$ **3.** $f'(x) = e^x - ex^{e-1}$

5. $y' = 3x^2e^{x^3}$ **7.** $y' = 36xe^{3x^2}$

9. $y' = 12x(x^2+1)^2e^{(x^2+1)^3}$ **11.** $y' = 3x^2$

13. $y' = e^{-1/x}/x^2$ **15.** $y' = \dfrac{2}{x^3}e^{-1/x^2} - 2xe^{-x^2}$

17. $ds/dt = te^t(t+2)$ **19.** $4x^3e^{x^4} - 4e^{4x}$

21. $\dfrac{4e^{4x}}{e^{4x}+2}$ **23.** $y' = e^{-3x}/x - 3e^{-3x}\ln(2x)$

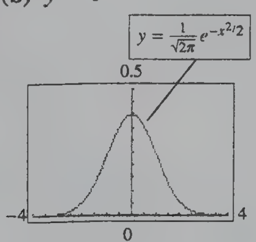
25. $y' = (2e^{5x} - 3)/e^{3x} = 2e^{2x} - 3e^{-3x}$

27. $y' = 30e^{3x}(e^{3x}+4)^9$ **29.** $y' = 6^x \ln 6$

31. $y' = 4^{x^2}(2x \ln 4)$

33. (a) $y'(1) = 0$ (b) $y = e^{-1}$

35. (a) $z = 0$ (b)



37. rel min at $x = 1, y = e$

39. rel max at $x = 0, y = -1$

41. (a) $(0.1)Pe^{0.1n}$ (b) $(0.1)Pe^{0.1}$
(c) Yes, because $e^{0.1n} > 1$ for any $n \geq 1$.

43. $\dfrac{dS}{dt} = -50{,}000e^{-0.5t}$ **45.** $1200e$

47. 29.107 **49.** $y' = \dfrac{98{,}990{,}100e^{-0.99t}}{(1+9999e^{-0.99t})^2}$

51. $y' = 100{,}000e^{-1/x}/x^2$

53. $\dfrac{dN}{dt} = N_0(1+r)^t \cdot \ln(1+r)$

55. $\dfrac{dI}{dR} = 10^R \ln 10$ **57.** 112.066 (\$billion/year)

59. (a) $d'(x) = 29 - 20.1891e^{x/100} + 0.044e^{x/10}$
(b) \$1.28 (billion/year), \$198.01 (billion/year)
(c) The United States could not tolerate exponential growth in its debt. The percent of federal expenditures devoted to payment of interest on the debt was too high.

61. $\dfrac{dP}{dx} = -0.044$ dollars per year

63. (a) $y = 6.8690\,(1.08514)^x$
(b) $y'(23) = 3.68$; \$3.68 per year

65. (a) $y = 604.9211\,(1.0820)^x$
(b) 126,186 per year

Exercise 11.3 (page 836)

1. $\frac{1}{2}$ **3.** $-\frac{1}{2}$ **5.** $-\frac{5}{3}$ **7.** $-x/(2y)$

9. $-(2x+4)/(2y-3)$ **11.** $-x/y$

13. $y' = \dfrac{-y}{2(x-1)}$ **15.** $\dfrac{dp}{dq} = \dfrac{p^2}{4-2pq}$

17. $\dfrac{dy}{dx} = \dfrac{x(3x^3-2)}{3y^2(1+y^2)}$ **19.** $\dfrac{dy}{dx} = \dfrac{4x^3+6x^2y-1}{-4x^3y-3y^2}$

21. $\dfrac{(4x^3+9x^2y^2-8x-12y)}{(18y+12x-6x^3y+10y^4)}$ **23.** undefined

25. 1 **27.** $y = \frac{1}{2}x + 1$ **29.** $y = 4x + 5$

31. $y' = \dfrac{1}{2xy}$ **33.** $y' = \dfrac{-y}{2x \ln x}$ **35.** -4

37. $-1/x$ **39.** $-y/x$

41. $ye^x/(1-e^x)$ **43.** 0 **45.** $y = -\frac{1}{3}x + 1$

47. horizontal: $(2, \sqrt{2}), (2, -\sqrt{2})$;
vertical: $(2+2\sqrt{2}, 0), (2-2\sqrt{2}, 0)$

51. (b) yes, because $x^2 + y^2 = 4$ **53.** $1/(2x\sqrt{x})$

55. max at $(0, 3)$; min at $(0, -3)$ **57.** $\frac{1}{2}$ **59.** $-\frac{243}{128}$

61. At $p = \$80$, $q = 49$ and $dq/dp = -\frac{5}{16}$, which means that if the price is increased to \$81, quantity demanded will decrease by approximately $\frac{5}{16}$ unit.

63. $-0.000436\,y$ **65.** $\dfrac{dh}{dt} = -\dfrac{3}{44} - \dfrac{h}{12}$

Exercise 11.4 (page 842)

1. 36 **3.** $\frac{1}{8}$ **5.** $-\frac{24}{5}$ **7.** $\frac{7}{6}$

9. -5 if $z = 5$, -10 if $z = -5$

11. -80 units/sec **13.** 12π ft^2/min

15. $\frac{16}{27}$ in./sec **17.** \$1798/day **19.** \$0.42/day

21. 430 units/month **23.** 36π mm^3/month

25. $\dfrac{\frac{dW}{dt}}{W} = 3\left(\dfrac{\frac{dL}{dt}}{L}\right)$ **27.** $\dfrac{\frac{dC}{dt}}{C} = 1.54\left(\dfrac{\frac{dW}{dt}}{W}\right)$

29. $\dfrac{1}{4\pi}$ micrometers/day **31.** $1/(20\pi)$ in./min

33. -0.75 ft/sec **35.** $-120\sqrt{6}$

37. 61.18 mph **39.** $\frac{1}{25}$ ft/hr

Exercise 11.5 (page 853)

1. (a) 1 (b) no change **3.** (a) 84
(b) Revenue will decrease.

5. (a) $\frac{100}{99}$ (b) elastic (c) decrease

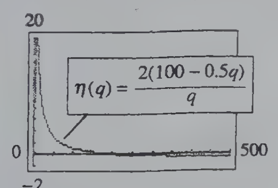
7. (a) 0.81 (b) inelastic (c) increase

9. (a) $\eta = 11.1$ (approximately) (b) elastic

11. (a) $\eta = \dfrac{375 - 3q}{q}$
(b) unitary: $q = 93.75$; inelastic: $q > 93.75$; elastic: $q < 93.75$
(c) As q increases over $0 < q < 93.75$, p decreases, so elastic demand means R increases. Similarly, R decreases for $q > 93.75$.
(d) Maximum for R when $q = 93.75$; yes.
13. \$12/item 15. $t =$ \$350 17. \$115/item
19. \$483 per item; \$40,100 21. \$1100/item

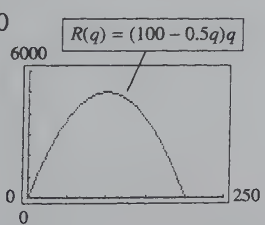
Chapter 11 Review Exercises (page 856)

1. $(6x - 1)e^{3x^2 - x}$ 2. $2x$ 3. $1/q - \left[\dfrac{2q}{(q^2 - 1)}\right]$
4. $dy/dx = e^{x^2}(2x^2 + 1)$ 5. $dy/dx = 3^{3x-3}\ln 3$
6. $dy/dx = \dfrac{10}{x}$ 7. $y' = \dfrac{1 - \ln x}{x^2}$
8. $dy/dx = -2e^{-x}/(1 - e^{-x})^2$
9. $y = 12ex - 8e$, or $y = 32.62x - 21.75$
10. $y = x - 1$ 11. $y' = \dfrac{-y}{x \ln x}$
12. $dy/dx = ye^{xy}/(1 - xe^{xy})$ 13. $dy/dx = 2/y$
14. $\dfrac{dy}{dx} = \dfrac{2(x + 1)}{3(1 - 2y)}$ 15. $y' = \dfrac{6x(1 + xy^2)}{y(5y^3 - 4x^3)}$
16. $d^2y/dx^2 = -(x^2 + y^2)/y^3 = -1/y^3$ 17. 5/9
18. $(-2, \pm\sqrt{\tfrac{2}{3}})$ 19. 3/4 20. 11 square units/min
21. 135.3 22. (a) 152.5 (b) 1.13 times faster
23. (a) $-0.00002876A_0$ (b) $-0.00002876A_0$
(c) less
24. $\$1200e \approx \$3261.94$ 25. $-\$603.48$
26. $1/(25\pi)$ mm/min 27. $\frac{48}{25}$ ft/min
28. $\dfrac{dS/dt}{S} = \dfrac{1}{3}\left(\dfrac{dA/dt}{A}\right)$ 29. yes
30. $t = 1446.67$ 31. \$880
32. (a) 1 (b) no change 33. $\frac{25}{12}$, elastic 34. 1
35. (a)



(b) $q = 100$

(c) max revenue at $q = 100$



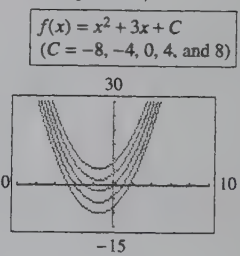
(d) Revenue is maximized where elasticity is unitary.

Chapter 11 Test (page 858)

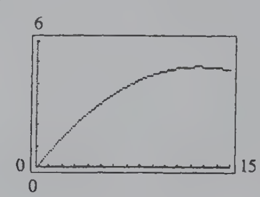
1. $y' = \dfrac{12x^2}{x^3 + 1}$ 2. $y' = \dfrac{12x^3}{x^4 + 1}$ 3. $y' = \dfrac{1 - \ln x}{x^2}$
4. $y' = 15x^2 e^{x^3} + 2x$ 5. $\dfrac{dS}{dt} = e^{t^4}(4t^4 + 1)$
6. $y' = \dfrac{e^{x^3+1}(3x^3 - 1)}{x^2}$ 7. $f'(x) = 20\,(3^{2x})\ln 3$
8. $g'(x) = \dfrac{8}{(4x + 7)\ln 5}$ 9. $y' = \dfrac{-3x^3}{y}$
10. $y' = \dfrac{-e^y}{xe^y - 10}$ 11. $-\frac{3}{2}$ 12. −586 per day
13. −0.05 unit per dollar 14. \$1349.50 per week
15. $n = 3.71$; decreases
16. (a) 16.82 (b) 25.96 17. 0.087%
18. 1020 (\$ billions/year) 19. \$540

Exercise 12.1 (page 870)

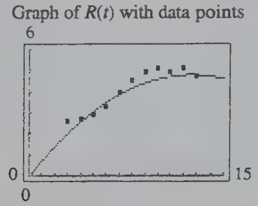
1. $x^4 + C$ 3. $\frac{1}{7}x^7 + C$ 5. $\frac{1}{8}x^8 + C$
7. $\frac{4}{3}x^6 + C$ 9. $27x + \frac{1}{14}x^{14} + C$
11. $3x - \frac{2}{5}x^{5/2} + C$ 13. $\frac{1}{5}x^5 - 3x^3 + 3x + C$
15. $2x + \frac{4}{3}x\sqrt{x} + C$ 17. $\frac{24}{5}x\sqrt[4]{x} + C$
19. $-5/(3x^3) + C$ 21. $\frac{3}{2}\sqrt[3]{x} + C$
23. $\dfrac{1}{4}x^4 - 4x - \dfrac{1}{x^5} + C$ 25. $\dfrac{1}{10}x^{10} + \dfrac{1}{2x^2} + 3x^{2/3} + C$
27. $\frac{1}{4}x^4 + \frac{10}{3}x^3 + \frac{25}{2}x^2 + C$ 29. $2x^8 - \frac{4}{3}x^6 + \frac{1}{4}x^4 + C$
31. $-1/x - 1/(2x^2) + C$ 33.



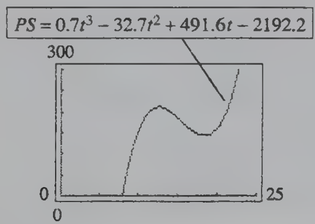
35. $\int(5 - \frac{1}{2}x)\,dx$ 37. $\int(3x^2 - 6x)\,dx$ 39. $R(x) = 3x$
41. $R(x) = 0.2x^2 + 3x$ 43. \$3800
45. $P(t) = \frac{1}{4}t^4 + \frac{4}{3}t^3 + 6t$
47. (a) $x = t^{7/4}/1050$ (b) 0.96 tons
49. (a) $x/4 + 100/x + 30$ (b) \$56
51. (a) $R = -0.031t^2 + 0.776t + 0.0639$



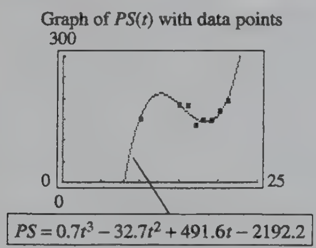
(b) Graph of $R(t)$ with data points



53. (a) $PS = 0.7t^3 - 32.7t^2 + 491.6t - 2192.2$

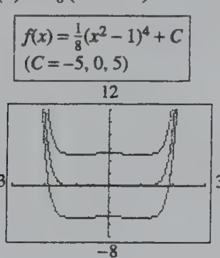


(b) Graph of $PS(t)$ with data points

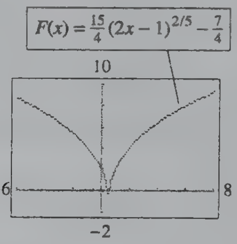


Exercise 12.2 *(page 850)*

1. $\frac{1}{4}(x^2 + 3)^4 + C$ **3.** $(15x^2 + 10)^5/5 + C$
5. $\frac{1}{3}(3x - x^3)^3 + C$ **7.** $\frac{1}{8}(x^2 + 5)^4 + C$
9. $\frac{1}{4}(4x - 1)^7 + C$ **11.** $-\frac{1}{4}(x^2 + 1)^{-2} + C$
13. $\frac{1}{10}(x^2 - 2x + 5)^5 + C$ **15.** $-\frac{1}{8}(x^4 - 4x + 3)^{-4} + C$
17. $\frac{7}{6}(x^4 + 6)^{3/2} + C$ **19.** $\frac{3}{8}x^8 + \frac{6}{5}x^5 + \frac{3}{2}x^2 + C$
21. $\frac{72}{7}x^7 - \frac{48}{5}x^5 + \frac{8}{3}x^3 + C$ **23.** $\frac{2}{9}(x^3 - 3x)^{3/2} + C$
25. $\frac{-1}{[3(x^3 - 1)]} + C$ **27.** $\frac{-1}{[10(2x^5 - 5)^3]} + C$
29. $\frac{-1}{[8(x^4 - 4x)^2]} + C$ **31.** $\frac{2}{3}\sqrt{x^3 - 6x^2 + 2} + C$
33. (a) $f(x) = \frac{1}{8}(x^2 - 1)^4 + C$
(b)



35. (a) $F(x) = \frac{15}{4}(2x - 1)^{2/5} + C$
(b)



(c) $x = \frac{1}{2}$
(d) vertical

37. $\int \frac{8x(x^2 - 1)^{1/3}}{3}\,dx$

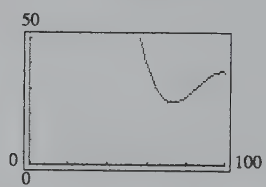
39. $R(x) = \frac{15}{2x + 1} + 30x - 15$

41. 3720 **43.** (a) $s = 10\sqrt{x + 1}$ (b) 50

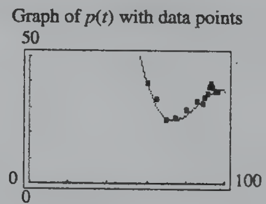
45. (a) $100/(t + 10) - 1000/(t + 10)^2$
(b) 2.5 million

47. 7400

49. (a) $p = -0.001545(t - 60)^3 + 0.1206(t - 60)^2 - 2.4165t + 184.26$
(a)

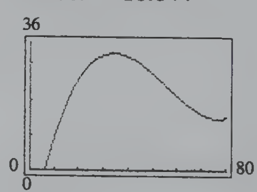


(b)

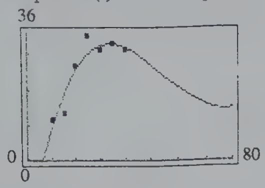


(c) a good fit

51. (a) $U = 0.4717(0.1t + 2)^3 - 10.6585(0.1t + 2)^2 + 7.3900t + 18.044$



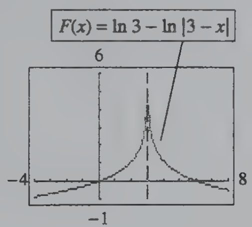
(b) Graph of $U(t)$ with data points



(c) The data fit fairly well, but the maximum points do not agree.

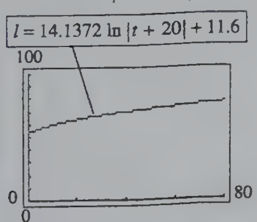
Exercise 12.3 *(page 890)*

1. $\ln|x^3 + 4| + C$ **3.** $\frac{1}{4}\ln|4z + 1| + C$
5. $\frac{1}{4}\ln|x^4 + 1| + C$ **7.** $2\ln|x^2 - 4| + C$
9. $\ln|x^3 - 2x| + C$ **11.** $\frac{1}{3}\ln|z^3 + 3z + 17| + C$
13. $\frac{1}{3}x^3 + \ln|x - 1| + C$ **15.** $x + \frac{1}{2}\ln|x^2 + 3| + C$
17. $e^{3x} + C$ **19.** $-e^{-x} + C$ **21.** $10{,}000\,e^{0.1x} + C$
23. $-1200\,e^{-0.7x} + C$ **25.** $\frac{1}{12}e^{3x^4} + C$
27. $-\frac{3}{2}e^{-2x} + C$ **29.** $\frac{1}{18}e^{3x^6 - 2} + C$
31. $\frac{1}{4}e^{4x} + 6/e^{x/2} + C$ **33.** $f(x) = h(x), \int f(x)\,dx = g(x)$
35. $F(x) = -\ln|3 - x| + C$

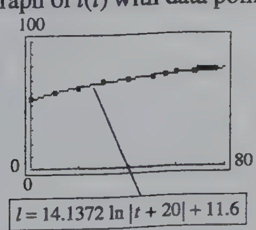


37. $\int\left(1 + \frac{1}{x}\right)dx$ **39.** $5e^{-x} - 5xe^{-x}$
41. $1030.97 **43.** $n = n_0e^{-Kt}$ **45.** 55
47. **(a)** $Pe^{0.1n}$ **(b)** approx. 7 yrs
49. **(a)** $p = 95e^{-0.491t}$ **(b)** ≈ 90.45
51. $l = 14.1372\ln|t + 20| + 11.6$

(a)

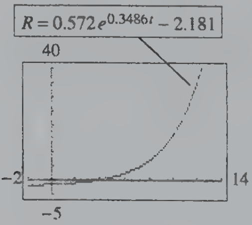


(b) Graph of $l(t)$ with data points



(c) They are a good match.

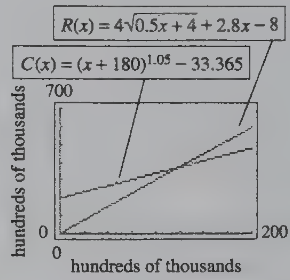
53. **(a)** $R = 0.572\,e^{0.3486t} + C$
(b)



(c) $24.29 billion

Exercise 12.4 *(page 900)*

1. $C(x) = x^2 + 100x + 200$
3. $C(x) = 2x^2 + 2x + 80$ **5.** $3750
7. **(a)** $x = 3$ units is optimal level
(b) $P(x) = -4x^2 + 24x - 200$ **(c)** loss of $164
9. **(a)** $3120 **(b)** 896
11. **(a)** $\overline{C}(x) = \frac{6}{x} + \frac{x}{6} + 8$ **(b)** $10.50
13. **(a)** and **(b)**



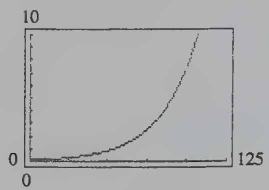
(c) Maximum profit is $114.743 thousand at $x = 200$ thousand units.

15. $C(y) = 0.4y + 0.6\sqrt{y} + 5$
17. $C(y) = 0.3y + 0.4\sqrt{y} + 8$
19. $C = 2\sqrt{y + 1} + 0.4y + 4$
21. $C = 0.7y + 0.5e^{-2y} + 5.15$
23. $C = 0.85y + 5.15$
25. $C = 0.8y + \frac{2\sqrt{3y + 7}}{3} + 4.24$

Exercise 12.5 *(page 910)*

1. $4y - 2xy' = 4x^2 - 2x(2x) = 0$ ✓
3. $2y\,dx - x\,dy = 2(3x^2 + 1)\,dx - x(6x\,dx) = 2\,dx$ ✓
5. $y = \frac{1}{2}e^{x^2 + 1} + C$ **7.** $y^2 = 2x^2 + C$
9. $y^3 = x^2 - x + C$ **11.** $y = e^{x - 3} - e^{-3} + 2$
13. $y = \ln|x| - \frac{x^2}{2} + \frac{1}{2}$ **15.** $\frac{y^2}{2} = \frac{x^3}{3} + C$
17. $\frac{1}{2x^2} + \frac{y^2}{2} = C$ **19.** $\frac{1}{x} + y + \frac{y^3}{3} = C$
21. $\frac{1}{y} + \ln|x| = C$ **23.** $x^2 - y^2 = C$
25. $y = C(x + 1)$ **27.** $y - \ln|y + 1| = -\frac{1}{2}e^{-2x} + C$
29. $3y^4 = 4x^3 - 1$ **31.** $2y = 3x + 4xy$ or $y = \frac{3x}{2 - 4x}$

33. $e^{2y} = x^2 - \frac{2}{x} + 2$ 35. $y^2 + 1 = 5x$
37. $y = Cx^k$
39. (a) $x = 10{,}000e^{0.06t}$ (b) \$10,618.37; \$13,498.59
(c) 11.55 years
41. $\approx$ 8.4 hours 43. $\approx$ 23,100 years
45. $x = 6(1 - e^{-0.05t})$ 47. $x = 20 - 10e^{-0.025t}$
49. $y = \frac{32}{(p+8)^{2/5}}$ 51. $V = 1.86e^{2-2e^{-0.1t}}$
53. $V = \frac{k^3t^3}{27}$ 55. $t \approx 4.5$ hours
57. (a) $E(t) = 0.1e^{0.043t}$
(b) The graph is a similar, but smooth, representation of the given data.



59. (a) $P(t) = 20{,}000e^{-0.05t}$
(b) $P(30) \approx$ \$4463. This result corresponds quite well to the graph.

Chapter 12 Review Exercises (page 914)

1. $\frac{1}{7}x^7 + C$ 2. $\frac{2}{3}x^{3/2} + C$
3. $\frac{1}{4}x^4 - x^3 + 2x^2 + 5x + C$
4. $\frac{1}{5}x^5 - \frac{2}{3}x^3 + x + C$
5. $\frac{1}{6}(x^2 - 1)^3 + C$ 6. $\frac{1}{6}(x^3 - 3x^2)^2 + C$
7. $\frac{1}{8}x^8 + \frac{8}{5}x^5 + 8x^2 + C$ 8. $\frac{1}{21}(x^3 + 4)^7 + C$
9. $\frac{1}{3}\ln|x^3 + 1| + C$ 10. $\frac{-1}{3(x^3+1)} + C$
11. $\frac{1}{2}(x^3 - 4)^{2/3} + C$ 12. $\frac{1}{3}\ln|x^3 - 4| + C$
13. $\frac{1}{2}x^2 - \frac{1}{x} + C$ 14. $\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x - \ln|x - 1| + C$
15. $\frac{1}{3}e^{y^3} + C$ 16. $x^3/3 - x^2 + x + C$
17. $\frac{1}{2}\ln|2x^3 - 7| + C$ 18. $\frac{-5}{4e^{4x}} + C$
19. $x^4/4 - e^{3x}/3 + C$ 20. $\frac{1}{2}e^{x^2+1} + C$
21. $\frac{-3}{40(5x^8+7)^2} + C$ 22. $-\frac{7}{2}\sqrt{1 - x^4} + C$
23. $\frac{1}{4}e^{2x} - e^{-2x} + C$ 24. $x^2/2 + 1/(x + 1) + C$
25. (a) $\frac{1}{10}(x^2 - 1)^5 + C$ (b) $\frac{1}{22}(x^2 - 1)^{11} + C$
(c) $\frac{3}{16}(x^2 - 1)^8 + C$ (d) $\frac{3}{2}(x^2 - 1)^{1/3} + C$
26. (a) $\ln|x^2 - 1| + C$ (b) $\frac{-1}{x^2 - 1} + C$
(c) $3\sqrt{x^2 - 1} + C$ (d) $\frac{3}{2}\ln|x^2 - 1| + C$
27. $y = C - 92e^{-0.05t}$
28. $y = 64x + 38x^2 - 12x^3 + C$
29. $(y - 3)^2 = 4x^2 + C$ 30. $(y + 1)^2 = 2\ln|t| + C$
31. $e^y = \frac{x^2}{2} + C$ 32. $y = Ct^4$
33. $3(y + 1)^2 = 2x^3 + 75$
34. $x^2 = y + y^2 + 4$ 35. 96 36. 472
37. $400[1 - 5/(t + 5) + 25/(t + 5)^2]$
38. $p = 1990.099 - 100{,}000/(t + 100)$
39. (a) $y = -60e^{-0.04t} + 60$ (b) 23%
40. $R = 800\ln(x + 1)$
41. (a) \$1000 (b) $C(x) = 3x^2 + 4x + 1000$
42. 80 units, \$440 43. $C = \sqrt{2y + 16} + 0.6y + 4.5$
44. $C = 0.8y - 0.05e^{-2y} + 7.85$ 45. $W = CL^3$
46. $\approx$ 10.7 million years 47. $x = 360(1 - e^{-t/30})$
48. $x = 600 - 500e^{-0.01t}$; $\approx$ 161 min

Chapter 12 Test (page 916)

1. $2x^3 + 4x^2 - 7x + C$ 2. $4x + \frac{2}{3}x\sqrt{x} + \frac{1}{x} + C$
3. $\frac{(4x^3 - 7)^{10}}{24} + C$ 4. $\frac{(3x^2 - 6x + 1)^{10}}{30} + C$
5. $\frac{\ln|2x^4 - 5|}{8} + C$ 6. $-10{,}000e^{-0.01x} + C$
7. $\frac{5}{8}e^{2y^4 - 1} + C$ 8. $e^x + 5\ln|x| - x + C$
9. $\frac{x^2}{2} - x + \ln|x + 1| + C$ 10. $6x^2 - 1 + 5e^x$
11. $y = x^4 + x^3 + 4$ 12. $y = \frac{1}{4}e^{4x} + \frac{7}{4}$
13. $y = \frac{4}{C - x^4}$ 14. 104,824
15. $P(x) = 450x - 2x^2 - 300$
16. $C(y) = 0.78y + \sqrt{0.5y + 1} + 5.6$ 17. 332.3 days

Exercise 13.1 (page 930)

1. 7 square units 3. 22 square units
5. 3 square units 7. 30 square units
9. $S_L(10) = 4.08$; $S_R(10) = 5.28$
11. Both equal 14/3.
13. It would lie between $S_L(10)$ and $S_R(10)$. It would equal 14/3.
15. 3 17. 42 19. -5 21. 180 23. 11,315
25. $3 - \frac{3(n+1)}{n} + \frac{(n+1)(2n+1)}{2n^2} = \frac{2n^2 - 3n + 1}{2n^2}$
27. (a) $S = (n - 1)/2n$ (b) 9/20 (c) 99/200
(d) 999/2000 (e) $\frac{1}{2}$
29. (a) $S = \frac{(n+1)(2n+1)}{6n^2}$
(b) $77/200 = 0.385$ (c) $6767/20{,}000 \approx 0.3384$
(d) $667{,}667/2{,}000{,}000 \approx 0.3338$ (e) $\frac{1}{3}$ 31. $\frac{20}{3}$

33. There are approximately 90 squares under the curve, each representing 1 second by $10 \text{ mph} = 10 \times \frac{1 \text{ hr}}{3600} \times \frac{1 \text{ mile}}{\text{hour}} = \frac{1}{360}$ mile.
The area under the curve is approximately $90\frac{1}{360} = \frac{1}{4}$ mile.
35. 1550 square feet

Exercise 13.2 (page 940)

1. 18 3. 2 5. 60 7. $12\sqrt[3]{25}$ 9. 98
11. $\frac{7}{3}$ 13. 12,960 15. 0 17. $8\sqrt{3} - \frac{7}{3}\sqrt{7}$
19. 2 21. $e^3/3 - 1/3$ 23. 4 25. 0
27. (a) $\frac{1}{6}\ln(112/31) \approx 0.2140853$ (b) 0.2140853
29. (a) $\frac{3}{2} + 3\ln 2 \approx 3.5794415$ (b) 3.5794415
31. (a) A, C (b) B
33. $\int_0^4 (2x - \frac{1}{2}x^2)\,dx$ (b) 16/3
35. (a) $\int_{-1}^0 (x^3 + 1)\,dx$ (b) 3/4
37. $\frac{1}{6}$ 39. $\frac{1}{2}(e^9 - e)$
41. same absolute values, opposite signs
43. 6 45. 20,405.39 47. 0.04 cm^3
49. 1222 (approximately) 51. $450,000
53. (a) $7007 (b) $19,649
55. (a) $y = 0.005963x^{0.701215}$
(b) 0.15 mile (approximately)

Exercise 13.3 (page 952)

1. (a) $\int_0^2 (4 - x^2)\,dx$ (b) $\frac{16}{3}$
3. (a) $\int_1^8 [\sqrt[3]{x} - (2 - x)]\,dx$ (b) 28.75
5. (a) $\int_1^2 [(4 - x^2) - (\frac{1}{4}x^3 - 2)]\,dx$ (b) 131/48
7. (a) $(-1, 1)$, $(2, 4)$ (b) $\int_{-1}^2 [(x + 2) - x^2]\,dx$
(c) 9/2
9. (a) $(0, 0)$, $(\frac{5}{2}, -\frac{15}{4})$
(b) $\int_0^{5/2} [(x - x^2) - (x^2 - 4x)]\,dx$ (c) $\frac{125}{24}$
11. (a) $(-2, -4)$, $(0, 0)$, $(2, 4)$
(b) $\int_{-2}^0 [(x^3 - 2x) - 2x]\,dx + \int_0^2 [2x - (x^3 - 2x)\,dx]$
(c) 8
13. $\frac{28}{3}$ 15. $\frac{1}{4}$ 17. $\frac{16}{3}$ 19. $\frac{1}{3}$ 21. $\frac{37}{12}$
23. $4 - 3\ln 3$ 25. $\frac{8}{3}$ 27. 6 29. 0 31. $-\frac{4}{9}$
33. $11.8\overline{3}$
35. 1980, 0.351; 1990, 0.377. The difference in incomes widened.
37. Whites, 0.391; Blacks, 0.439. In 1996, income was more equally distributed among whites.
39. average profit $= \dfrac{1}{x_1 - x_0}\displaystyle\int_{x_0}^{x_1} (R(x) - C(x))\,dx$
41. (a) $1402 (b) $535,333.33
43. (a) 102.5 units (b) 100 units
45. 7.40% 47. 147 milligrams

Exercise 13.4 (page 964)

1. $120,000 3. $346,664 (nearest dollar)
5. $506,000 (nearest thousand)
7. $18,660 (nearest dollar)
9. $82,155 (nearest dollar)
11. PV = $265,781 (nearest dollar), FV = $377,161 (nearest dollar)
13. PV = $190,519 (nearest dollar), FV = $347,148 (nearest dollar)
15. Gift shop, $151,024; Video rental, $141,093. Gift shop is a better buy.
17. $83.33 19. $161.89 21. (5, 56); $83.33
23. $11.50 25. $204.17 27. $2766.67
29. $17,839.58 31. $133.33 33. $2.50
35. $103.35

Exercise 13.5 (page 970)

1. $\frac{1}{8}\ln|(4 + x)/(4 - x)| + C$
3. $\frac{1}{3}\ln[(3 + \sqrt{10})/2]$ 5. $w(\ln w - 1) + C$
7. $\frac{1}{3} + \frac{1}{4}\ln(\frac{3}{7})$ 9. $3^x \log_3 e + C$ or $3^x/\ln 3 + C$
11. $\frac{1}{2}[7\sqrt{24} - 25\ln(7 + \sqrt{24}) + 25\ln 5]$
13. $\dfrac{(6w - 5)(4w + 5)^{3/2}}{60} + C$ 15. $\frac{1}{2}(5^{x^2})\log_5 e + C$
17. $\frac{1}{3}(13^{3/2} - 8)$ 19. $-\frac{5}{2}\ln\left|\dfrac{2 + \sqrt{4 - 9x^2}}{3x}\right| + C$
21. $\frac{1}{3}\ln|3x + \sqrt{9x^2 - 4}| + C$ 23. $\frac{1}{8}\ln(\frac{9}{5})$
25. $\frac{1}{3}\ln|3x + 1 + \sqrt{(3x + 1)^2 + 1}| + C$
27. $\frac{1}{4}[10\sqrt{109} - \sqrt{10} + 9\ln(10 + \sqrt{109}) - 9\ln(1 + \sqrt{10})]$
29. $-\frac{1}{6}\ln|7 - 3x^2| + C$
31. $\frac{1}{2}\ln|2x + \sqrt{4x^2 + 7}| + C$
33. $2(e^{\sqrt{2}} - e) \approx 2.7899$
35. $\frac{1}{32}[\ln(9/5) - 4/9] \approx .004479$ 37. $3391.10
39. (a) $C = \frac{1}{2}x\sqrt{x^2 + 9} + \frac{9}{2}\ln\left|\dfrac{x + \sqrt{x^2 + 9}}{3}\right| + 300$
(b) $314.94
41. $3882.9 thousand

Exercise 13.6 (page 977)

1. $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$ 3. $\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$
5. $\dfrac{104\sqrt{2}}{15}$ 7. $-(1 + \ln x)/x + C$ 9. 1
11. $\dfrac{x^2}{2}\ln(2x - 3) - \dfrac{1}{4}x^2 - \dfrac{3}{4}x - \dfrac{9}{8}\ln(2x - 3) + C$
13. $\frac{1}{5}(q^2 - 3)^{3/2}(q^2 + 2) + C$ 15. 282.4
17. $-e^{-x}(x^2 + 2x + 2) + C$ 19. $(3e^4 + 1)/2$
21. $\frac{1}{4}x^4\ln^2 x - \frac{1}{8}x^4\ln x + \frac{1}{32}x^4 + C$

23. $\frac{2}{15}(e^x + 1)^{3/2}(3e^x - 2) + C$ 25. $\frac{1}{2}e^{x^2} + C$
27. $\frac{2}{3}(e^x + 1)^{3/2} + C$ 29. $-5e^{-4} + 1$ 31. \$2794.46
33. \$34,836.73 35. 0.264

Exercise 13.7 (page 985)

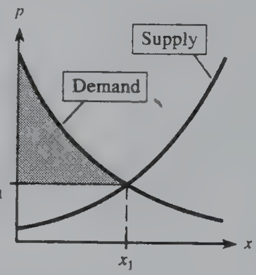
1. 1/5 3. 2 5. $1/e$ 7. diverges 9. diverges
11. 10 13. diverges 15. diverges 17. diverges
19. 0 21. 0.5 23. $1/(2e)$ 25. $\frac{3}{2}$
27. $\int_{-\infty}^{\infty} f(x)\,dx = 1$ 29. $c = 1$ 31. $c = \frac{1}{4}$
33. 20 35. area $= \frac{8}{3}$ 37. $\int_0^{\infty} Ae^{-rt}\,dt = A/r$
39. \$2,400,000 41. \$700,000
43. (a) $500\left[\frac{e^{-0.03b} + 0.03b - 1}{0.0009}\right]$ (b) limit $= \infty$
45. 0.147

Chapter 13 Review Exercises (page 989)

1. 212 2. $\frac{3(n+1)}{2n^2}$ 3. $\frac{91}{72}$ 4. 1 5. 1
6. 14 7. $\frac{248}{5}$ 8. $-\frac{205}{4}$ 9. $\frac{825}{4}$ 10. $\frac{125}{3}$
11. -2 12. $\frac{1}{6}\ln 47 - \frac{1}{6}\ln 9$ 13. $\frac{9}{2}$
14. $\ln 4 + \frac{14}{3}$ 15. $\frac{26}{3}$ 16. $\frac{1}{2}\ln 2$ 17. $(1 - e^{-2})/2$
18. $(e - 1)/2$ 19. 95/2 20. 36 21. $\frac{1}{4}$ 22. $\frac{1}{2}$
23. $\frac{1}{2}x\sqrt{x^2 - 4} - 2\ln\left|x + \sqrt{x^2 - 4}\right| + C$
24. $2\log_3 e$ 25. $\frac{1}{2}x^2(\ln x^2 - 1) + C$
26. $\frac{1}{2}\ln|x| - \frac{1}{2}\ln|3x + 2| + C$
27. $\frac{1}{6}x^6\ln x - \frac{1}{36}x^6 + C$
28. $(-xe^{-2x}/2) - (e^{-2x}/4) + C$
29. $2x\sqrt{x + 5} - \frac{4}{3}(x + 5)^{3/2} + C$ 30. 1 31. ∞
32. -100 33. $\frac{5}{3}$ 34. $-\frac{1}{2}$ 35. \$28,000
36. $e^{-2.8} \approx 0.061$ 37. \$1297.44 38. \$76.60
39. (a) $(7, 6)$ (b) \$7.33 40. \$24.50
41. \$1,621,803 42. (a) \$403,609 (b) \$602,114
43. \$217.42 44. \$86,557.41
45. $-\frac{x^2}{4} + \frac{7}{2}x + \frac{x^2 - 1}{2}\ln(x + 1) + 2000$
46. $e^{-1.4} \approx 0.247$
47. \$4000 thousand, or \$4 million

Chapter 13 Test (page 990)

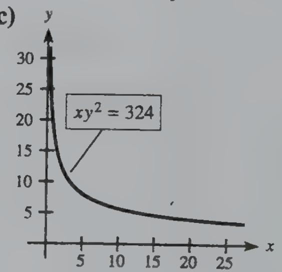
1. 3.496 (approximately) 2. (a) $5 - \frac{n+1}{n}$ (b) 4
3. $\int_0^6 (12 + 4x - x^2)\,dx$; 72
4. (a) 4 (b) 3/4 (c) $\frac{5}{4}\ln 5$ (d) 7
5. (a) $3xe^x - 3e^x + C$ (b) $\frac{x^2}{2}\ln(2x) - \frac{x^2}{4} + C$
6. -8
7. (a) $x[\ln(2x) - 1] + C$
   (b) $\frac{2(9x + 14)(3x - 7)^{3/2}}{135} + C$
8. 16.089
9. (a) \$4000 (b) \$16,000/3
10. (a) \$961.18 thousand (b) \$655.68 thousand
    (c) \$1062.5 thousand
11. 125/6 12.



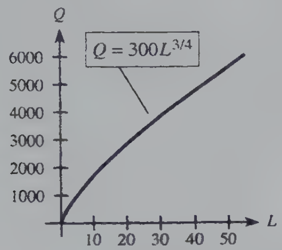
13. Before, 0.446; After, 0.19. The change decreases the difference in income.
14. (a) 567.357 million barrels per year
    (b) 641.589 million barrels per year

Exercise 14.1 (page 1002)

1. $\{(x, y)$: x and y are real numbers$\}$
3. $\{(x, y)$: x and y are real numbers and $y \neq 0\}$
5. $\{(x, y)$: x and y are real numbers and $2x - y \neq 0\}$
7. $\{(p_1, p_2)$: p_1 and p_2 are real numbers and $p_1 \geq 0\}$
9. -2 11. $\frac{5}{3}$ 13. 2500 15. 36 17. 3
19. $\frac{1}{25}\ln(12)$ 21. $\frac{13}{3}$
23. \$6640.23; the amount that results when \$2000 is invested for 20 years
25. 500; if the cost of placing an order is \$200, the number of items ordered is 625, and the weekly holding cost per item is \$1, then the most economical order size is 500.
27. Max: $S \approx 112.5°F$; $A \approx 106.3°F$
    Min: $S \approx 87.4°F$; $A \approx 77.6°F$
29. (a) \$752.80; when \$90,000 is borrowed for 20 years at 8%, the monthly payment is \$752.80.
    (b) \$1622.82; when \$160,000 is borrowed for 15 years at 9%, the monthly payment is \$1622.82.
31. (a) $x = 4$ (b) $y = 2$
    (c)



33. **(a)** 37,500
(b) $30(2K)^{1/4}(2L)^{3/4} = 30(2^{1/4})(2^{3/4})K^{1/4}L^{3/4} = 2[30\,K^{1/4}L^{3/4}]$
(c)



35. **(a)** 7200 **(b)** 5000 **37.** \$284,000

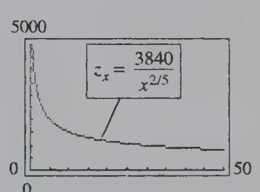
Exercise 14.2 (page 1013)

1. $\dfrac{\partial z}{\partial x} = 4x^3 - 10x + 6 \quad \dfrac{\partial z}{\partial y} = 9y^2 - 5$
3. $z_x = 3x^2 + 8xy \quad z_y = 4x^2 + 12y$
5. $\dfrac{\partial f}{\partial x} = 9x^2(x^3 + 2y^2)^2 \quad \dfrac{\partial f}{\partial y} = 12y(x^3 + 2y^2)^2$
7. $f_x = 2x(2x^2 - 5y^2)^{-1/2} \quad f_y = -5y(2x^2 - 5y^2)^{-1/2}$
9. $\dfrac{\partial C}{\partial x} = -4y + 20xy \quad \dfrac{\partial C}{\partial y} = -4x + 10x^2$
11. $\dfrac{\partial Q}{\partial s} = \dfrac{2(t^2 + 3st - s^2)}{(s^2 + t^2)^2} \quad \dfrac{\partial Q}{\partial t} = \dfrac{3t^2 - 4st - 3s^2}{(s^2 + t^2)^2}$
13. $z_x = 2e^{2x} + \dfrac{y}{x} \quad z_y = \ln x$
15. $\dfrac{\partial f}{\partial x} = 100ye^{xy} \quad \dfrac{\partial f}{\partial y} = 100xe^{xy}$ **17.** 2
19. 7 **21.** 0
23. **(a)** 0 **(b)** $-2xz + 4$ **(c)** $2y$ **(d)** $-x^2$
25. **(a)** $8x_1 + 5x_2$ **(b)** $5x_1 + 12x_2$ **(c)** 1
27. **(a)** 2 **(b)** 0 **(c)** 0 **(d)** $-30y$
29. **(a)** $2y$ **(b)** $2x - 8y$ **(c)** $2x - 8y$ **(d)** $-8x$
31. 0 **33.** -6
35. **(a)** $\frac{188}{4913}$ **(b)** $\frac{-188}{4913}$ **37.** $2 + 2e$
39. **(a)** $2 + y^2e^{xy}$ **(b)** $xye^{xy} + e^{xy}$
(c) $xye^{xy} + e^{xy}$ **(d)** x^2e^{xy}
41. **(a)** $1/x^2$ **(b)** 0 **(c)** 0 **(d)** $2 + 1/y^2$
43. **(a)** $24x$ **(b)** $24x$ **(c)** 0
45. **(a)** For a mortgage of \$100,000 and an 8% interest rate, the monthly payment is \$1289.
(b) The rate of change of the payment with respect to the interest rate is \$62.51. That is, if the rate goes from 8% to 9% on a \$100,000 mortgage, the approximate increase in the monthly payment is \$62.51.

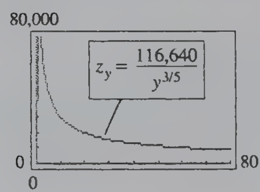
47. **(a)** If the number of items sold per week changes by 1, the most economical order quantity should also increase. $\dfrac{\partial Q}{\partial M} = \sqrt{\dfrac{K}{2Mh}} > 0$
(b) If the weekly storage costs change by 1, the most economical order quantity should decrease.
$\dfrac{\partial Q}{\partial h} = -\sqrt{\dfrac{KM}{2h^3}} < 0$
49. **(a)** $65e^{-0.01x}$ **(b)** $70e^{-0.02y}$
51. **(a)** $2xy^2$ **(b)** $2x^2y$
53. $\dfrac{\partial Q}{\partial K} = 44\frac{4}{9}$; If labor hours are held constant at 1728 and K changes by \$1 (thousand) to \$730,000, Q will change by about $44\frac{4}{9}$ thousand units. $\dfrac{\partial Q}{\partial L} = 37\frac{1}{2}$;
If capital expenditures are held constant at \$729,000 and L changes by 1 hour (to 1729), Q will change by about $37\frac{1}{2}$ thousand units.
55. **(a)** $\dfrac{\partial WC}{\partial s} = -0.020t - 1.85 + \dfrac{0.152t}{\sqrt{s}} - \dfrac{13.87}{\sqrt{s}}$
(b) at $t = 10$, $s = 25$, $\dfrac{\partial WC}{\partial s} = -0.20 - 1.85 + 0.304 - 2.774 = -4.52$
This means that the rate of change of WC with respect to s is -4.52 if $t = 10°F$, $s = 25$ mph. An increase in speed causes a decrease in wind chill temperature.

Exercise 14.3 (page 1021)

1. 57 **3.** **(a)** $2 + y/50$ **(b)** $4 + x/50$
5. **(a)** 20.18 **(b)** 70.80 **7.** **(a)** 36 **(b)** 19
9. **(a)** $\sqrt{y^2 + 1}$ **(b)** $xy/\sqrt{y^2 + 1}$
11. **(a)** $1200y/(xy + 1)$ **(b)** $1200x/(xy + 1)$
13. **(a)** $\sqrt{y/x}$ **(b)** $\sqrt{x/y}$
15. **(a)** $\ln(y + 1)/(2\sqrt{x})$ **(b)** $\sqrt{x}/(y + 1)$
17. $z = 1092$ (approximately) **19.** $z_x = 3.6$
21. **(a)** $z_x = \dfrac{240y^{2/5}}{x^{2/5}}$ **(b)**



(c) $z_y = \dfrac{160x^{3/5}}{y^{3/5}}$ **(d)**



(e) Both z_x and z_y are positive, so increases in both capital investment and work-hours result in increases in productivity. However, both are decreasing, so such increases have a diminishing effect on productivity. Also, z_y decreases more slowly than z_x, so that increases in work-hours have a more significant impact on productivity than do increases in capital investment.

23. $q_1 = 188$; $q_2 = 270$

25. any values for p_1 and p_2 that satisfy $6p_2 - 3p_1 = 100$ and that make q_1 and q_2 nonnegative, such as $p_1 = 10$, $p_2 = 21\frac{2}{3}$

27. (a) -3 (b) -2 (c) -6 (d) -5
(e) complementary

29. (a) -50 (b) $600/(p_B + 1)^2$
(c) $-400/(p_B + 4)^2$
(d) $400/(p_A + 4)^2$ (e) competitive

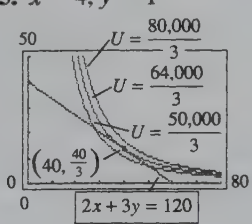
Exercise 14.4 *(page 1032)*

1. max (0, 0, 9) **3.** min(0, 0, 4) **5.** min(1, −2, 0)
7. saddle (1, −3, 8) **9.** saddle (0, 0, 0)
11. max(12, 24, 456) **13.** min(−8, 6, −52)
15. saddle (0, 0, 0); min(2, 2, −8)
17. $\hat{y} = 5.7x - 1.4$
19. $x = 5000, y = 128$ **21.** $x = \frac{20}{3}, y = \frac{10}{3}$
23. $x = 28, y = 100$ **25.** $x = 0, y = 10$
27. length = 100, width = 100, height = 50
29. $x = 3, y = 0$
31. (a) $\hat{y} = 2547.6x + 44{,}770$ (b) \$146,674
33. (a) $\hat{y} = 0.334x + 3.235$ (b) \$14.925
(c) Slope = 0.334 means that hourly earnings change at the rate of \$0.334 per year.

Exercise 14.5 *(page 1041)*

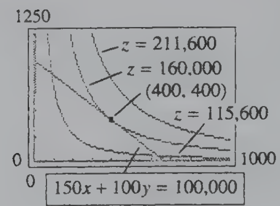
1. 18 at (3, 3) **3.** 35 at (3, 2) **5.** 32 at (4, 2)
7. −28 at $(3, \frac{5}{2})$ **9.** $\frac{1}{5}$ at $(-\frac{2}{5}, -\frac{1}{5})$ **11.** 3 at (1, 1, 1)

13. 1 at (0, 1, 0) **15.** $x = 4, y = 1$
17. $x = 40, y = \frac{40}{3}$



19. (a) $x = 400, y = 400$
(b) $-\lambda = 1.6$ means that each additional dollar spent on production results in approximately 1.6 additional units produced.

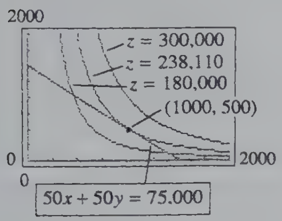
(c)



21. $x = 900, y = 300$
23. $x = \$10{,}003.33, y = \$19{,}996.67$
25. length = 100 cm, width = 100 cm, height = 50 cm

Chapter 14 Review Exercises *(page 1043)*

1. $\{(x, y)$: x and y are real numbers and $y \neq 2x\}$
2. $\{(x, y)$: x and y are real numbers with $y \geq 0$ and $(x, y) \neq (0, 0)\}$
3. −5 **4.** 896,000
5. $15x^2 + 6y$ **6.** $24y^3 - 42x^3y^2$
7. $z_x = 8xy^3 + 1/y$; $z_y = 12x^2y^2 - x/y^2$
8. $z_x = x/\sqrt{x^2 + 2y^2}$; $z_y = 2y/\sqrt{x^2 + 2y^2}$
9. $z_x = -2y/(xy + 1)^3$; $z_y = -2x/(xy + 1)^3$
10. $z_x = 2xy^3e^{x^2y^3}$; $z_y = 3x^2y^2e^{x^2y^3}$
11. $z_x = ye^{xy} + y/x$; $z_y = xe^{xy} + \ln x$
12. $z_x = y$; $z_y = x$ **13.** −8 **14.** 8
15. (a) $2y$ (b) 0 (c) $2x - 3$ (d) $2x - 3$
16. (a) $18xy^4 - 2/y^2$ (b) $36x^3y^2 - 6x^2/y^4$
(c) $36x^2y^3 + 4x/y^3$ (d) $36x^2y^3 + 4x/y^3$
17. (a) $2e^{y^2}$ (b) $4x^2y^2e^{y^2} + 2x^2e^{y^2}$
(c) $4xye^{y^2}$ (d) $4xye^{y^2}$
18. (a) $-y^2/(xy + 1)^2$ (b) $-x^2/(xy + 1)^2$
(c) $1/(xy + 1)^2$ (d) $1/(xy + 1)^2$
19. max(−8, 16, 208)
20. saddle(0, 0, 0); min(1, 1, −1)
21. 80 at (2, 8) **22.** 11,664 at (6, 3)
23. 3 units **24.** 20
25. (a) 62.8 km (approximately)
(b) If the surface temperature is 287 K, the Concorde's altitude is 17.1 km, and the vertical temperature gradient is 4.9 K/km, then the width of the sonic boom is about 63.3 km.
(c) $\dfrac{\partial f}{\partial h} \approx 1.87$ If the surface temperature is 293 K and the temperature gradient is 5 K/km, then a change of 1 km in the Concorde's altitude would result in a change in the width of the sonic boom of about 1.87 km.
(d) $\dfrac{\partial f}{\partial d} \approx -6.28$ If the surface temperature is 293 K and the Concorde's altitude is 16.8 km, then a change of 1 K/km in the temperature gradient would result in a change in the width of the sonic boom of about −6.28 km.

26. (a) 280 (b) 2400/7
27. $\partial Q/\partial K = 81.92$; $\partial Q/\partial L = 37.5$
28. (a) -2 (b) -6 (c) complementary
29. competitive 30. $x = 20, y = 40$
31. $x = 10, y = 4$
32. (a) $x = 1000, y = 500$
(b) $-\lambda \approx 3.17$ means that each additional dollar spent on production results in approximately 3 additional units.
(c) 

33. (a) $\hat{y} = 0.127x + 8.926$ (b) \$897.926 billion
34. $\hat{q} = 34{,}726 - 55.09p$

Chapter 14 Test *(page 1046)*

1. (a) all pairs (x, y) with $y < x^2$ (b) 14
2. $z_x = 5 + 10y(xy + 1)^4$ $z_y = -18y + 10x(xy + 1)^4$
$z_{xx} = 40y^2(xy + 1)^3$ $z_{yy} = -18 + 40x^2(xy + 1)^3$
$z_{xy} = z_{yx} = 10(5xy + 1)(xy + 1)^3$
3. (0, 2), a relative minimum; (4, −6) and (−4, −6), saddle points
4. (a) \$1625 thousand
(b) 73.11 means that if capital investment increases from \$10,000 to \$11,000, the expected change in monthly production value will be \$73.11 thousand, if labor hours remain at 1590.
(c) 0.56 means that if labor hours increase by 1 to 1591, the expected change in monthly production value will be \$0.56 thousand, if capital investment remains at \$10,000.
5. (a) When \$94,500 is borrowed for 25 years at 7%, the monthly payment is \$667.91
(b) If the percent goes from 7% to 8%, the expected change in the monthly payment is \$49.76, if the loan amount remains at \$94,500 for 25 years.
(c) Negative. If the loan amount remains at \$94,500 and the percent remains at 7%, increasing the time to pay off the loan will decrease the monthly payment, and vice versa.
6. $8xy\,e^{x^2y^2}(x^2y^2 + 1)$
7. Find $\dfrac{\partial q_1}{\partial p_2}$ and $\dfrac{\partial q_2}{\partial p_1}$ and compare their signs.
Both positive means competitive. Both negative means complementary. These products are complementary.
8. $x = \$7, y = \$11$
9. $x = 200, y = 100$
10. (a) $\hat{y} = 573.57x + 7652.05$
(b) \$15,682 (nearest dollar)
(c) No; a child cannot be 35 years old.

Index

Credits

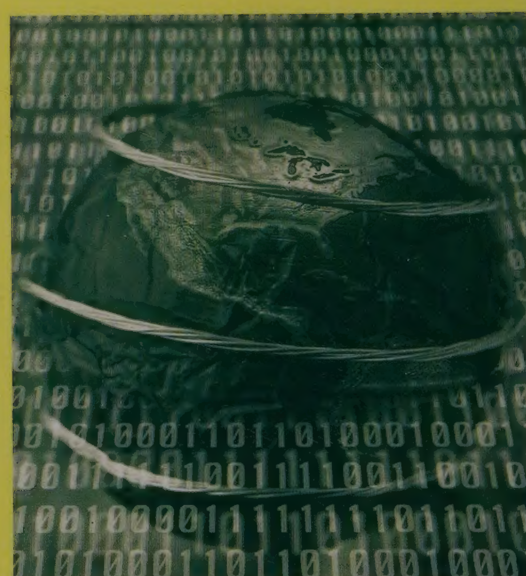
Chapter 1 "The Dow Jones Averages" chart: Republished by permission of Dow Jones, Inc. *via* Copyright Clearance Center, Inc. (c) Dow Jones and Company, Inc. All Rights Reserved Worldwide. "Dow Jones Activity: AP/Wide World Photos. **Chapter 2** "U.S. Trade Deficit" chart: Republished by permission of Dow Jones, Inc. *via* Copyright Clearance Center, Inc. (c) Dow Jones and Company, Inc. All Rights Reserved Worldwide. "E-mail Usage Chart": *Newsweek* (c) January 26, 1998 Newsweek, Inc. All rights reserved. Reprinted by permission. **Chapter 3** "Tardiness" and "Time" charts: Adapted from Bobrowki, P.M. and P.S. Part, "An evaluation of labor assignment rules," *Journal of Operations Management,* Vol. II Sept., 1993. **Chapter 5** "Price per share 1978–1993" chart: Republished by permission of Federal Signal Corporation. Metal Processors chart: Reprinted by permission of *Investor's Business Daily* "For People Who Chose To Succeed." "Basic Cable Rate" chart: Source: Nielsen Media Research. "World HIV Infections" chart: World Health Organization and the Global AIDS Policy Coalition, as reported in the *El Paso Times,* December 6, 1992. "Tax Foundation" chart: Copyright 1998, USA TODAY. Reprinted by permission. "First-Class Postage Rates": AP/Wide World Photos. "Rally Attendance" Chart: (c) 1997 Time Inc. Reprinted by permission. **Chapter 8** "Military Troops in Asia" chart: Reprinted with permission. Knight-Ridder/Tribune Information Services. **Chapter 9** "DIJA" at 5-Min Intervals Yesterday" chart: Republished by permission of Dow Jones, Inc. *via* Copyright Clearance Center, Inc. (c) Dow Jones and Company, Inc. All Rights Reserved Worldwide. "First-Class Postage Rates": AP/Wide World Photos. "Mutual Fund Growth History" chart: Reprinted with permission of VALIC (The Variable Annuity Life Insurance Company) from *Investment Digest* Vol 5 No 2. **Chapter 10** "Trade Deficit" chart: Republished by permission of Dow Jones, Inc. *via* Copyright Clearance Center, Inc. (c) Dow Jones and Company, Inc. All Rights Reserved Worldwide. "Dow Jones Closing Averages" and "Dow Jones Activity on March 31, 1994": AP/Wide World Photos. "Crime Statistics" chart: Copyright 1998, USA TODAY. Reprinted by permission. "Barometric Pressure" chart: By permission of the *Statesboro Herald.* **Chapter 12** "Revenue from Wireless Technology" chart: Copyright 1998, USA TODAY. Reprinted by permission. "National Health Expenditures" chart: *Newsweek* 4 October 1993 (c) Newsweek, Inc. All rights reserved. Reprinted by permission. "Fossil-Fuel Emissions" chart: *American Scientist,* Vol. 79, July-August 1990, p. 313. "Mutual Fund Growth History" chart: Reprinted with permission of VALIC (The Variable Annuity Life Insurance Company) from *Investment Digest* Vol 5 No 2. "Pension Dollars" graph: Reprinted with permission of VALIC (The Variable Annuity Life Insurance Company) from *Viewpoints* Summer 1993.

Social Science

Finance

Life Science

Miscellaneous



HOUGHTON MIFFLIN

New Ways to Know

ISBN 0-618-25801-9

90000

9 780618 258017

N-00959